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Second order difference scheme for singularly perturbed boundary turning point problems

Govindarajan Janani Jayalakshmi, Ayyadurai Tamilselvan*

Department of Mathematics, Bharathidasan University, Tiruchirappalli 620 024, Tamilnadu, India
Email(s): jananimath26@gmail.com, mathats@bdu.ac.in

Abstract. A singularly perturbed convection diffusion equation with boundary turning point is considered in this paper. A higher order method on piecewise uniform Shishkin mesh is suggested to solve this problem. We prove that this method is of order $O(N^{-2}(\ln N)^2)$. Numerical results are given which validate the analytical results.

Keywords: Singular perturbation, boundary turning point, hybrid difference scheme, Shishkin.
AMS Subject Classification 2010: 65L10; 65L11; 65L12; 65L20

1 Introduction

Many phenomena in biology, chemistry, engineering, physics, etc., can be described by boundary value problems associated with various types of differential equations or systems. Singular perturbation problems with turning points arise as mathematical models for various phenomena. The numerical analysis of singular perturbation problems has always been far from trivial because of the boundary layer behavior of the solution. Solutions of singular perturbation problems undergo rapid changes within very thin layers near the boundary or inside the problem domain [4, 11, 13, 14, 19]. It is well known that standard numerical methods for solving such problems are unstable and fail to give accurate results when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods to these problems whose accuracy does not depend on the parameter ε, i.e., methods that are parameter uniform convergent. For various approaches on the numerical solution of differential equations with steep, continuous solutions one may refer to [4, 11, 12, 19].

Boundary turning point problems arise in geophysics, where it models the heat flow and mass transport near an oceanic rise [6] and in modeling thermal boundary layers in laminar

*Corresponding author.
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flow [20, Chap 12]. Liseikin [9] have discussed the qualitative behavior of the solutions of singularly perturbed problems with arbitrary convection coefficients, as well as the exponential or power boundary layers. Numerical methods for singularly perturbed boundary turning point problem have been studied by many authors [2,3,10,15,16]. In [22], the authors have considered singularly perturbed multiple boundary turning point problem of the form

$$P_k^{\pm}[t,1] := -\varepsilon y'' \pm x^k b(x)y' + c(x)y = f(x), \qquad y(t) \text{ and } y(1) \text{ is given,} \tag{1}$$

where $t=0$ or $t=-1$ and $b(x)>0,\ c(x)\geq 0,\ \ x\in[t,1]$. The problem (1) discusses all possible cases modeling turning point behavior for t and k. Uniform methods for semilinear problems with an attractive boundary point have been studied by Linβ and Vulanovic in [8].

In [1, 17, 18, 21], the authors have studied higher order methods for singularly perturbed convection diffusion problems. In [15], the authors have proposed a parameter uniform numerical method for some linear and nonlinear singularly perturbed convection diffusion boundary turning point problems. They have proved that their method is of almost first order convergent. In this paper we have suggested a higher order method for linear singularly perturbed boundary turning point problems which is of almost second order. The constant C used throughout is generic and positive. We assume that $\varepsilon \leq CN^{-1}$ is generally the case for discretization of convection diffusion equations. Moreover, the maximum norm $\|u\|_D = \max\limits_{x\in\bar{D}}|u(x)|$ is used in error analysis.

Consider the following singularly perturbed convection diffusion problem with boundary turning point:

$$\begin{cases} L_\varepsilon y_\varepsilon(x) = \varepsilon y_\varepsilon''(x) + b_\varepsilon(x)y_\varepsilon'(x) - c(x)y_\varepsilon(x) = f(x), \quad x\in D=(0,1), \\ y_\varepsilon(0)=y_0, \quad y_\varepsilon(1)=y_n, \end{cases} \tag{2}$$

where $0<\varepsilon<<1,\ b_\varepsilon(x)\geq 0,\ c(x)\geq\delta>0$ and $f(x)$ are sufficiently smooth functions on $\bar{D}$ with the following assumptions on convection coefficient:

$$\begin{cases} b_\varepsilon(0)=0, \\ b_\varepsilon(x)\geq\beta_\varepsilon(x) := \theta(1-e^{-\frac{r}{\varepsilon}x}), \quad r\geq 2\theta>0, \\ \displaystyle\int_{t=0}^{x} |b_\varepsilon'(t)|dt \leq C, \\ d_\varepsilon(x) := b_0(x)-b_\varepsilon(x) \ \text{ satisfies } |d_\varepsilon(x)|\leq|d_\varepsilon(0)|e^{-\frac{\theta}{2\varepsilon}x}, \end{cases} \tag{3}$$

where $b_0(x) := \lim\limits_{\varepsilon\to 0} b_\varepsilon(x),\ b_0(0) := \lim\limits_{x\to 0} b_0(x)$ and $b_0\in C^2(\bar{D})$. This same condition on convection coefficient $b(x)\geq\beta(x)>0$ has been studied in semilinear convection difussion problem [8]. The differential operator defined in problem (2) satisfies the following minimum principle.

