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Advances in induced optimal partition invariancy analysis in uni-parametric linear optimization

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Abstract. In this study, we consider a family of uni-parametric linear optimization problems that the objective function, the right, and the left hand side of constraints are linearly perturbed with an identical parameter. We are interested in studying the effect of this variation on a given optimal solution and the behavior of the optimal value function on its domain. This problem has several applications, such as in linear time dynamical systems. A prototype example is provided in dynamical systems as a justification for the practicality of the study results. Based on the concept of induced optimal partition, we identify the intervals for the parameter value where optimal induced partitions are invariant. We show that the optimal value function is piecewise fractional continuous in the interior of its domain, while it is not necessarily to be continuous at the endpoints. Some concrete examples depict the results of the analysis.

Keywords: Uni-parameter linear optimization, induced optimal partition invariancy analysis, change point, Moore-Penrose inverse, realization theory.
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1 Introduction

Optimization-based techniques have demonstrated their undeniable capability in the design, control, and operating of many engineering systems [20]. Financial markets are in a sense, dynamical systems; and evolving market variables would affect the optimal decision-making process. Identifying an optimal tax-free bond portfolio by exploiting the price differential stream of cash flows would be an instance. We will discuss a prototype example later in detail.

The existence of inaccuracy and variability in the parameters of an optimization problem may imply a deviation from a predetermined optimal situation and lead either the system fails at being an entirely beneficial one, or additional burden of problem-solving for other values of

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parameters is required. Investigation of these possibilities and their consequences has attracted the primary concern of many researchers.

Parametric programming has denoted competence when uncertainty appears on the problem data, or the process states corresponding to the parameters. This approach economically identifies the exact mapping of the optimal solution in the space of system parameters. Moreover, unnecessary several problems solving is avoided, and the optimal solution can be immediately adjusted to the system dynamics.

Linear programming, with its parametric version, is a mature field of optimization and has proved its efficiency in simplifying many dynamical systems. For instance, when the input data of a linear program only depends on a single parameter, the parameter would be considered as "time" in a "time-developing system", and analyzing the system behavior over a period could be the aim.

To be specific, let $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ be fixed data, and $x, s \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ as unknown vectors. Here, vectors b and c are usually referred to as *rim data* and A as technological data. Moreover, let $t \in \mathbb{R}$ be a parameter, and $\triangle c \in \mathbb{R}^n$, $\triangle b \in \mathbb{R}^m$ and $\triangle A \in \mathbb{R}^{m\times n}$ be components of a perturbing direction denoted by $\triangle = (\triangle A, \triangle b, \triangle c)$ for the sake of brevity in notation. In this way, a uni-parametric linear optimization problem could be defined as

$$P_t(\triangle) \qquad \min\Big\{(c+t\triangle c)^T x :\ (A+t\triangle A)x = b+t\triangle b,\ x \geq 0\Big\},$$

with its dual as

$$D_t(\triangle) \qquad \max\Big\{(b+t\triangle b)^T y :\ (A+t\triangle A)^T y \leq c+t\triangle c\Big\}.$$

Special cases may occur when $\triangle c = 0$ or $\triangle b = 0$, which are denoted by $P_t(\triangle A, \triangle b)$, $P_t(\triangle A, \triangle c)$ and $P_t(\triangle A)$ (when both are zero), along with their dual problems, respectively.

As a practical example, consider the problem of identifying an optimal tax-free bond portfolio using the price differential stream of cash flows. This objective could be accomplished by purchasing at the ask price "underpriced" bonds, while simultaneously selling at the bid price "overpriced" bonds. The following model was proposed in [21] as a tax-specific version at which the objective is achieved by analogous exercising for a given tax bracket and the price differential of an after-tax stream of cash flows (See also [23]).

$$\begin{aligned}
\max \quad & \sum_{j=1}^{n} p_j^b x_j^b - \sum_{j=1}^{n} p_j^a x_j^a \\
s.t. \quad & c_1 = \sum_{j=1}^{n} a_j^1 x_j^a - \sum_{j=1}^{n} a_j^1 x_j^b, \\
& c_t = (1+\rho)c_{t-1} + \sum_{j=1}^{n} a_j^t x_j^a - \sum_{j=1}^{n} a_j^t x_j^b, \quad t = 2,\dots,T, \\
& c_t \geq 0, \quad t = 1,\dots,T, \\
& x_j^a, x_j^b \geq 0, \quad j = 1,\dots,n, \\
& x_j^a, x_j^b \leq 1, \quad j = 1,\dots,n,
\end{aligned} \tag{1}$$

where $\{1, 2, \dots, n\}$ is the set of riskless bonds, p_j^a is the ask price, p_j^b is the bid price, and a_j^t is the coupon and/or principal payment on bond j at time t. The natural assumption is $p_j^a > p_j^b$. The variables x_j^a and x_j^b are respectively the amount of bond j bought and sold short. Here, c_t's are future cash flows of the portfolio, and ρ is the exogenous riskless reinvestment rate, while $c_t \geq 0$ guaranties that the portfolio is riskless. The final restriction on variables guaranties of having a finite solution. The objective function makes sense since the long side of an arbitrage position must be established at ask prices while the short side of the position must be established at bid prices.

