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# Alpha power Maxwell distribution: Properties and application

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**Abstract.** In this study, alpha power Maxwell (APM) distribution is obtained by applying alpha power transformation, a reparametrized version of the Exp-G family of distributions, to the Maxwell distribution. Some tractable properties of the APM distribution are provided as well. Parameters of the APM distribution are estimated by using the maximum likelihood method. The APM distribution is used to model a real data set and its modeling capability is compared with different distributions, which can be considered its strong alternatives.

*Keywords*: Alpha Power Transformation, exp-G family of distribution, maximum likelihood, Maxwell distribution.

*AMS Subject Classification 2010*: 62E15, 60E05, 62F10.

## 1 Introduction

Statistical distributions are used for modeling data in almost every field of science. Maxwell distribution is one of these distributions, particularly in studies involved with Physics and Chemistry. It was first formed for describing speeds of molecules in thermal equilibrium by Maxwell [20]. In addition to applications in Physics and Chemistry, Tyagi and Bhattacharya [29,30] used the Maxwell distribution for the first time in Statistics for modeling lifetime data. The Maxwell distribution is a submodel of the generalized Weibull (GW) distribution proposed by Al-Mutairi and Agarwal [1]. Thus, it shows some useful features of the GW distribution in terms of lifetime testing; see Beker and Roux [5]. Some theoretical properties of the Maxwell distribution are studied in previous works, e.g., see Arslan et al. [3] and references therein. For the sake of brevity, these are not provided here.

The probability density function (pdf) and cumulative distribution function (cdf) of the

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Maxwell distribution are

$$f_Y(y;\sigma) = \frac{4}{\sigma\Gamma(1/2)}\left(\frac{y}{\sigma}\right)^2 \exp\left(-\left(\frac{y}{\sigma}\right)^2\right), \quad y>0, \quad \sigma>0, \tag{1}$$

and

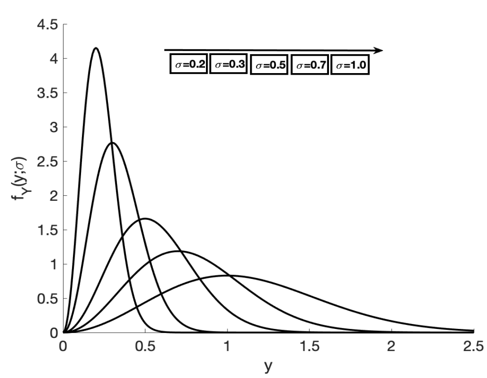
$$F_Y(y;\sigma) = \frac{1}{\Gamma(3/2)}\Gamma\left[\left(\frac{y}{\sigma}\right)^2, \frac{3}{2}\right], \quad y>0, \quad \sigma>0, \tag{2}$$

respectively. Here, $\sigma$ is the scale parameter, $\Gamma(\cdot)$ is the Gamma function and $\Gamma(\cdot,\cdot)$ is the incomplete Gamma function defined as

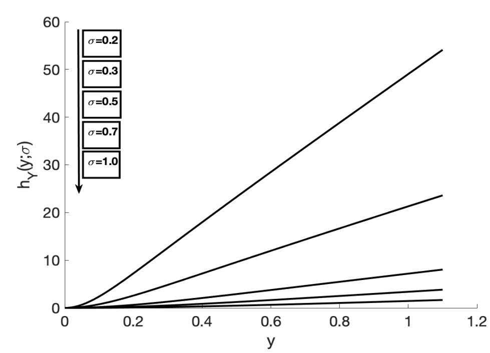
$$\Gamma(x,s) = \int_0^x u^{s-1}e^{-u}du.$$

Hereinafter, random variable $Y$ following the Maxwell distribution will be denoted by $Y \sim Maxwell(\sigma)$.

As it is known, skewness and kurtosis measures are functions of the shape parameter(s). If a distribution does not have a shape parameter, skewness and kurtosis measures take only constant values. This result delimits the modeling capability of a distribution. For example, the Maxwell distribution only takes values 2.0589 and 0.3217 for the skewness and kurtosis measures, respectively. The shape parameter also gives extra flexibility to the hazard rate function (hrf) of a distribution used in lifetime data analysis. The shape of the hrf of a distribution can be variate based on the shape parameter(s). For example, the hrf of Maxwell distribution, $h_Y(y;\sigma) = f_Y(y;\sigma)/(1-F_Y(y;\sigma))$, can only be increasing function for different values of the scale parameter $\sigma$. The Maxwell distribution density and hrf plots for certain values of $\sigma$ are given in Figure 1.




(a) pdf




(b) hrf

Figure 1: The plots of density and hrf of the Maxwell distribution for certain values of $\sigma$.

Different forms of the Maxwell distribution have been studied in the literature. For example, Voda [32] derived generalized Maxwell distribution by using a modified Weibull hazard rate. Kazmi et al. [18] used a two-component mixture of the Maxwell distribution for censored data.