2 Analytical results

Theorem 1. *Let L_ε be the differential operator defined in (2) and $\psi(x)\in C^2(D)\cap C^0(\bar{D})$. If $\psi(0)\geq 0,\ \psi(1)\geq 0$ and $L_\varepsilon\psi(x)\leq 0$ for $x\in D,$ then $\psi(x)\geq 0, \forall\ x\in\bar{D}$.*

To derive error estimates, we need sharper bounds on the derivatives of the solution y_ε. We derive this using the following decomposition of the solution y_ε into regular component v_ε and layer component w_ε. Since b_ε does not satisfy the bound $b_\varepsilon \geq C > 0$ for all $x \in \bar{D}$, we study the problem

$$\begin{cases} L_* v_\varepsilon(x) = \varepsilon v_\varepsilon''(x) + b_0(x) v_\varepsilon'(x) - c(x) v_\varepsilon = f(x), \ x \in D, \\ v_\varepsilon(0) = \sum\limits_{i=0}^{3} \varepsilon^i v_i(0) \quad \text{and} \quad v_\varepsilon(1) = y_\varepsilon(1), \end{cases} \tag{4}$$

where $v_\varepsilon = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \varepsilon^3 v_3(x)$ and $v_0(x),\ v_1(x),\ v_2(x),\ v_3(x)$ satisfy

$$\begin{aligned} b_0(x) v_0'(x) - c(x) v_0(x) &= f(x), & v_0(1) &= y_n, \\ b_0(x) v_1'(x) - c(x) v_1(x) &= -v_0''(x), & v_1(1) &= 0, \\ b_0(x) v_2'(x) - c(x) v_2(x) &= -v_1''(x), & v_2(1) &= 0, \\ L_* v_3(x) &= -v_2''(x), & v_3(0) &= v_3(1) = 0. \end{aligned}$$

Note that in problem (4), the coefficient $b_\varepsilon(x)$, of the first derivative, has been replaced by $b_0(x)$. We incorporate the error $(L_* - L_\varepsilon) v_\varepsilon$ into the layer component w_ε, defined as the solution of

$$\begin{cases} L_\varepsilon w_\varepsilon(x) = d_\varepsilon(x) v_\varepsilon'(x), \quad x \in D, \\ w_\varepsilon(0) = y_\varepsilon(0) - v_\varepsilon(0) \quad \text{and} \quad w_\varepsilon(1) = 0. \end{cases} \tag{5}$$

The following theorem generalizes a result from [15].

Theorem 2. *For each integer k, satisfying $0 \leq k \leq 4$, the derivatives of the solutions $v_\varepsilon(x)$ and $w_\varepsilon(x)$ of (4) and (5), respectively, satisfy the following bounds:*

$$|v_\varepsilon^{(k)}(x)| \leq C(1 + \varepsilon^{3-k}) \quad \text{and} \quad |w_\varepsilon^{(k)}(x)| \leq C \varepsilon^{-k} e^{-\theta x / 2\varepsilon}, \quad x \in \bar{D}. \tag{6}$$

Proof. Since $v_0(x)$, $v_1(x)$, $v_2(x)$ are independent of ε and $v_3(x)$ is the solution of the problem (4), we have $\|v_3\| \leq C$ and so $\|v_\varepsilon\| \leq C(1 + \varepsilon^3)$. To bound the derivatives of v_3, integration by parts gives

$$\int_0^x b_0(t) v_3'(t) dt = [b_0(t) v_3(t)]_0^x - \int_0^x b_0'(t) v_3(t) dt. \tag{7}$$