In the sequel, to have a concrete description, we set $T = 1$, which means that there is no opportunity for reinvestment. One may consider that inputs p_j^a, p_j^b, and a_j^1 are are changing continuously over the developing time. For instance, $p_j^b(\theta_1) = p_j^b(0)(1 + t\theta_1)$, $p_j^a(\theta_2) = p_j^a(0)(1 + t\theta_2)$, and $a_j^1(\theta_3) = a_j^0(1+t\theta_3)$, where a_j^0, $p_j^b(0)$ and $p_j^a(0)$ are corresponding prices at the beginning of the period, and θ_1, θ_2, and θ_3 are potential respective different change rates over time. Moreover, this positive cash flow must be enough to repay a commitment amount that is b at the first of period and will change over this time period as $b + t\theta_4$. This means that not only ask and bid prices are varying but also coupon payment on bond j and commitment value are evolving over time. Moreover, suppose that you are free to exercise all acts at any time of the period, but simultaneously. In this way, we have the following parametric program.

$$\begin{array}{lll} \max & \displaystyle\sum_{j=1}^{n} p_j^b(0)(1+t\theta_1)x_j^b - \sum_{j=1}^{n} p_j^a(0)(1+t\theta_2)x_j^a & \\ s.t. & \displaystyle\sum_{j=1}^{n} a_j^0(1+t\theta_3)x_j^a - \sum_{j=1}^{n} a_j^0(1+t\theta_3)x_j^b \geq (b+t\theta_4), & \\ & 0 \leq x_j^a, x_j^b \leq 1, \quad j = 1, \dots, n, & \end{array} \tag{2}$$

at which the inequality $p_j^b(0)(1+t\theta_1) < p_j^a(0)(1+t\theta_2)$ must be satisfied for all $t \geq 0$. Observe that the standard form of this prototype instance is a dynamical system presented as a parametric problem of type $P_t(\triangle c, \triangle A, \triangle b)$. We will provide a simple numerical instance for this problem in Section 8.

Now, suppose that the Problems $P_t(\triangle)$ and $D_t(\triangle)$ are feasible at $t = 0$. We denote the feasible solution sets of these problems by $\mathcal{P}_t$ and $\mathcal{D}_t$, and their optimal solution sets by $\mathcal{P}_t^*$ and $\mathcal{D}_t^*$, respectively. A primal-dual feasible solution $(x^*(t), y^*(t), s^*(t))$ is optimal if it satisfies the well-known complementary condition $x^{*T}(t)s^*(t) = 0$. Let us denote the index set $\{1, 2, \dots, n\}$ by $\mathcal{I}$. For a nonsingular submatrix $A_\mathcal{B}$ of the columns of A, where $\mathcal{B}$ is a set of m indices in $\mathcal{I}$ and $\mathcal{N} = \mathcal{I} \backslash \mathcal{B}$, $x^T(t) = (x_\mathcal{B}^T(t), x_\mathcal{N}^T(t))$ is a primal basic feasible solution when $x_\mathcal{B}(t) = A_\mathcal{B}^{-1}b \geq 0$, and $x_\mathcal{N}(t) = 0$. Further, $(y^T(t), s^T(t))$ is a dual basic feasible solution, when $y(t) = A_\mathcal{B}^{-T}c_\mathcal{B}$ and $s(t) = c - A^T A_\mathcal{B}^{-T} c_\mathcal{B} \geq 0$. Note that a basic optimal solution characterizes a partition of the index set, known as basic optimal partition.

The primal-dual optimal solution $(x^*(t), y^*(t), s^*(t))$ is strictly complementary when $x_j^*(t) + s_j^*(t) > 0$ for all $j \in \mathcal{I}$. By the Goldman-Tucker's theorem [10], the existence of a strictly complementary solution is guaranteed when for a given t both problems $P_t(\triangle)$ and $D_t(\triangle)$ are feasible. An interior point method starts from a solution and leads to a strictly complementary optimal solution [22]. Recall that an interior point method's success to provide a strictly complementary optimal solution strongly depends on having a full row rank coefficient matrix in a linear optimization problem. There are efficient methods to remove redundant rows and columns of a matrix. Here, without loss of generality, we assume that the matrix A has full row rank. As a result, a primal-dual strictly complementary optimal solution $(x^*(t), y^*(t), s^*(t))$ partitions the index set $\mathcal{I}$ as $\pi_t = (B_t, N_t)$, where

$$\begin{aligned} B_t &:= \{j | x_j^*(t) > 0, \text{ for some } x^*(t) \in \mathcal{P}_t^*\}, \\ N_t &:= \{j | s_j^*(t) > 0, \text{ for some } (y^*(t), s^*(t)) \in \mathcal{D}_t^*\}. \end{aligned}$$

This partition is referred to as optimal partition, hereafter.

Parametric analysis studies the effect of data perturbation along a direction (or some directions) according to some parameters. The aim of a study could be to identify the region for the parameters where specific properties of the current optimal solution are invariant. More clearly, when the current optimal solution is strictly complementary (basic), one might be interested in identifying the region for the parameter(s) at which the known optimal partition (optimal basis) is invariant for every parameter(s) values in this region.

Sometimes, we only have an optimal solution without knowing that it is weather basic or strictly complementary. In this case, one may be interested in determining the region for parameter values where the support set of the in-hand optimal solution is invariant [9]. From the support set of a nonnegative vector v, we mean $\sigma(v) = \{j \in \mathcal{I}, v_j > 0\}$. In this way, expansion and restriction of a support set can be studied, too [19]. These points of view have been studied extensively during the last decade. Especially in an interesting recent application in game theory [15], the author considered interval bimatrix games and focused on three kinds of support set invariancy. In such games, support of a mixed strategy consists of pure strategies having positive probabilities. Given an interval-valued bimatrix game and supports for both players, the paper aims to answer some questions. Questions like "Does every bimatrix game instance have an equilibrium with the prescribed support?" and "Has every bimatrix game instance an equilibrium being a subset/superset of the prescribed support?". It was shown that answering these questions is computationally challenging.

Practically, the degeneracy of a primal or dual basic solution leads to having multiple dual or primal optimal solutions. This fact shows that the basic optimal partition may not be unique in general, while the convexity of optimal solution sets implies the uniqueness of the optimal partition. The optimal partition and the basic optimal partition are identical when the primal problem has a unique nondegenerate optimal solution.

Due to this important property of the optimal partition, we consider Problem $P_t(\triangle)$ and its special version $P_t(\triangle A, \triangle c)$, and aim to identify the region where the known optimal partition is invariant. Our approach is to generalize the methodology introduced in [18] for identifying the optimal partition invariancy region of $P_t(\triangle A, \triangle b)$. We first convert these problems into equivalent forms in which only their left-hand sides are perturbed. In this way, some free variables are introduced, which lead to defining the induced optimal partition and the notion of induced optimal partition invariancy analysis. As a result, in addition to the transition point, the change point concept is introduced. Furthermore, the representation of the optimal value function of such problems is presented in this study.