Iriarte and Astorga [13] and Iriarte et al. [14] proposed the transmuted Maxwell (TM) and Gamma-Maxwell (GM) distributions, respectively. They also provided the method of moment (MoM) and maximum likelihood (ML) estimators of the unknown distribution parameters and performed an application on the wind speed data. Sharma et al. [27] introduced the extended Maxwell distribution and provided some statistical inference. Yadav et al. [33] proposed a new generalization of the Maxwell distribution using the power transformation of the Maxwell random variable. Sindhu et al. [28] obtained the inverted Maxwell mixture distribution. They provided the ML and Bayesian estimations of reliability function as well. Acitas et al. [2] derived slash Maxwell distribution and compared its modeling performance with its rivals.

There are several methods for extending/generating distribution. Lee et al. [19] summarized these methods under some topics such as the method of differential equations, quantile function, and transformation of random variables. Recently, Mahdavi and Kundu [21] used distribution generating method called alpha power transformation (APT). Let $F_T(t)$ be a cdf of a random variable $T$. The APT of the $F_T(t)$ for $t \in \mathbb{R}$ is

$$F_{APT}(t) = \begin{cases} \dfrac{\alpha^{F_T(t)} - 1}{\alpha - 1}, & \alpha > 0, \quad \alpha \neq 1, \\ F_T(t), & \alpha = 1. \end{cases}$$

Then, the corresponding pdf is

$$f_{APT}(t) = \begin{cases} \dfrac{\ln \alpha}{\alpha - 1} f_T(t) \alpha^{F_T(t)}, & \alpha > 0, \quad \alpha \neq 1, \\ f_T(t), & \alpha = 1. \end{cases}$$

The idea of the APT method is based on adding an extra parameter to the existing distribution. This method has some useful properties and features due to its easy applicability; see Mahdavi and Kundu [21] for further information.

It should be emphasized that the APT method is obtained by reparametrization $\alpha = \exp(-\lambda)$ of the exp-$G$ family of distributions introduced by Barreto-Souza and Simas [4]. It is also equivalent to the "truncated-exponential skew-symmetric" family of distributions proposed by Nadarajah et al. [22] as stated in Jones [16]. Despite the fact that the APT method is a reparametrized version of the exp-$G$ family of distributions, the APT terminology will be used in remaining of this study to be parallel to the recent literature.

In the literature, there exist many different distributions obtained by using APT method. For example; Nassar et al. [23], Dey et al. [7], Mahdavi and Kundu [21], Dey et al. [9], Unal et al. [31], Hassan et al. [12], Dey et al. [10], Ihtisham et al. [15] used APT method on the Weibull, generalized Exponential, Exponential, Lindley, inverted Exponential, extended Exponential, inverse Lindley, Pareto distributions, respectively. See also Dey et al. [8], in the context of the Weibull distribution.

The motivation of this study comes from the improvement of the modeling capability of the Maxwell distribution in terms of the skewness and kurtosis measures via the APT method. The resulting distribution is called alpha power Maxwell (APM) distribution. Some properties of the APM distribution, such as weighted representation, mixture representation, and stochastic ordering, are shown. The ML estimations of the unknown parameters of the APM distribution are provided. A real data set is used to show the implementation of the APM distribution and compare its modeling performance with its strong rivals. Note that the earlier version of this study was presented at International Conference on Data Science, Machine Learning and Statistics (DMS-2019); see Erdogan et al. [11].

The rest of this study is organized as follows. Section 2 includes the derivation of the APM distribution and some properties of it. Estimation of its parameters via the ML method is provided in Section 3. A real data set is used to show the implementation of the APM distribution in Section 4. The paper is finalized with some concluding remarks in Section 5.

## 2 The APM distribution

In this section, the APT method is applied to the Maxwell distribution. The resulting distribution is called alpha power Maxwell (APM), and from now on, the random variable $X$ following the APM distribution will be denoted by $X \sim APM(\alpha, \sigma)$.

The pdf, cdf, survival functions, hrf, and quantile function of the APM distribution are obtained. Then, moments of the APM distribution are given. Also, some tractable properties, such as mixing representation, weighted representation, and stochastic ordering, are provided.