Now,

$$\begin{aligned} \left| \int_0^x -b_0(t) v_3'(t) + c(t) v_3(t) - v_2''(t) \right| &= \left| \int_0^x -b_0(t) v_3'(t) dt + \int_0^x c(t) v_3(t) dt - \int_0^x v_2''(t) dt \right| \\ &\leq |b_0(x) v_3(x)| + |b_0(0) v_3(0)| + \int_0^x \left| b_0'(t) v_3(t) \right| dt \\ &\quad + \int_0^x |c(t) v_3(t)| \, dt + \int_0^x \left| v_2''(t) \right| dt \\ &\leq C \|v_3\|, \end{aligned} \tag{8}$$

where C depends on $\|b_0\|$, $\|b_0'\|$ and $\|c\|$. By the mean value theorem, there exists $z \in (0, \varepsilon)$ such that

$$\left|\varepsilon v_3'(z)\right| \leq 2\|v_3\|. \tag{9}$$

Now integrating the differential equation $L_* v_3(x)$ defined in (4), we get

$$\varepsilon v_3'(x) - \varepsilon v_3'(0) = \int_0^x v_2''(t) - b_0(t)v_3'(t) + c(t)v_3(t)dt, \quad \forall\, x \in D. \tag{10}$$

Letting $x = z$ and substituting (8) and (9) in (10), we obtain

$$\left|\varepsilon v_3'(0)\right| \leq C\|v_3\|. \tag{11}$$

Using Eqs. (11) in (10), we deduce that

$$\left|\varepsilon v_3'(x)\right| \leq C\|v_3\|, \quad \forall\, x \in \bar{D}, \tag{12}$$

which is the required bound for $k = 1$. Then, from (2), we have

$$\varepsilon v_3'' = -v_2'' - b_0 v_3' + c v_3 \quad \text{and} \quad \varepsilon v_3''' = (-v_2'' - b_0 v_3' + c v_3)',$$

from which we obtain successively the required bounds on the second and third derivatives. A similar argument holds for $k = 4$. Therefore, we have

$$|v_\varepsilon^{(k)}(x)| \leq C(1 + \varepsilon^{3-k}), \quad 0 \leq k \leq 4.$$

For the bound on layer component w_ε, we consider the functions

$$\phi^{\pm}(x) = |w_\varepsilon(0)| e^{-\frac{1}{2\varepsilon}\int_0^x \beta_\varepsilon(t)dt} \pm w_\varepsilon(x), \;\; x \in D.$$

Clearly $\phi^{\pm}(0) \geq 0$ and $\phi^{\pm}(1) \geq 0$. Using the assumptions on the coefficient $b_\varepsilon(x)$, we have

$$\frac{1}{2}\beta_\varepsilon(x)^2 + \varepsilon\beta_\varepsilon'(x) \geq \frac{\theta^2}{2}.$$

Now,

$$L_\varepsilon \phi^{\pm}(x) \leq -\frac{1}{2\varepsilon}\left(\left(\frac{1}{2}\beta_\varepsilon^2 + \varepsilon\beta_\varepsilon'\right)(x) - 2\varepsilon|d_\varepsilon(0)|\|y_\varepsilon'\|\right) e^{\frac{-\theta x}{2\varepsilon}} \leq 0.$$

Then, by Theorem 1, we get

$$|w_\varepsilon(x)| \leq |w_\varepsilon(0)| e^{-\frac{1}{2\varepsilon}\int_0^x \beta_\varepsilon(t)dt} \leq C e^{\frac{-\theta x}{2\varepsilon}}. \tag{13}$$

Using the mean value theorem and (13), there exists a point $n \in (1 - \varepsilon, 1)$ such that

$$\varepsilon|w_\varepsilon'(n)| \leq |w_\varepsilon(1 - \varepsilon)| \leq C e^{\frac{-\theta x}{2\varepsilon}}. \tag{14}$$

Integrating (5) on $[\eta, 1]$, for any $\eta > 0$ and using $w_\varepsilon(1) = 0$ from (2), we have

$$\varepsilon |w_\varepsilon'(\eta) - w_\varepsilon'(1)| \leq (\|b_\varepsilon\| + \|c\|)|w_\varepsilon(\eta)| + \frac{C}{\theta}|d_\varepsilon(0)|\varepsilon e^{\frac{-\theta}{2\varepsilon}\eta} + Ce^{\frac{-\theta}{2\varepsilon}\eta}. \tag{15}$$