The rest of the paper is organized as follows. Section 2 is devoted to reviewing some related findings in parametric linear programming. In Section 3, first, we state some necessary presumptions for guaranteeing the convexity of the optimal partition invariancy region on Problem $P_t(\triangle)$. Then, some preliminary concepts on Moore-Penrose inverse and Realization theory are reviewed. We formalize the methodology of induced optimal partition invariancy analysis on Problems $P_t(\triangle)$ and $P_t(\triangle A, \triangle c)$, in Section 4. Moreover, the concept of change point is clarified, and distinguished from the transition point. This section ends with a description of the concept of free variables on this parametric analysis. The process of finding all transition points, change points, and invariancy intervals is expressed in Section 5. An explicit formula of the optimal value function on each invariancy interval is presented in Section 6. In Section 7, we briefly explain the adaptation of the presented algorithm in [18] with the notations of this paper. Some concrete examples illustrate the results in Section 8. The final section contains some concluding remarks.

2 Literature review

Let us first review some findings related to the parametric linear program. It is proved that in a parametric linear optimization problem when either $\triangle b$ or $\triangle c$ is a nonzero vector, optimal partition invariancy intervals are open if they are not singletons. The optimal value function is a continuous piecewise linear function over these intervals [6, 22]. Singleton regions are referred to as breakpoints since they separate the invariancy intervals, and the optimal value function fails to be differentiable at these points. Moreover, this function has constant slopes an invariancy intervals, which suggests one to refer to these intervals as linearity intervals [22]. In [13], the results of sensitivity and parametric analysis of single-parametric linear programming were extended to the case when there are multiple parameters in the objective function and the right-hand side of constraints. The author described the set of admissible parameters under the support set and optimal partition invariancies and compared them with the classical optimal basis invariancy.

Similarly, when both the right-hand side of the constraints and the objective function are perturbed with identical parameters, each invariancy region is again an open interval if it is not a singleton. However, the optimal value function is a continuous piecewise quadratic function on these invariancy intervals [6, 8], while optimal partitions are different on them and their separating points. Therefore, these points would be referred to as transition points instead of breakpoints. In this case, the representation of the optimal value function is different at each invariancy interval; it is continuous at transition points and fails to have the first or the second derivative at them. In general, and as a result, optimal partition invariancy analysis in these cases aims to identify those subintervals where the optimal value function has different representations [8, 22]. This latter case has also been studied when the problem is in canonical form [12]. Though the concept of the invariancy region in this study is somehow dissimilar, and the interpretation of the notion differs, the results are almost identical.

In another point of view in [14], the aim was to compute tolerances (intervals) for the objective function coefficients and the right-hand side values. The tolerance means that having an optimal solution, they can independently and simultaneously vary inside their tolerances while preserving some optimality criterion such as optimal basis, support set, and optimal partition invariancy. In this paper, the tolerance analysis was put in a unified framework convenient for algorithmic processing, which is applicable not only in linear programming but also in other linear systems. Moreover, the known results were surveyed, and an improvement was proposed while taking into account the proportionality. This improvement refers to optimality in some sense; that is the resulted tolerances are maximal. It is worth mentioning that this approach is useful not only for the simplex method solvers but also for the interior points methods. Time complexity has also been discussed, and it is showed that determining the maximal tolerances in an NP-hard problem.

Investigation of the case when only the coefficient matrix A is perturbed alongside a direction $\triangle A$ has a long story. Here, we only mention the findings from the perspective of optimal partition invariancy. The problem in the canonical form with a perturbing direction of rank one, when a primal-dual strictly complementary optimal solution is in hand, has been studied in [11]. For a linear program with inequality constraints, at which a single parameter perturbs only their left-hand side, a solution algorithm has been presented in [17]. The main obstacle in this algorithm is that the inversion techniques of perturbed matrices cause several computational complexities.

Another algorithm has been introduced for finding the exact solution of multi-parametric linear programming problems with inequality constraints when simultaneously, the coefficients of the objective function, the right-hand-side, and the left-hand-side of the constraints are parameterized [5]. This algorithm is based on symbolic manipulation and semi-algebraic geometry. By semi-algebraic geometry, one can identify the critical regions at which the optimal value function is fractional, and active constraint sets are invariant. Note that these regions are neither necessarily convex nor connected. Without having to determine the inverse of parametric matrices, the entire parametric space can be explored implicitly within this algorithm. As the authors acknowledged, the implementation of this algorithm is highly dependent on mathematical software. Thus, it is not adequate for large-scale problems due to the complexity of computing.

The optimal partition invariancy analysis of Problem $P_t(\triangle A)$, with $\triangle A$ of arbitrary rank and a known optimal partition, has been investigated in [7]. The authors presented a computational procedure that identifies all invariancy intervals. Besides, the optimal value function on its domain has been characterized, and some of its properties have been investigated. More recently, this argument is continued by adapting the methodology in [7] to the case when the perturbed problem is in the form $P_t(\triangle A, \triangle b)$. The existence of change point in induced optimal partition invariancy analysis is one of the most important feature that distinguishes it from the optimal partition invariancy analysis.

3 Preliminaries

Let t run throughout a nonempty set $\Lambda \subseteq \mathbb{R}$ for which Problems $P_t(\triangle)$ and $D_t(\triangle)$ have optimal solutions. This set is nonempty since it is assumed that these problems are feasible at $t = 0$. Since optimal solutions set $\mathcal{P}_t^*$ is a subset of feasible solution set of $\mathcal{P}_t$, thus connectivity of Λ refers back to the continuity of the solution sets of $(A + t\triangle A)x = b + t\triangle b$ with respect to t. If $A + t\triangle A$ is invertible over a dense open subset $U \subseteq \mathbb{R}$ or rank $A + t\triangle A$ is constant for all $t \in \mathbb{R}$, the continuity is a straightforward result in standard linear algebra. When this system is underdetermined and rank $A + t\triangle A$ varies for different values of t, to guarantee the continuity, one must impose some regularity conditions on the problem. Thus, the region for the parameter value is not necessarily connected in general. Here, we are only interested in finding the largest connected set that includes the origin $t = 0$.