**Definition 1.** *Let $X \sim APM(\alpha, \sigma)$, then the random variable $X$ has the pdf*

$$f_X(x;\alpha,\sigma) = \begin{cases} \dfrac{\ln\alpha}{\alpha-1}\dfrac{4}{\sigma\Gamma(1/2)}\left(\dfrac{x}{\sigma}\right)^2 \exp\left[-\left(\dfrac{x}{\sigma}\right)^2\right]\alpha^{\frac{1}{\Gamma(3/2)}\Gamma\left[\left(\frac{x}{\sigma}\right)^2,\frac{3}{2}\right]}, & x>0, \quad \alpha>0, \quad \alpha\neq 1, \\ f_Y(x;\sigma), & x>0, \quad \alpha=1, \end{cases} \tag{3}$$

*and the corresponding cdf is*

$$F_X(x;\alpha,\sigma) = \begin{cases} \dfrac{\alpha^{\frac{1}{\Gamma(3/2)}\Gamma\left[\left(\frac{x}{\sigma}\right)^2,\frac{3}{2}\right]}-1}{\alpha-1}, & x>0, \quad \alpha>0, \quad \alpha\neq 1, \\ F_Y(x;\sigma), & x>0, \quad \alpha=1. \end{cases} \tag{4}$$

*Here, $\sigma(>0)$ is the scale parameter, and $\alpha$ is the shape parameter.*

**Proposition 1.** *Let $\alpha \to 1$, then the APM distribution tends to the Maxwell distribution, i.e.*

$$\lim_{\alpha\to 1} f_X(x;\alpha,\sigma) \longrightarrow f_Y(x;\sigma).$$

*Proof.* It follows from $\lim_{\alpha\to 1}\dfrac{\ln\alpha}{\alpha-1}=1$ and $\lim_{\alpha\to 1}\alpha^{F_Y(y;\sigma)}{=}1$. See also Barreto-Souza and Simas [4], p. 86, in the context of the exp-$G$ famiy of distributions. □

**Definition 2.** *The survival function and hrf of the APM distribution, i.e. $S_X(x;\alpha,\sigma)$ and $h_X(x;\alpha,\sigma)$, are*

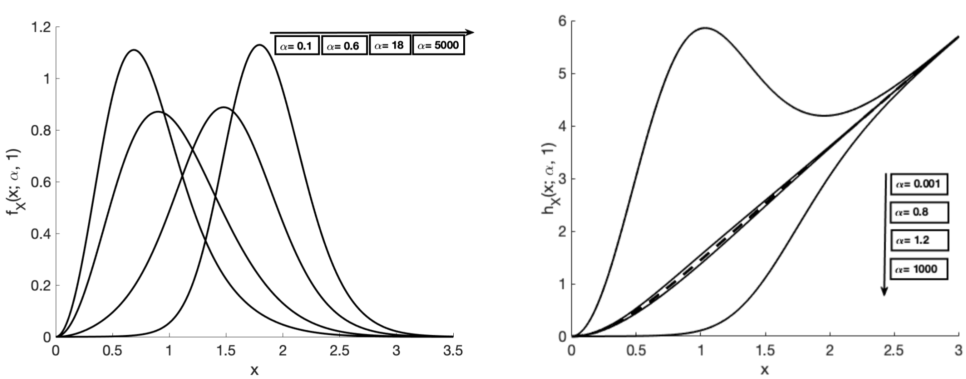
$$S_X(x;\alpha,\sigma) = \begin{cases} \frac{\alpha}{\alpha-1}\left(1-\alpha^{\frac{1}{\Gamma(3/2)}\Gamma\left[\left(\frac{x}{\sigma}\right)^2,\frac{3}{2}\right]-1}\right), & x>0, \quad \alpha>0, \quad \alpha\neq 1, \\ S_Y(x;\sigma), & x>0, \quad \alpha=1 \end{cases}$$

*and*

$$h_X(x;\alpha,\sigma) = \begin{cases} \frac{4}{\sigma\Gamma(1/2)} \ln(\alpha) \left(\frac{x}{\sigma}\right)^2 \exp\left[-\left(\frac{x}{\sigma}\right)^2\right] \frac{\alpha^{\frac{1}{\Gamma(3/2)}\Gamma\left[\left(\frac{x}{\sigma}\right)^2, \frac{3}{2}\right]-1}}{1-\alpha^{\frac{1}{\Gamma(3/2)}\Gamma\left[\left(\frac{x}{\sigma}\right)^2, \frac{3}{2}\right]-1}}, & x>0, \quad \alpha>0, \quad \alpha\neq 1, \\ \frac{f_Y(x;\sigma)}{S_Y(x;\sigma)}, & x>0, \quad \alpha = 1, \end{cases}$$

*respectively. Here,* $S_Y(x;\sigma)$ *is survival function of the Maxwell distribution, i.e.* $1 - F_Y(x;\sigma)$.

In Figure 2, the pdf and hrf of the APM($\alpha$, $\sigma$) distribution are plotted for the different values of shape parameter $\alpha$, where $\sigma$ is taken to be 1 for the sake of simplicity.




(a) pdf (b) hrf

Figure 2: The shape of density and hrf of the APM($\alpha$, $\sigma$) distribution for different values of the shape parameter; $\sigma = 1$.

Note that, in Figure 2(b), the Maxwell distribution is represented by dashed line ($--$). It can be easily seen from Figure 2(b) that the shape parameter provides flexibility to the hrf of the APM distribution. The shape of the hrf of the APM distribution can be concave and convex for certain values of $\alpha$. It should be realized that the hrf of the APM distribution tends to Maxwell's counterpart when $\alpha \to 1$.