Letting $\eta = n$, and then using (14) with (15), we get $\varepsilon|w_\varepsilon'(1)| \leq Ce^{\frac{-\theta}{2\varepsilon}\eta}$. Similarly, letting $\eta = x \in \bar{D}$, we get $\varepsilon|w_\varepsilon'(x)| \leq Ce^{\frac{-\theta}{2\varepsilon}x}$, which is the required bound for $k = 1$. From the differential equation, we have

$$\varepsilon w_\varepsilon'' = -b_\varepsilon w_\varepsilon' + cw_\varepsilon \quad \text{and} \quad \varepsilon w_\varepsilon''' = (-b_\varepsilon w_\varepsilon' + cw_\varepsilon)',$$

from which we obtain successively the required bounds on the second and third derivatives. A similar arguments holds for $k = 4$. Therefore, we have

$$|w_\varepsilon^{(k)}(x)| \leq C\varepsilon^{-k}e^{-\theta x/2\varepsilon}, \quad 0 \leq k \leq 4,$$

which completes the proof. □

3 Numerical analysis

3.1 Discretization of mesh

The fitted piecewise uniform mesh is constructed by dividing $\bar{D}$, into two subintervals $[0, \sigma]$ and $[\sigma, 1]$, for some transition point $\sigma = \min\{\frac{1}{2}, \frac{4\varepsilon}{\theta}\ln N\}$. On each subinterval, a uniform mesh with $N/2$ mesh intervals is placed. We define the piecewise uniform mesh:

$$\bar{D}_\varepsilon^N = \left\{ x_i : x_i = \begin{cases} 2i\sigma/N, & 0 \leq i \leq N/2, \\ \sigma + 2(i - N/2)(1-\sigma)/N, & N/2 < i \leq N \end{cases} \right\}, \ D_\varepsilon^N := \bar{D}_\varepsilon^N \backslash \{x_0, x_N\},$$

condensing at the boundary point at $x_0 = 0$. The mesh widths are given by

$$h_i = x_i - x_{i-1} = \begin{cases} H_1 = 2\sigma/N, & i = 1, 2, \ldots, N/2, \\ H_2 = 2(1-\sigma)/N, & i = N/2 + 1, \ldots, N. \end{cases}$$

3.2 Hybrid difference scheme

We discretize (2) by the hybrid difference scheme, where we use central difference scheme

$$\begin{aligned} L_c^N Y_\varepsilon(x_i) =& \frac{2\varepsilon}{h_i + h_{i+1}}\left[\frac{Y_\varepsilon(x_{i+1}) - Y_\varepsilon(x_i)}{h_{i+1}} - \frac{Y_\varepsilon(x_i) - Y_\varepsilon(x_{i-1})}{h_i}\right] \\ &+ b_\varepsilon(x_i)\left[\frac{Y_\varepsilon(x_{i+1}) - Y_\varepsilon(x_{i-1})}{h_i + h_{i+1}}\right] - c(x_i)Y_\varepsilon(x_i) = f(x_i), \end{aligned} \tag{16}$$

in the fine mesh region and midpoint scheme

$$\begin{aligned} L_m^N Y_\varepsilon(x_i) =& \frac{2\varepsilon}{h_i + h_{i+1}}\left[\frac{Y_\varepsilon(x_{i+1}) - Y_\varepsilon(x_i)}{h_{i+1}} - \frac{Y_\varepsilon(x_i) - Y_\varepsilon(x_{i-1})}{h_i}\right] \\ &+ \bar{b}_\varepsilon(x_i)\left[\frac{Y_\varepsilon(x_{i+1}) - Y_\varepsilon(x_i)}{h_{i+1}}\right] - \left[\frac{c(x_i)Y_\varepsilon(x_i) + c(x_{i+1})Y_\varepsilon(x_{i+1})}{2}\right] = \bar{f}(x_i), \end{aligned} \tag{17}$$

in the coarse region where $\bar{b}_\varepsilon(x_i) = (b_\varepsilon(x_i) + b_\varepsilon(x_{i+1}))/2$; similarly for $\bar{f}(x_i)$. Thus the hybrid difference scheme for the boundary value problem (2) is

$$\begin{cases} L_\varepsilon^N Y_\varepsilon(x_i) = \begin{cases} L_c^N Y_\varepsilon(x_i) = f_i, & \text{for } \ 1 \le i \le N/2 - 1, \\ L_m^N Y_\varepsilon(x_i) = \bar{f}_i, & \text{for } \ N/2 \le i \le N - 1, \end{cases} \\ Y_\varepsilon(x_0) = y_0, \quad Y_\varepsilon(x_N) = y_n. \end{cases} \tag{18}$$