First, we refer to the perturbing direction $\triangle$ as *change direction*. For a $t \in \mathbb{R}$, $t\triangle$ is said an *admissible change* if problem $P_t(\triangle)$ has an optimal solution, or equivalently

$$\exists (x, y), x \geq 0 : \ (A + t\triangle A)x = b + t\triangle b, \ (A + t\triangle A)^T y \leq c + t\triangle c.$$

It is not hard to see that $t\triangle$ is not generally an admissible change for all $t \in (0, t^*)$, just because $t^*\triangle$ is an admissible change. A change direction $\triangle$ is an *admissible direction* if there exists $t^* > 0$, such that $t\triangle$ is an admissible change for all $t \in [0, t^*)$ [11]. Let us denote the set of all admissible changes by $\mathcal{A}$. Unlike the variation at rim data, this set is not convex when the left-hand-side of constraints is additionally perturbed [11]. For an admissible direction $\triangle$, let $t^*(\triangle) := \sup\{t^* : \ t\triangle \in \mathcal{A}, \forall t \in [0, t^*)\}$, and $\Lambda(\triangle) := \{t : \ t\triangle \in \mathcal{A}\}$.

Analogous to [11], it can be proved that when $\mathcal{A} = \bigcup_{k=1}^{K} \{\mathcal{P}_k\}$, and each $\mathcal{P}_k$ is a polyhedron containing the origin, then $\Lambda(\triangle)$ is simply an interval. In this paper, we assume that these assumptions are fulfilled.

Now, we adapt the concept of optimal partition in [7] on Problem $P_t(\triangle)$, the relationship of this concept with the optimal partition defined in Page 148 is stated. Consider $1 \leq l \leq n$, and let

$$\tau : \{1, \ldots, l\} \rightarrow \{1, \ldots, n\}, \tau' : \{1, \ldots, n - l\} \rightarrow \{1, \ldots, n\}, \tag{3}$$

be injective and increasing functions, so that

$$\text{Range}(\tau) \cup \text{Range}(\tau') = \{1, \ldots, n\}. \tag{4}$$

Let A_τ and c_τ denote the columns of A and c corresponding to the indices identified by the map τ. Analogous notation will be used later. Furthermore, let $\pi_{t_0} = (B_{t_0}, N_{t_0})$ be the known optimal partition corresponding to Problems $P_{t_0}(\triangle)$ and $D_{t_0}(\triangle)$, and $\Lambda_0 \subseteq \mathbb{R}$ be a set of parameters such that for each $t \in \Lambda_0$, the Problem $P_t(\triangle)$ has an optimal solution $(x^*(t), y^*(t), s^*(t))$ with the optimal partition π_{t_0}. Recall that corresponding to any parameter $t \in \Lambda_0$, every strictly complementary optimal solution implies $B_t = \{\tau(1), \ldots, \tau(l)\}$ and $N_t = \{\tau'(1), \ldots, \tau'(n - l)\}$ [7]. Moreover, according to (4), (B_t, N_t) is a partitioning of $\mathcal{I}$. Therefore, for $t \in \Lambda_0$,

Property 1. $A_\tau(t) = A_\tau + t\triangle A$ has a pseudo-inverse, (See the next paragraph for the definition of pseudo-inverse),

Property 2. $x_\tau(t) = A_\tau(t)^\dagger(b + t\triangle b) > 0$,

Property 3. $s_{\tau'}(t) > 0$ (equivalently $(c + t\triangle c)_{\tau'}^T - (c + t\triangle c)_\tau^T A_\tau(t)^\dagger A_{\tau'}(t) > 0$),

hold if and only if $(x(t), y(t), s(t))$ is a strictly complementary optimal solution of problems $P_t(\triangle)$ and $D_t(\triangle)$ with optimal partition π_{t_0}. It is of importance to mention that Property 1 holds without any restriction on t and it will be used in other two properties. Recall that the aim of optimal partition invariancy analysis is to find the region $\Lambda_0 \subseteq \Lambda$, where for every $t \in \Lambda_0$, optimal partition of the associated problem is π_{t_0}. This is equivalent to establishment of Properties 1-3 for such a parameter t.

Observe that the matrix A_τ is not an invertible matrix in general, and therefore we need another tool known as the Moore-Penrose inverse. The Moore-Penrose inverse (or simply the pseudo-inverse) of a real matrix $A \in \mathbb{R}^{m\times n}$ is a unique matrix $A^\dagger \in \mathbb{R}^{n\times m}$, where

$$\begin{aligned} A^\dagger A A^\dagger &= A^\dagger, \\ A A^\dagger A &= A, \\ (A^\dagger A)^T &= A^\dagger A, \\ (A A^\dagger)^T &= A A^\dagger. \end{aligned}$$

In general, $AA^\dagger$ is not necessarily an identity matrix, while it maps all column vectors of A to themselves, and $(A^\dagger)^\dagger = A$.

It is worth mentioning that the pseudo-inverse of a matrix always exists and unique, and can be efficiently calculated using Singular Value Decomposition (SVD) method. Let A be an $m \times n$ real matrix. Then $A = U\Sigma V$ is the SVD of A where Σ is an $m \times n$ diagonal matrix with Σ_{ii} $i = 1, \ldots, r$ as square roots of eigenvalues of $A^T A$, and $r = rank(A)$. Further, U and V are orthogonal matrices of dimension m and n, respectively. In this case, $A^\dagger = V^T \Sigma^\dagger U^T$, where $\Sigma^\dagger$ is an $n \times m$ matrix with $\Sigma^\dagger_{ii} = \frac{1}{\Sigma_{ii}}$ for $i = 1, \ldots, r$ and zero otherwise. In special cases, when A has full-row rank, $A^\dagger = (AA^T)^{-1}A^T$ and when it has full-column rank, $A^\dagger = A^T(AA^T)^{-1}$. We refer to [2] for more details.