**Remark 1.** *Note that the APM distribution tends to the Maxwell distribution when shape paramter* $\alpha$ *equals 1. Therefore, Definitions 3-7 and Theorem 1 are obtained for* $\alpha \neq 1$, *i.e., not for the Maxwell distribution, for the sake of brevity.*

**Definition 3.** *The p-th quantile of the APM distribution is*

$$x_p = \sigma \left( \Gamma^{-} \left[ \Gamma(3/2) \frac{\ln(1+p(\alpha-1))}{\ln\alpha}, \frac{3}{2} \right] \right)^{\frac{1}{2}}, \quad 0<p<1, \quad \alpha>0, \quad \alpha\neq 1,$$

*where* $\Gamma^{-}(\cdot,\cdot)$ *is the inverse of the incomplete Gamma function.*

**Definition 4.** *The APM distribution is expressed as a weighted distribution given below*

$$f_X(x) = \frac{f_Y(x)w(x;\alpha)}{c},$$

*where* $w(x) = \alpha^{F_Y(x)}$ *and* $c = E[w(X)]$*. Here,* $c$ *represents a normalizing constant and* $E[\cdot]$ *denotes the expected value of it. The weight function* $w(X)$ *can be increasing or decreasing according to* $\alpha > 1$ *or* $\alpha < 1$*, respectively.*

**Definition 5.** *The APM distribution has the following mixture representation for* $\alpha > 1$*:*

$$X = \begin{cases} Y, & \text{with probability} \quad \left(\frac{\ln\alpha}{\alpha-1}\right), \\ Z, & \text{with probability} \quad 1-\left(\frac{\ln\alpha}{\alpha-1}\right), \end{cases}$$

*where* $Y \sim Maxwell(\sigma)$ *and* $Z$ *have the following pdf,*

$$f_Z(z) = \frac{\ln\alpha}{\alpha-1-\ln\alpha} f_Y(z)\left[\alpha^{F_Y(z)}-1\right].$$

**Definition 6.**

$$\begin{aligned} E(X^r) =& \frac{\ln\alpha}{\alpha-1}\int_0^\infty x^r f_y(x;\sigma)\alpha^{F_Y(x;\sigma)}dx \\ =& \frac{\ln\alpha}{\alpha-1}\frac{4}{\sigma\Gamma(1/2)}H(x), \end{aligned}$$

*where*

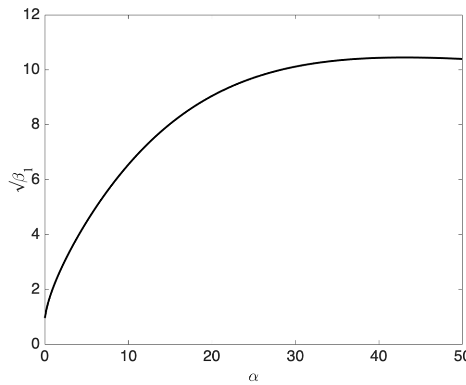
$$H(x) = \int_0^\infty x^r \left(\frac{x}{\sigma}\right)^2 \exp\left[-\left(\frac{x}{\sigma}\right)^2\right]\alpha^{\frac{1}{\Gamma(3/2)}\Gamma\left[\left(\frac{x}{\sigma}\right)^2,\frac{3}{2}\right]}.$$

Note that the closed-form of the expression for $E(X^r)$ may not be straightforward. Therefore, the values of variance ($Var[X]$), expected value ($E[X]$), skewness ($\sqrt{\beta_1}$) and kurtosis ($\beta_2$) measures of the APM distribution are tabulated for the certain values of shape parameter $\alpha$; see Table 1.

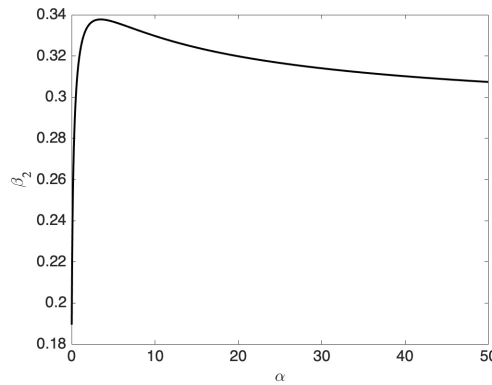
Table 1: The $E[X], V[X], \sqrt{\beta_1}$ and $\beta_2$ values of the APM distribution for the certain values of $\alpha$; $\sigma = 0.5$.