From [5], we have the following truncation error for (18)

$$|L_\varepsilon^N (Y_\varepsilon - y_\varepsilon)(x_i)| \le \begin{cases} \varepsilon H_1^2 \|y_\varepsilon^{(4)}\| + H_1^2 \|b_\varepsilon\| \|y_\varepsilon^{(3)}\|, & i = 1, \dots, N/2 - 1, \\ \varepsilon H_2 \|y_\varepsilon^{(3)}\| + C_{(\|b_\varepsilon\|, \|b'_\varepsilon\|)} H_2^2 (\|y_\varepsilon^{(3)}\| + \|y_\varepsilon^{(2)}\|), & i = N/2, \dots, N - 1. \end{cases} \tag{19}$$

To guarantee the monotonicity property of the difference operator L_ε^N, we impose the following mild assumption on the minimum number of mesh points

$$\frac{N}{\ln \ N} \ge 4 \frac{\|b_\varepsilon\|}{\theta}. \tag{20}$$

3.3 Error analysis

Theorem 3. *Assume that the inequality (20) holds. Then the operator L_ε^N defined by (18) satisfies a discrete minimum principle, i.e., if ϕ_i and ψ_i are mesh functions that satisfy $\phi_0 < \psi_0$, $\phi_N < \psi_N$ and $L_\varepsilon^N \phi_i \le L_\varepsilon^N \psi_i$ for $i = 1, \dots, N - 1$, then $\phi_i \le \psi_i$ for $1 \le i \le N - 1$.*

Proof. Refer [7, Lemma 3.1]. □

Theorem 4. *The solution $Y_\varepsilon(x_i)$ satisfies the bound*

$$\|Y_\varepsilon\|_{\bar{D}_\varepsilon^N} \le C \ \max \Big\{ |Y_\varepsilon(x_0)|, \ |Y_\varepsilon(x_N)|, \|L_\varepsilon^N Y_\varepsilon\|_{D_\varepsilon^N} \Big\}.$$

Proof. Refer [4, Lemma 2.9] □

As in the continuous case, we decompose the solution Y_ε into sum of a discrete regular component V_ε and discrete layer component W_ε. We define the regular component as the solution of the following problem:

$$\begin{cases} L_*^N V_\varepsilon(x_i) = f(x_i), \ x_i \in D_\varepsilon^N, \\ V_\varepsilon(x_0) = v_\varepsilon(0) \ \text{ and } \ V_\varepsilon(x_N) = v_\varepsilon(1), \end{cases} \tag{21}$$

and the layer component is defined as:

$$\begin{cases} L_\varepsilon^N W_\varepsilon(x_i) = (L_*^N - L_\varepsilon^N) V_\varepsilon(x_i), \\ W_\varepsilon(x_0) = y_\varepsilon(0) - v_\varepsilon(0) \ \text{ and } \ W_\varepsilon(x_N) = w_\varepsilon(1). \end{cases} \tag{22}$$

Theorem 5. *Let V_ε and v_ε be the solution of the problems (21) and (4), respectively. Then the error of the regular component satisfies the bound*

$$|(V_\varepsilon - v_\varepsilon)(x_i)| \le CN^{-2}, \ x_i \in \bar{D}_\varepsilon^N.$$

Proof. Using (19), $\varepsilon \le CN^{-1}$ and bounds on the derivatives of v_ε, we have

$$|L_\varepsilon^N(V_\varepsilon - v_\varepsilon)(x_i)| \le \begin{cases} \varepsilon H_1^2 \|v_\varepsilon^{(4)}\| + H_1^2\|b_\varepsilon\|\|v_\varepsilon^{(3)}\|, & i = 1,\dots,N/2-1, \\ \varepsilon H_2\|v_\varepsilon^{(3)}\| + C_{(\|b_\varepsilon\|,\|b'_\varepsilon\|)} H_2^2(\|v_\varepsilon^{(3)}\| + \|v_\varepsilon^{(2)}\|), & i = N/2,\dots,N-1, \end{cases}$$
$$\le \begin{cases} CN^{-2}, & \text{for } i = 1,2,\dots,N/2-1, \\ CN^{-1}(\varepsilon + N^{-1}), & \text{for } i = N/2,\dots,N-1, \end{cases}$$
$$\le CN^{-2},\ x_i \in D_\varepsilon^N.$$