Let $a_{i,j}$ denote the elements of the matrix $A \in \mathbb{R}^{m\times n}$. For an integer $0 < r \leq n$, the set of increasing sequences of r elements from $\{1, \ldots, n\}$, is denoted by $Q_{r,n} = \{(i_1, i_2, \cdots, i_r) : 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$. In this way, for $I \in Q_{r,m}$ and $J \in Q_{r,n}$, the submatrix

$$A_{IJ}: \text{ the } r \times r \text{ submatrix of } A \text{ with elements } a_{i,j} \text{ where } i \in I, j \in J,$$

and the associated indices set

$$\mathcal{H}(A) = \{(I, J) : I \in Q_{r,m}, J \in Q_{r,n}, A_{IJ} \text{ is nonsingular}\},$$

are defined. It was shown [1] that the pseudo-inverse $A^\dagger$ is a convex combination of ordinary inverses $\{A^{-1}_{IJ} \; : \; (I, J) \in \mathcal{H}(A)\}$. Moreover, for the $Ax = b$, $x = A^\dagger b$ is a solution [3]. In other words, the Euclidean norm least squares solution of the linear system $Ax = b$ is a convex combination of basic solutions $A^{-1}_{IJ} b_I$, where b_I stands for the subvector of $b \in \mathbb{R}^m$ with indices in I.

Finally, let $b, c \in \mathbb{R}^l$ and C be an $l \times l$ matrix. Based on the realization theory [26], a rational function $f(t) = 1 + tc^T(I_l + tC)^{-1}b$ can be described completely in terms of eigenvalues of two matrices C and $C^\times = C + bc^T$ as

$$\begin{aligned} f(t) &= \det f(t) = \det(1 + tc^T(I_l + tC)^{-1}b) = \det(I_l + tbc^T(I_l + tC)^{-1}) \\ &= \frac{\det(I_l + t(C + bc^T))}{\det(I_l + tC)} = \frac{\det(I_l + tC^\times)}{\det(I_l + tC)}, \end{aligned}$$

where I_l is an $l\times l$ identity matrix. More clearly,

$$f(t) = \prod_{j=1}^{l} \frac{1 + t\alpha_j^\times}{1 + t\alpha_j}, \tag{5}$$

where $\alpha_1, \ldots, \alpha_l$ and $\alpha_1^\times, \ldots, \alpha_l^\times$ are eigenvalues of C and $C^\times$, counted according to their multiplicities. The number l of factors in numerator and denominator on the right-hand-side of (5) is minimal when C and $C^\times$ do not have common eigenvalues.

4 Induced optimal partition invariancy analysis

In this section, we generalize the methodology of induced optimal partition invariancy analysis in [18] for the Problem $P_t(\triangle)$. To do this, we first convert $P_t(\triangle)$ to an equivalent one with only perturbation in the left-hand-side of its constraints. It is necessary to mention that the equivalency of two problems with different numbers of variables and constraints implies the similar features for them and enables one to identify a solution of one by the others. [4].

Consider the substitutions $x_0 = (c + t\triangle c)^T x$ and $z = \triangle Ax - \triangle b$, which converts $P_t(\triangle)$ to

$$\min\Big\{x_0 : (c + t\triangle c)^T x - x_0 = 0,\ Ax + tz = b, \triangle Ax - z = \triangle b,\ x \geq 0\Big\}. \tag{6}$$

By simplifying notations

$$\tilde{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tilde{A} = \begin{pmatrix} c^T & 0 & -1 \\ A & 0 & 0 \\ \triangle A & -I_m & 0 \end{pmatrix}, \triangle\tilde{A} = \begin{pmatrix} \triangle c^T & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tilde{b} = \begin{pmatrix} 0 \\ b \\ \triangle b \end{pmatrix},$$

$\tilde{x}_j := x_j, j = 1, \ldots, n$, $\tilde{x}_{n+j} := z_j, j = 1, \ldots, m$, and $\tilde{x}_{m+n+1} := x_0$, Problem (6) can be squeezed as

$$P_t(\triangle\tilde{A}) \qquad \min\Big\{\tilde{c}^T\tilde{x}\ :\ (\tilde{A} + t\triangle\tilde{A})\tilde{x} = \tilde{b}, \tilde{x}^T = (x^T, z^T, x_0),\ x \geq 0\Big\},$$

where I_m is the $m \times m$ identity matrix, and zeros are of appropriate sizes. Its dual is

$$D_t(\triangle\tilde{A}) \quad \max\Big\{\tilde{b}^T\tilde{y}\ :\ (\tilde{A} + t\triangle\tilde{A})^T\tilde{y} + \tilde{s} = \tilde{c}, \tilde{y}^T = (v, y^T, \bar{y}^T), \tilde{s}^T = (s^T, \mathbf{0}, 0), s \geq 0\Big\},$$

where $v \in \mathbb{R}$, $y, \bar{y} \in \mathbb{R}^m$, and in $(s^T, \mathbf{0}, 0)$, $s \in \mathbb{R}^n$, $\mathbf{0}$ is an m-vector of zeros, and the latter is simply the real number zero. Let $\mathcal{P}_t^*(\triangle\tilde{A})$ and $\mathcal{D}_t^*(\triangle\tilde{A})$, respectively denote optimal solution sets of $P_t(\triangle\tilde{A})$ and $D_t(\triangle\tilde{A})$, and $\tilde{\mathcal{I}} = \{1, \ldots, m+n+1\}$ be the index set of variables in $\mathcal{P}_t^*(\triangle\tilde{A})$.