| | $\alpha$ | | | | | |
|---|---|---|---|---|---|---|
| | 0.1 | 0.5 | 0.99 | 1.001 | 2 | 5 |
| $E[X]$ | 0.4252 | 0.5185 | 0.5635 | 0.5643 | 0.6107 | 0.6704 |
| $Var[X]$ | 0.0404 | 0.0530 | 0.0566 | 0.0567 | 0.0585 | 0.0580 |
| $\sqrt{\beta_1}$ | 1.1018 | 1.6107 | 2.0508 | 2.0597 | 2.7772 | 4.4553 |
| $\beta_2$ | 0.2378 | 0.2997 | 0.3215 | 0.3218 | 0.3347 | 0.3366 |

It can be seen from the Table 1 that when $\alpha$ approaches to 1, the corresponding measures of the APM distribution converges to those of the Maxwell distribution; i.e. $E[X] = 0.5642$, $Var[X] = 0.0567$, $\sqrt{\beta_1} = 2.0589$, $\beta_2 = 0.3217$. Also, for an illustration, the skewness and kurtosis measures of the APM distribution are plotted in Figure 3(a) - 3(b), respectively.




(a) skewness




(b) kurtosis

Figure 3: The plots for the skewness and kurtosis measures of the APM distribution for different values of the shape parameter $\alpha$.

**Remark 2.** *Notice that the Maxwell distribution has only a scale parameter, i.e., it does not have a shape parameter; therefore, it has deficiencies in terms of the skewness and kurtosis measures. Thus, the Maxwell distribution cannot adequately model data having more skewness and/or kurtosis values than its. In contrast to the Maxwell, the APM distribution has one shape parameter; therefore, its skewness and kurtosis measures take values in a broader range than the Maxwell counters.*

**Definition 7.** *The pdf, cdf, hrf, and mean residual life function(mrlf) of a positive random variable $X$ are represented by $f_X(.), F_X(.), h_X(.)$ and $m_X(.)$, respectively, and those of another positive random variable $Y$ are represented by $f_Y(.), F_Y(.), h_Y(.)$ and $m_Y(.)$, respectively. Then, the following definitions are given.*

- *The stochastic order $(X \leq_{(sto)} Y)$ if $F_X(x) \leq F_Y(x)$ for all $x$;*
- *The hazard rate order $(X \leq_{(hro)} Y)$ if $h_X(x) \leq h_Y(x)$ for all $x$;*
- *The mean residual life order $(X \leq_{(mrlo)} Y)$ if $m_X(x) \leq m_Y(x)$ for all $x$;*
- *The likelihood ratio order $(X \leq_{(lro)} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decrease in $x$.*

**Remark 3.** *Let $X$ and $Y$ be random variables following any distributions. Then, $[X \leq_{(lro)} Y] \Rightarrow [X \leq_{(hro)} Y] \Rightarrow [X \leq_{(mrlo)} Y] \Rightarrow [X \leq_{(sto)} Y]$ ; see Shaked and Shanthikumar [26].*

**Theorem 1.** *Let $X_1 \sim APM(\alpha_1, \sigma)$ and $X_2 \sim APM(\alpha_2, \sigma)$. If $\alpha_1 < \alpha_2$, then $[X_1 \leq_{(lro)} X_2]$, $[X_1 \leq_{(hro)} X_2]$, $[X_1 \leq_{(mrlo)} X_2]$, and $[X_1 \leq_{(sto)} X_2]$.*

*Proof.* For any $x$, the likelihood ratio order is

$$\frac{f_{X_1}(x)}{f_{X_2}(x)} = \left(\frac{\alpha_2 - 1}{\alpha_1 - 1}\right)\left(\frac{\ln\alpha_1}{\ln\alpha_2}\right)\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{\Gamma(3/2)}\Gamma\left[\left(\frac{x}{\sigma}\right)^2, \frac{3}{2}\right]},$$

$$r(x) = \frac{d}{dx}\ln\left(\frac{f_{X_1}(x)}{f_{X_2}(x)}\right) = \frac{4}{\sigma\Gamma(1/2)}\left(\frac{x}{\sigma}\right)^2 exp\left(-\left(\frac{x}{\sigma}\right)^2\right)\ln\left(\frac{\alpha_1}{\alpha_2}\right).$$

Now, $r(x) < 0$ if $\alpha_1 < \alpha_2$ and hence $X_1 \leq_{(lro)} X_2$. The other orderings are immediately following from the Remark 3. □

## 3 Parameter estimation

In this section, the ML estimations for parameters $\alpha$ and $\sigma$ of the APM distribution are provided. The ML methodology is based on the maximization of the likelihood or log-likelihood ($\ln L$) function. Let $x_1, x_2, \ldots, x_n$ be a random sample from the APM distribution. Then, the $\ln L$ function of the APM distribution is

$$\begin{aligned}\ln L = {} & n\ln\left(\frac{4}{\sqrt{\pi}}\right) + n\ln\left(\frac{\ln\alpha}{\alpha-1}\right) - n\ln\sigma + 2\sum_{i=1}^{n}\ln\left(\frac{x_i}{\sigma}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^2 \\ & + \frac{\ln\alpha}{\Gamma(3/2)}\sum_{i=1}^{n}\Gamma\left[\left(\frac{x_i}{\sigma}\right)^2, \frac{3}{2}\right].\end{aligned} \tag{5}$$