Applying Theorem 4 to the mesh function $(V_\varepsilon - v_\varepsilon)(x_i)$, we get the required result. □

Theorem 6. *Let W_ε be the numerical solution of the problem (22) and w_ε be the solution of (5). Then the error of the layer component satisfies*

$$|(W_\varepsilon - w_\varepsilon)(x_i)| \le CN^{-2}(\ln N)^2,\ x_i \in \bar{D}_\varepsilon^N. \tag{23}$$

Proof. The mesh is piecewise uniform and $\sigma = 4\varepsilon \ln N/\theta$. The mesh spacing in the subinterval $(0,\sigma)$ is $H_1 = 2\sigma/N$ and in the subinterval $(\sigma,1)$ is $H_2 = 2(1-\sigma)/N$.

If $x_i \in [\sigma,1)$, then by using the triangle inequality and the bound on w_ε given in (6), we have

$$|(W_\varepsilon - w_\varepsilon)(x_i)| \le |W_\varepsilon(x_i)| + |w_\varepsilon(x_i)| \le CN^{-2}.$$

Now, for $x_i \in [0,\sigma)$, we conclude that

$$\begin{aligned} L_\varepsilon^N(W_\varepsilon - w_\varepsilon)(x_i) &= L_\varepsilon^N W_\varepsilon(x_i) - L_\varepsilon^N w_\varepsilon(x_i) \\ &= d_\varepsilon(x_i)D^0V_\varepsilon - d_\varepsilon(x_i)D^0 v_\varepsilon \\ &= d_\varepsilon(x_i)D^0(V_\varepsilon - v_\varepsilon). \end{aligned}$$

Then from (3) and [17, Lemma 5.2], we have $|L_\varepsilon^N(W_\varepsilon - w_\varepsilon)(x_i)| \le CN^{-2}(\ln N)^2,\ x_i \in [0,\sigma)$. Applying Theorem 4, we get

$$|(W_\varepsilon - w_\varepsilon)(x_i)| \le CN^{-2}(\ln N)^2,\ x_i \in \bar{D}_\varepsilon^N,$$

which completes the proof. □

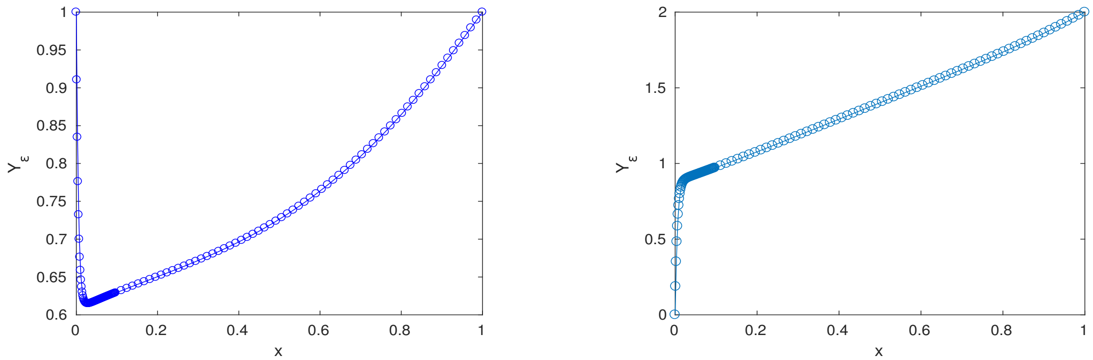
Theorem 7. *Let $y_\varepsilon(x)$ be the solution of the problem (2) and $Y_\varepsilon(x_i)$ be the corresponding numerical solution of (18). Then for sufficiently large N, the maximum pointwise error satisfies the following error bound*

$$\sup_{0<\varepsilon\le 1} \|Y_\varepsilon - y_\varepsilon\|_{\bar{D}_\varepsilon^N} \le CN^{-2}(\ln N)^2. \tag{24}$$

Proof. This follows from

$$(Y_\varepsilon - y_\varepsilon)(x_i) = ((V_\varepsilon - v_\varepsilon) + (W_\varepsilon - w_\varepsilon))(x_i),\quad \forall x_i \in \bar{D}_\varepsilon^N,$$

and Theorems 5 and 6. □



(a) Example 1 (b) Example 2

Figure 1: Graph of the numerical solution of for $N = 2^7$ and $\varepsilon = 10^{-2}$.