We define the partition $\tilde{\pi}_t = (\tilde{B}_t, \tilde{N}_t)$, as $\tilde{B}_t := B_t \cup B_t^- \cup B_t^+$ with

$$\begin{aligned} B_t &:= \{j | x_j^*(t) > 0,\ 1 \leq j \leq n, \text{for some } x^*(t) \in \mathcal{P}_t^*(\triangle\tilde{A})\}, \\ B_t^- &:= \{j | x_j^*(t) < 0,\ n+1 \leq j \leq m+n+1, \text{for some } x^*(t) \in \mathcal{P}_t^*(\triangle\tilde{A})\}, \\ B_t^+ &:= \{j | x_j^*(t) > 0,\ n+1 \leq j \leq m+n+1, \text{for some } x^*(t) \in \mathcal{P}_t^*(\triangle\tilde{A})\}, \end{aligned}$$

and $\tilde{N}_t = N_t \cup N_t^\circ$ with

$$\begin{aligned} N_t &:= \{j | s_j^*(t) > 0,\ 1 \leq j \leq n, \text{for some } \ (y^*(t), s^*(t)) \in \mathcal{D}_t^*(\triangle\tilde{A})\}, \\ N_t^\circ &:= \{j | x_j^*(t) = 0,\ n+1 \leq j \leq m+n+1, \forall x^*(t) \in \mathcal{P}_t^*(\triangle\tilde{A})\}. \end{aligned}$$

Observe that in this partition, B_t is the same as in the optimal partition extracted from a strictly complementary optimal solution of Problems $P_t(\triangle)$ and $D_t(\triangle)$. Moreover, considering the equivalence of two Problems $P_t(\triangle)$ and $P_t(\triangle\tilde{A})$, an optimal solution $(\tilde{x}^*(t), \tilde{y}^*(t), \tilde{s}^*(t))$ can be induced by an optimal solution $x^*(t)$ of Problem $P_t(\triangle)$ and vice versa. Therefore, an optimal solution $\tilde{x}^*(t)$ of $P_t(\triangle\tilde{A})$ can be considered as an induced optimal solution, and its corresponding partition as an induced optimal partition.

It is worth mentioning that an induced optimal solution is an induced strictly complementary optimal solution when for all $j \in \{1,\ldots,n\}$, $\tilde{x}_j^* + \tilde{s}_j^* > 0$, and for $n+1 \leq j \leq m+n+1$, $\tilde{x}_j^*(t)$ and $\tilde{s}_j^*(t)$ are not simultaneously zero. More clearly from this property, $s_j^*(t)$ is positive for $1 \leq j \leq n$ only when $x_j^*(t)$ is zero for these indices. This means that at a parameter value when optimal partition changes, some positive $s_j^*(t)$ vanishes while corresponding $x_j^*(t)$'s become positive or vice versa. To imitate this property when $s_j^*(t) = 0$ for all $j \in \{n+1,\ldots,m+n+1\}$, we also considered corresponding $x_j^*(t)$'s as zero in N_t°. Thus, a change in the induced optimal partition at a parameter value may happen by moving some variables with indices in N_t° out of zero. In this case, these indices move from N_t° to either B_t^- or B_t^+ and vice versa.

Let π_{t_0} be the known optimal partition of Problems $P_{t_0}(\triangle)$ and $D_{t_0}(\triangle)$ and $\tilde{\pi}_{t_0}$ be the induced optimal partition of $P_{t_0}(\triangle\tilde{A})$ and $D_{t_0}(\triangle\tilde{A})$. Let $l = |B_{t_0}| \leq n$ and $l \leq \tilde{l} = |\tilde{B}_{t_0}| \leq m+n+1$. Let us adapt the notations and concepts in Section 3 to the Problem $P_t(\triangle\tilde{A})$. We first assume

$$\begin{aligned} \tilde{\tau} &: \{1,\ldots,l,\ldots,\tilde{l}\} \to \{1,\ldots,m+n+1\}, \\ \tilde{\tau}' &: \{1,\ldots,m+n+1-\tilde{l}\} \to \{1,\ldots,m+n+1\}, \end{aligned}$$

are injective and increasing functions, $\text{Range}(\tilde{\tau}) \cup \text{Range}(\tilde{\tau}') = \{1,\ldots,m+n+1\}$, and $\text{Range}(\tilde{\tau}) \cap \text{Range}(\tilde{\tau}') = \varnothing$. Observe that an induced optimal solution with the corresponding induced optimal partition $\tilde{\pi}_{t_0}$ implies

$$\tilde{B}_{t_0} = \{\tilde{\tau}(1),\ldots,\tilde{\tau}(\tilde{l})\}, \tilde{N}_{t_0} = \{\tilde{\tau}'(1),\ldots,\tilde{\tau}'(m+n+1-\tilde{l})\}.$$

Moreover, $\{j | n+1 \leq j \leq m+n+1, v_j \neq 0\}$ is empty when $l = \tilde{l}$. That is, $\tilde{B}_{t_0} = \{\tau(1),\ldots,\tau(l)\}$.

For a fixed $t_0 \in \Lambda$, when $l = \tilde{l}$, it holds

$$\tilde{x}_j^*(t_0) \begin{cases} > 0, & \text{if } j \in \{\tilde{\tau}(1),\ldots,\tilde{\tau}(l)\}, \\ = 0, & \text{if } j \in \{\tilde{\tau}'(1),\ldots,\tilde{\tau}'(m+n+1-l)\}, \end{cases}$$

and when $l < \tilde{l}$,

$$\tilde{x}_j^*(t_0) \begin{cases} > 0, & \text{if } j \in \{\tilde{\tau}(1),\ldots,\tilde{\tau}(l)\}, \\ \neq 0, & \text{if } j \in \{\tilde{\tau}(l+1),\ldots,\tilde{\tau}(\tilde{l})\}, \\ = 0, & \text{if } j \in \{\tilde{\tau}'(1),\ldots,\tilde{\tau}'(m+n+1-\tilde{l})\}. \end{cases}$$

Further, for the induced optimal solution $(\tilde{y}^*(t_0), \tilde{s}^*(t_0))$ of Problem $D_{t_0}(\triangle\tilde{A})$, it holds

$$\tilde{s}_j^*(t_0) \begin{cases} > 0, & \text{if} \quad j \in N_{t_0}, \\ = 0, & \text{otherwise.} \end{cases}$$

The necessary and sufficient conditions for a primal-dual optimal solution of $P_t(\triangle\tilde{A})$ and $D_t(\triangle\tilde{A})$ being a strictly induced optimal solution are stated in the following theorem. We restate that the pseudo-inverse exists and is unique for a matrix of an arbitrary size, and in special case, it is identical with the common inverse. This means that in our study, $\tilde{A}^{\dagger}_{\tilde{\tau}}(t)$ and $A^{\dagger}_{\tau}(t)$ exist for any possible parameter value t.