The estimate values of the $\alpha$ and $\sigma$, points in which the $\ln L$ function attains its maximum, are the solutions of likelihood equations

$$\frac{\partial lnL}{\partial\alpha} = \frac{n}{\alpha\ln\alpha} - \frac{n}{\alpha-1} + \frac{1}{\alpha}\frac{1}{\Gamma(3/2)}\sum_{i=1}^{n}\Gamma\left[\left(\frac{x_i}{\sigma}\right)^2, \frac{3}{2}\right] = 0, \tag{6}$$

and

$$\frac{\partial lnL}{\partial\sigma} = -\frac{3n}{\sigma} + \frac{2}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^2 - \frac{2\ln\alpha}{\sigma^4\Gamma(3/2)}\sum_{i=1}^{n}x_i^3\exp\left[-\left(\frac{x_i}{\sigma}\right)^2\right] = 0, \tag{7}$$

respectively.

Likelihood equations (6) and (7) can not be solved explicitly since they are nonlinear functions of parameters $\alpha$ and $\sigma$. Therefore, iterative methods such as Newton-Raphson (NR) should be utilized to obtain the solution of these equations simultaneously.

The two-dimensional NR algorithm consists of the following steps.

Step 1. Set $k = 0$, then give the initial values of the parameters, i.e., $\alpha_0$ and $\sigma_0$.

Step 2. Obtain the values $\alpha_{k+1}$ and $\sigma_{k+1}$ by using the equation

$$\begin{bmatrix}\alpha_{k+1} \\ \sigma_{k+1}\end{bmatrix} = \begin{bmatrix}\alpha_k \\ \sigma_k\end{bmatrix} - \begin{bmatrix}\frac{\partial^2\ln L}{\partial\alpha^2}(\alpha_k,\sigma_k) & \frac{\partial^2\ln L}{\partial\alpha\partial\sigma}(\alpha_k,\sigma_k) \\ \frac{\partial^2\ln L}{\partial\sigma\partial\alpha}(\alpha_k,\sigma_k) & \frac{\partial^2\ln L}{\partial\sigma^2}(\alpha_k,\sigma_k)\end{bmatrix}^{-1}\begin{bmatrix}\frac{\partial\ln L}{\partial\alpha}(\alpha_k,\sigma_k) \\ \frac{\partial\ln L}{\partial\sigma}(\alpha_k,\sigma_k)\end{bmatrix}.$$

Step 3. Repeat Step 1 and Step 2 for $(k = 1, 2, \ldots)$ until $|\alpha_{k+1} - \alpha_k| \leq \varepsilon$ and $|\sigma_{k+1} - \sigma_k| \leq \varepsilon$.

Here, elements of the Hessian matrix which includes second partial derivates of the $\ln L$ concerning the $\alpha$ and $\sigma$ are

$$\frac{\partial^2 lnL}{\partial\alpha^2} = \frac{n}{(\alpha-1)^2} - \frac{n}{\alpha^2(\ln\alpha)^2} - \frac{n}{\alpha^2\ln\alpha} - \frac{1}{\alpha^2\Gamma(3/2)}\sum_{i=1}^{n}\Gamma\left[\left(\frac{x_i}{\sigma}\right)^2, \frac{3}{2}\right],$$

$$\frac{\partial^2 lnL}{\partial\sigma^2} = \frac{3n}{\sigma^2} - \frac{6}{\sigma^2}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^2 + \frac{8\ln\alpha}{\sigma^5\Gamma(3/2)}\sum_{i=1}^{n}x_i^3\exp\left[-\left(\frac{x_i}{\sigma}\right)^2\right]$$
$$-\frac{4\ln\alpha}{\sigma^7\Gamma(3/2)}\sum_{i=1}^{n}x_i^5\exp\left[-\left(\frac{x_i}{\sigma}\right)^2\right],$$

and

$$\frac{\partial^2 lnL}{\partial\sigma\partial\alpha} = -\frac{2}{\alpha\sigma^4\Gamma(3/2)}\sum_{i=1}^{n}x_i^3\exp\left[-\left(\frac{x_i}{\sigma}\right)^2\right].$$

**Remark 4.** *To find the ML estimates of parameters $\alpha$ and $\sigma$ which the $\ln L$ function in (5) attains its maximum, optimization functions such as* `fminunc` *and* `fminsearch` *that are available in* `MATLAB` *can also be used. However, as stated in Jones and Naufaliy [17], "the log-likelihood surface is not always very well behaved since one or more parameters of the distributions are made to depend on covariates. Therefore optimisation functions should be run from several starting points in order to ensure that the global maximum of the likelihood is found." Using a population-based method, e.g., the genetic algorithm, particle swarm, simulated annealing, can also be recommended to alleviate this problem. These optimization functions are also available in* `MATLAB`.