4 Experimental analysis

In this section, the theoretical results obtained in the earlier sections are verified experimentally through Examples 1 and 2.

Example 1.

$$\begin{cases} \varepsilon y''_\varepsilon(x) + 2\tanh\left(\frac{4x}{\varepsilon}\right) y'_\varepsilon(x) - (1-\cos(3x))y_\varepsilon(x) = \frac{1}{2} - x, \\ y_\varepsilon(0) = 1, \quad y_\varepsilon(1) = 1. \end{cases}$$

Example 2.

$$\begin{cases} \varepsilon y''_\varepsilon(x) + 2(1-e^{-4x/\varepsilon})y'_\varepsilon(x) - (1+x)y_\varepsilon(x) = \cos(2x), \\ y_\varepsilon(0) = 0, \quad y_\varepsilon(1) = 2. \end{cases}$$

The graphs of numerical solution of Examples 1 and 2 are plotted in the Figure 1. The computed maximum pointwise errors E^N and the uniform rates of convergence p^N, using the double mesh principle (see [4]) are displayed in Tables 1 and 2. The computed rates are in line with the theoretical rates of convergence established in Theorem 7. Example 1 is the test problem discussed in [15], where they have proved their method is of almost first order convergent. Error plot in given in Figure 3 which shows that the maximum pointwise error E^N decreases as N increases.

5 Discussion

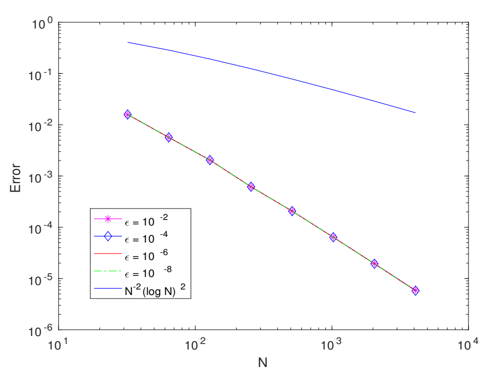
A higher order method was proposed for a class of linear singularly perturbed convection diffusion equations with boundary turning point. Riordan and Quinn [15] have proposed an almost first order convergent numerical method for some linear and nonlinear singularly perturbed convection diffusion boundary turning point problems. Example 1 is the test problem discussed in [15]. But we have proved that our numerical method is of almost order 2 which is evident from Table 1 and also supported by loglog plot and error plot given in Figures 2 and 3. One

Table 1: Computed maximum pointwise errors E^N and order of convergence p^N of Example 1 for various values of ε and N.

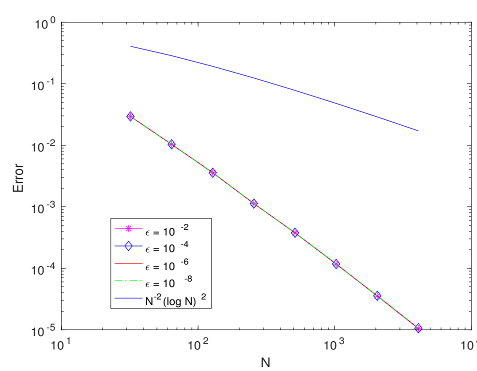
$\varepsilon \backslash N$	32	64	128	256	512	1024	2048	4096
10^{-1}	6.1305e-3	1.4809e-3	3.7066e-4	9.2927e-5	2.3488e-5	6.0077e-6	1.5709e-6	5.6965e-7
10^{-2}	1.5770e-2	5.6421e-3	2.0428e-3	6.1557e-4	2.0669e-4	6.4173e-5	1.9585e-5	5.8978e-6
10^{-3}	1.5857e-2	5.6667e-3	2.0499e-3	6.1692e-4	2.0664e-4	6.3909e-5	1.9437e-5	5.8037e-6
10^{-4}	1.5865e-2	5.6690e-3	2.0505e-3	6.1696e-4	2.0657e-4	6.3839e-5	1.9402e-5	5.7820e-6
10^{-5}	1.5866e-2	5.6692e-3	2.0505e-3	6.1697e-4	2.0656e-4	6.3832e-5	1.9398e-5	5.7797e-6
10^{-6}	1.5866e-2	5.6692e-3	2.0505e-3	6.1697e-4	2.0656e-4	6.3831e-5	1.9397e-5	5.7795e-6
10^{-7}	1.5866e-2	5.6692e-3	2.0505e-3	6.1697e-4	2.0656e-4	6.3831e-5	1.9397e-5	5.7794e-6
10^{-8}	1.5866e-2	5.6692e-3	2.0505e-3	6.1697e-4	2.0656e-4	6.3831e-5	1.9397e-5	5.7794e-6
E^N	1.5866e-2	5.6692e-3	2.0505e-3	6.1697e-4	2.0669e-4	6.4173e-5	1.9585e-5	5.8978e-6
p^N	**1.4847**	**1.4672**	**1.7327**	**1.5777**	**1.6874**	**1.7122**	**1.7315**	-