Theorem 1. *For $t \in \Lambda$, let $\pi_t = (B_t, N_t)$ and $\tilde{\pi}_t = (\tilde{B}_t, \tilde{N}_t)$. Then, $(\tilde{x}^*(t), \tilde{y}^*(t), \tilde{s}^*(t))$ is a strictly induced complementary optimal solution of $P_t(\triangle\tilde{A})$ and $D_t(\triangle\tilde{A})$ if and only if*

Cond. 1 *For $1 \le q \le \tilde{l}$, $\tilde{x}^*_{\tilde{\tau}(q)}(t) = e_q^T \tilde{A}^{\dagger}_{\tilde{\tau}(q)}(t)\tilde{b}$ is positive when $\tilde{\tau}(q) \in B_t \cup B_t^+$, negative when $\tilde{\tau}(q) \in B_t^-$, and zero otherwise,*

Cond. 2 *For $p \in \mathrm{Range}(\tau')$, $\tilde{s}^*_p(t) = (c + t\triangle c)_p - (c + t\triangle c)_\tau^T A^{\dagger}_{\tau}(t) A_p(t)$ is positive, and it is zero otherwise.*

Proof. Recall that in problem $P_t(\triangle\tilde{A})$, for $1 \le q \le \tilde{l}$ we have $\tilde{x}_{\tilde{\tau}(q)}(t) = e_q^T \tilde{A}^{\dagger}_{\tilde{\tau}(q)}(t)\tilde{b}$ as a strictly feasible solution when it is positive for $\tilde{\tau}(q) \in B_t \cup B_t^+$, negative for $\tilde{\tau}(q) \in B_t^-$, and zero otherwise. Further, the strictly feasibility of $D_t(\triangle\tilde{A})$, i.e., $\tilde{s}(t) > 0$ is identical with the strictly feasibility of $D_t(\triangle)$, i.e., $s(t) > 0$. To be clear, let us consider the constraints of $D_t(\triangle\tilde{A})$ as

$$\begin{pmatrix} c + t\triangle c & A^T & \triangle A^T \\ 0 & tI_m & -I_m \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ y \\ \bar{y} \end{pmatrix} + \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{7}$$

From the first row of (7), we have $s = -(c+t\triangle c)^T v - A^T y - \triangle A^T \bar{y}$, while from the second row $\bar{y} = ty$ and from the last row, $v = -1$. By replacing the values of v and $\bar{y}$ in s, the inequality $\tilde{s}^T = (s^T, 0, 0) > 0$ leads to $s = (c + t\triangle c) - (A + t\triangle A)^T y > 0$. For being an induced primal strictly complementary optimal solution of $\tilde{x}^*(t)$, we must have $\tilde{x}^*_{\tilde{\tau}(q)}(t) > 0$ for $\tilde{\tau}(q) \in B_t \cup B_t^+$, and $\tilde{x}^*_{\tilde{\tau}(q)}(t) < 0$ for $\tilde{\tau}(q) \in B_t^-$. Further, strictly induced complementarity imposes to have $\tilde{x}^*_{\tilde{\tau}'(p)}(t) = 0$ for $\tilde{\tau}'(p) \in N_t$ as well as for $\tilde{\tau}'(p) \in N_t^\circ$, where $p \in \mathrm{Range}(\tilde{\tau}')$. On the other hand, strictly complementarity of $P_t(\triangle)$ and $D_t(\triangle)$ implies for $p \in \mathrm{Range}(\tau')$, $s_p(t) > 0$ and $s_\tau(t) = 0$. Equivalently for $p \in \mathrm{Range}(\tau')$, $(c + t\triangle c)_p^T - (c + t\triangle c)_\tau^T A^{\dagger}_{\tau}(t) A_p(t) > 0$. Validity of the inverse of statement is immediate and omitted. □

Now, let $\tilde{\pi}_{t_0} = (\tilde{B}_{t_0}, \tilde{N}_{t_0})$ be the induced optimal partition of Problem $P_{t_0}(\triangle\tilde{A})$. Respecting the concept of induced optimal partition invariancy, the aim in this study is to identify the region $\tilde{\Lambda}_0 \subseteq \Lambda$, where for all $t \in \tilde{\Lambda}_0$, the induced optimal partition of Problem $P_t(\triangle\tilde{A})$ is identical with $\tilde{\pi}_{t_0}$. Note that this invariancy region contains all such t's at which Conds 1 and 2 in Theorem 1 hold.

To clarify the differences between a change point and a transition point, we have to mention some facts. Recall that, when $\triangle\tilde{A}_{\tilde{\tau}}$ is an admissible direction (See page 150) the induced optimal partition invariancy region is an interval. At the endpoints of this interval, the problem $P_t(\triangle\tilde{A})$ may fail to be bounded or feasible. In this case, there is no induced optimal partition. Otherwise, induced optimal partitions are changed. Variation in an induced partition would be the result of two possibilities. First, some indices interchange between $\tilde{B}_t$ and $\tilde{N}_t$ when the parameter reaches to one of the endpoints of the interval. More clearly, this transition may happen between B and N, or between B_t^+, B_t^-, and N_t°. We refer to the point in the former as a *transition point*, and in the latter as a *change point*.