# 4 Application

In this section, a real data set is modeled by using the APM distribution. Then, the modeling performance of the APM distribution is compared with its strong alternatives, which can be the TM proposed by Iriarte and Astorga [13] and the GM introduced by Iriarte et al. [14] distributions. Note that alpha power transformed extended exponential (APEE) distribution from Hassan et al. [12] is also considered in the application to make comparisons complete.

In the comparisons, information criterion $\ln L$, and goodness-of-fit measures, e.g., Kolmogorov-Smirnov (KS), Cramér-von Mises (CvM), Anderson Darling (AD), root-mean-squared error (RMSE), and coefficient of determination ($R^2$), are considered. According to these criteria, the smaller values of the KS, CvM, AD, and RMSE, and bigger values of the $\ln L$ and $R^2$ imply better modeling performance. The ML method is performed to obtain estimates of the unknown distribution parameters via the NR procedure given in Section 3 and optimization function `fminunc` available in `MATLAB`.

In the application part of the study, data set including 100 observations on breaking stress of carbon fibres (in GBA) is considered. The breaking stress data set is taken from the Nichols and Padgett [24] and given in Table 2.

Table 2: The values for the brekaing stress of carbon fibres (in GBA).

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.39 | 0.81 | 0.85 | 0.98 | 1.08 | 1.12 | 1.17 | 1.18 | 1.22 | 1.25 | 1.36 | 1.41 | 1.47 | 1.57 | 1.57 |
| 1.59 | 1.59 | 1.61 | 1.61 | 1.69 | 1.69 | 1.71 | 1.73 | 1.80 | 1.84 | 1.84 | 1.87 | 1.89 | 1.92 | 2.00 |
| 2.03 | 2.03 | 2.05 | 2.12 | 2.17 | 2.17 | 2.17 | 2.35 | 2.38 | 2.41 | 2.43 | 2.48 | 2.48 | 2.50 | 2.53 |
| 2.55 | 2.55 | 2.56 | 2.59 | 2.67 | 2.73 | 2.74 | 2.76 | 2.77 | 2.79 | 2.81 | 2.81 | 2.82 | 2.83 | 2.85 |
| 2.87 | 2.88 | 2.93 | 2.95 | 2.96 | 2.97 | 2.97 | 3.09 | 3.11 | 3.11 | 3.15 | 3.15 | 3.19 | 3.19 | 3.22 |
| 3.22 | 3.27 | 3.28 | 3.31 | 3.31 | 3.33 | 3.39 | 3.39 | 3.51 | 3.56 | 3.6 | 3.65 | 3.68 | 3.68 | 3.68 |
| 3.7 | 3.75 | 4.2 | 4.38 | 4.42 | 4.7 | 4.9 | 4.91 | 5.08 | 5.56 | | | | | |

The ML estimates of the unknown parameters for the APM, TM, and GM distributions are given in Table 3. The values of the $\ln L$ and goodness-of-fit measures of the APM, TM, and GM distributions are also given in Table 3.

Table 3: The ML estimates of the parameters along with the values of the $\ln L$ and goodness-of-fit measures for each of the distribution.

| | Parameter estimates | | | | | | Comparison criteria | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Distribution | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\gamma}$ | $\hat{\theta}$ | $\hat{\lambda}$ | $\hat{\sigma}$ | $\ln L$ | KS | CvM | AD | RMSE | $R^2$ |
| APM | 2.7401 | — | — | — | — | 2.0751 | -141.2245 | 0.0593 | 0.0611 | 0.3739 | 0.0241 | 0.9929 |
| GM | 1.1814 | — | — | 0.2194 | — | — | -141.4539 | 0.0722 | 0.0884 | 0.4482 | 0.0284 | 0.9902 |
| TM | — | — | — | 0.4473 | -0.4155 | — | -141.2552 | 0.0633 | 0.0681 | 0.3879 | 0.0253 | 0.9922 |
| APEE | 319.9195 | 143.0375 | 1.4600 | — | — | — | -142.6702 | 0.0874 | 0.1349 | 0.6510 | 0.0352 | 0.9850 |