Table 2: Computed maximum pointwise errors E^N and order of convergence p^N of Example 2 for various values of ε and N.

$\varepsilon \backslash N$	32	64	128	256	512	1024	2048	4096
10^{-1}	1.1017e-2	2.8613e-3	6.9589e-4	1.6365e-4	3.5804e-5	1.4265e-5	7.1177e-6	3.5552e-6
10^{-2}	2.9427e-2	1.0359e-2	3.5811e-3	1.1252e-3	3.7628e-4	1.1690e-4	3.5326e-5	1.0400e-5
10^{-3}	2.9398e-2	1.0352e-2	3.5765e-3	1.1263e-3	3.7676e-4	1.1732e-4	3.5628e-5	1.0573e-5
10^{-4}	2.9395e-2	1.0351e-2	3.5760e-3	1.1263e-3	3.7678e-4	1.1735e-4	3.5646e-5	1.0584e-5
10^{-5}	2.9395e-2	1.0351e-2	3.5759e-3	1.1263e-3	3.7678e-4	1.1735e-4	3.5648e-5	1.0585e-5
10^{-6}	2.9395e-2	1.0351e-2	3.5759e-3	1.1263e-3	3.7678e-4	1.1735e-4	3.5648e-5	1.0585e-5
10^{-7}	2.9395e-2	1.0351e-2	3.5759e-3	1.1263e-3	3.7678e-4	1.1735e-4	3.5648e-5	1.0585e-5
10^{-8}	2.9395e-2	1.0351e-2	3.5759e-3	1.1263e-3	3.7678e-4	1.1735e-4	3.5648e-5	1.0585e-5
E^N	2.9427e-2	1.0359e-2	3.5811e-3	1.1263e-3	3.7678e-4	1.1735e-4	3.5648e-5	1.0585e-5
p^N	**1.5063**	**1.5324**	**1.6688**	**1.5798**	**1.6829**	**1.7189**	**1.7518**	-

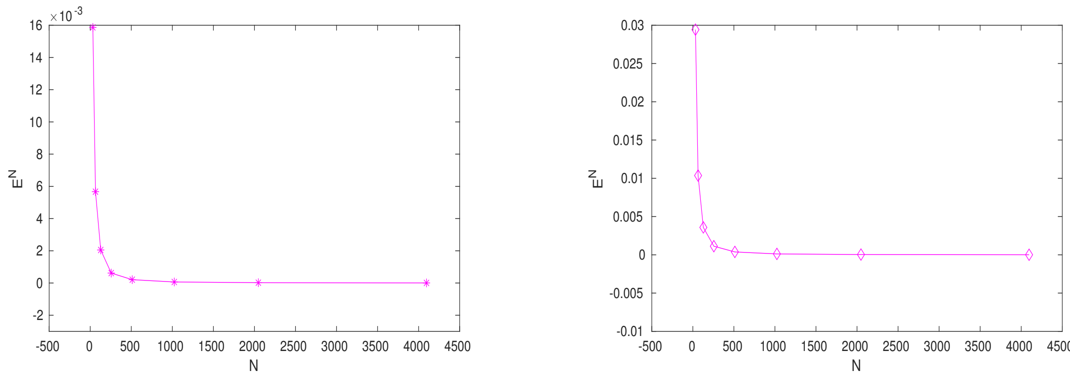


(a) Example 1



(b) Example 2

Figure 2: Loglog plot of pointwise error calculated in Tables 1 and 2 for different values of ε.



(a) Example 1 (b) Example 2

Figure 3: Error plot of computed maximum pointwise error calculated in Tables 1 and 2 for different values of N.

more example and its corresponding numerical results are given through Table 2, Figures 1, 2 and 3 which further validate our theoretical results.

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