The Hessian matrix of the model at the estimated parameter values $\hat{\alpha}$ and $\hat{\sigma}$ is

$$\begin{bmatrix} \dfrac{\partial^2 \ln L}{\partial \alpha^2}(\hat{\alpha},\hat{\sigma}) & \dfrac{\partial^2 \ln L}{\partial \alpha \partial \sigma}(\hat{\alpha},\hat{\sigma}) \\ \dfrac{\partial^2 \ln L}{\partial \sigma \partial \alpha}(\hat{\alpha},\hat{\sigma}) & \dfrac{\partial^2 \ln L}{\partial \sigma^2}(\hat{\alpha},\hat{\sigma}) \end{bmatrix} = \begin{bmatrix} -1.0556 & -11.5910 \\ -11.5910 & -176.1922 \end{bmatrix}, \tag{8}$$

and determinant of the Hessian matrix equals to 51.6295. As stated Example 7.2.12 p. 322 in Casella and Berger [6], $\ln L(\alpha,\sigma)$ has a local maximum at $(\hat{\alpha},\hat{\sigma})$ if following conditions hold

a. $\dfrac{\partial}{\partial \alpha} \ln L(\alpha,\sigma)\Big|_{\alpha=\hat{\alpha},\sigma=\hat{\sigma}} = 0$ and $\dfrac{\partial}{\partial \sigma} \ln L(\alpha,\sigma)\Big|_{\alpha=\hat{\alpha},\sigma=\hat{\sigma}} = 0$;

b. $\dfrac{\partial^2}{\partial \alpha^2} \ln L(\alpha,\sigma)\Big|_{\alpha=\hat{\alpha},\sigma=\hat{\sigma}} < 0$ and $\dfrac{\partial^2}{\partial \sigma^2} \ln L(\alpha,\sigma)\Big|_{\alpha=\hat{\alpha},\sigma=\hat{\sigma}} < 0$;

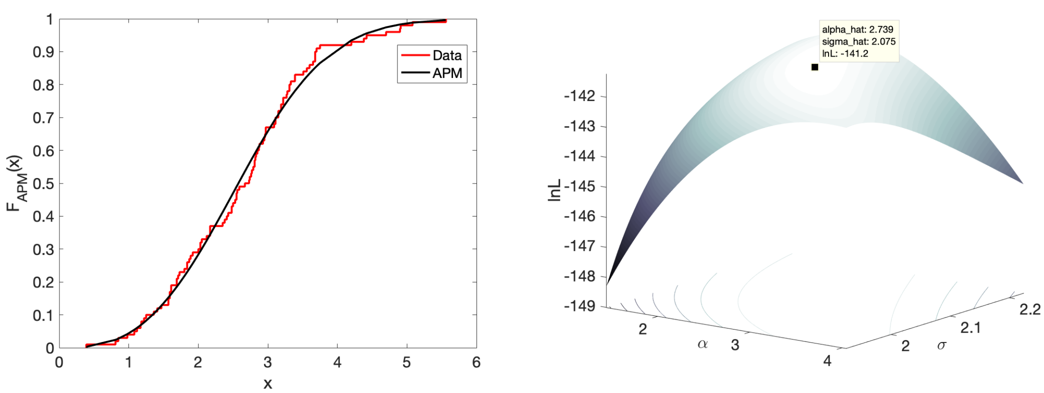
c. $\dfrac{\partial^2}{\partial \alpha^2} \ln L(\alpha,\sigma) \dfrac{\partial^2}{\partial \sigma^2} \ln L(\alpha,\sigma) - \left(\dfrac{\partial^2}{\partial \alpha \partial \sigma} \ln L(\alpha,\sigma)\right)^2 \Big|_{\alpha=\hat{\alpha},\sigma=\hat{\sigma}} > 0$.

It is clear from the Hessian matrix given in (8) that conditions (a)-(c) have been satisfied. Note that the NR procedure given in Section 3 and also optimization function `fminunc` available in

`MATLAB` are run from several starting points to ensure that the value of $\ln L(\hat{\alpha}, \hat{\sigma})$, $-141.2245$, is the global maximum.

According to the $\ln L$ values given in Table 3, the APM distribution can be considered as an alternative to the TM and GM distributions since it has larger $\ln L$ values than its rivals. Also, it can be concluded that the APM distribution is preferable over the TM and GM distributions when goodness-of-fit measures are taken into account. Note that Qian [25] used two-parameter exponentiated exponential (EE) and three-parameter exponentiated Weibull (EW) distributions to model breaking stress data. The $\ln L$ values of the EE and EW distributions are obtained as -146.1823 and -141.3320, respectively.

The fitting performance of the APM distribution is illustrated in Figure 4(a). The surface plot for the $\ln L$ function of the APM distribution for the data set considered in the application is given in Figure 4(b). It is clear from Figure 4(b) that the ML estimates of parameters $\alpha$ and $\sigma$ are the points on which the $\ln L$ function attains its maximum.




(a) fitted cdf plot

(b) $\ln L$ surface plot

Figure 4: The fitted cdf and $\ln L$ surface plots of the APM distribution.

## 5 Conclusions

In this paper, the APM distribution, which is flexible for modeling unimodal positive skew data sets, is obtained. Actually, this study is motivated by the extensive use of the Maxwell distribution in Physics and Statistics. It is clear that the generalization of the Maxwell distribution, as the APM distribution, provides more flexibility to analyze lifetime data. Some statistical properties of the APM distribution are obtained. The ML method is used in estimating unknown parameters of the APM distribution. Results in an application show that the APM distribution is preferable over its strong rivals.

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