Let us mention some facts about the behavior of the optimal value function at change and transition points. First we need to mention some properties of the feasible solution set $\mathcal{P}_t$. By the continuity of the objective function and constraints (in fact they are linear), the feasible set mapping $\mathcal{P}_t$ of $P_t(\triangle)$ is always closed. We also remind the concept of local compactness as follows. The set $\mathcal{P}_{\bar{t}}$ is locally compact at $\bar{t}$ if there exists $\delta > 0$ and a compact set C_0 such that

$$\bigcup_{\|t-\bar{t}\|\leq\delta} \mathcal{P}_t \subseteq C_0.$$

It is not hard to prove that a system of a parametric linear equations system satisfy this condition. Thus, with these two conditions, the set-valued map $t \to \mathcal{P}_t$ is upper semi-continuous at $\bar{t} \in \Lambda$. Moreover, with the constraint qualification property at $\bar{t}$, that is the interior of the feasible set for every $t \in \Lambda$ is nonempty and it is fulfilled in a linear optimization problem when solving with an interior point method, the lower semi-continuity of this map is also guaranteed. As a result, $\mathcal{P}_t$ is continuous w.r.t. $t \in \Lambda$.

Lower semi-continuity of the optimal value function needs further condition than just continuity of the feasible solution sets $\mathcal{P}_{\bar{t}}$. , i.e., a weak upper semi-continuity, saying that for t near $\bar{t}$ at least one point $x_t^* \in \mathcal{P}_t^*$ can be approached by elements in a compact subset of $\mathcal{P}_{\bar{t}}$. Since of closedness of the feasible set $\mathcal{P}_{\bar{t}}$, this condition is also fulfilled. To assure upper semi-continuity of the optimal value function at $\bar{t}$, we only need lower semi-continuity condition at one point $x^* \in \mathcal{P}_{\bar{t}}^*$, and this is fulfilled by the constraint qualification property at $\bar{t}$ (See [24] for more detail). Considering these facts, the optimal value function is continuous for all $t \in \Lambda$, including the transition and change points.

Note that when a parameter value is a change point only, indices of free variables interchange only between their index sets. Since they are absent in the objective function, then this variation does not affect its representation. The representation of the optimal value function on the neighborhoods of a change point does not alter when it is not a transition point, simultaneously. On the other hand, the representation of the optimal value function changes at a transition point, and it fails to have the first derivative.

Let us elaborate on the concept of a change point more clearly. Let the problem $P_t(\triangle\tilde{A})$ correspond to a production plan, and without loss of generality $t_0 = 0$ be a change point of this problem. Recall that the optimal value function in Problem $P_t(\triangle)$ is a free variable x_0 in Problem $P_t(\triangle\tilde{A})$. Thus, a possibility of a situation in a change point is the change in the sign of x_0, which means that the sign of parametric optimal value function changes at this change point. Here, the objective function stands as the production cost in a manufacturing plan, and therefore, it may increase when the parameter value decreases. Alternatively, the production cost decreases when x_0 goes from negative to positive at this change point.

Other possibility is the change in a free variable in some constraints of Problem $P_t(\triangle\tilde{A})$. Again without loss of generality, let the free variable in one constraint is zero in an optimal solution of the problem at this change point, and its sign changes passing through this point. In this case we have an equation like

$$(a_{i1} + \triangle a_{i1}t)x_1 + \cdots + (a_{in} + \triangle a_{in}t)x_n + x_{n+i} = b_i + \triangle b_i t,$$

where b_i is the amount of available amount of source i, and a unit of product j needs a_{ij} amount of the source i. Here, x_{n+i} stands for the amount of excess source in the production plan.

Therefore, $\triangle a_{ij} > 0$ might translate as an increase in the production time of j when the quality of this product increases. This promotion in the quality may necessitate a change on b_i, the available time source, while this may not be the only reason.
An equivalent form of this constraint is

$$\begin{array}{rrrrl} a_{i1}x_1 + \cdots + & a_{in}x_n + & x_{n+i} + & tz & = b_i \\ \triangle a_{i1}x_1 + \cdots + & \triangle a_{in}x_n + & - & z & = \triangle b_i, \end{array}$$

where t could be considered as the degree of quality. Thus, $z = \triangle a_{i1}x_1 + \cdots + \triangle a_{in}x_n - \triangle b_i = 0$ for $t_0 = 0$ by the above-mentioned assumption, while we may have $z > 0$ for some $t > 0$, and $z < 0$ for some $t < 0$. This means that increasing of the quality, $(t > 0)$, implies in slack in total considered extra time $\triangle b_i$ versus the necessary time $\triangle a_{i1}x_1 + \cdots + \triangle a_{in}x_n$ in an optimal solution at $t > 0$ (i.e. $z > 0$), and vis versa. Observe that these two possibilities may occur simultaneously as well.

As illustrated above, the optimal value function is differentiable at mere change points because its representation does not alter. However, it fails to be differentiable at transition points, since the representation of the optimal value function is calculated based on eigenvalues of different matrices (See Sections 6 and 8).

4.1 Special case

Let us consider the special case when $\triangle b = 0$. By substitution $x_0 = (c + t\triangle c)^T x$, the Problem $P_t(\triangle A, \triangle c)$ is converted to

$$\min \Big\{ x_0 : (c + t\triangle c)^T x - x_0 = 0,\ (A + t\triangle A)x = b,\ x \geq 0 \Big\},$$

where $x \in \mathbb{R}^n$ and $x_0 \in \mathbb{R}$. This problem can be summarized as

$$P_t(\triangle \hat{A}) \qquad \min \Big\{ \hat{c}^T \hat{x} \ : \ (\hat{A} + t\triangle \hat{A})\hat{x} = \hat{b}, \hat{x}^T = (x^T, x_0),\ x \geq 0 \Big\},$$

where

$$\hat{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \hat{A} = \begin{pmatrix} c^T & -1 \\ A & 0 \end{pmatrix}, \triangle \hat{A} = \begin{pmatrix} \triangle c^T & 0 \\ \triangle A & 0 \end{pmatrix}, \hat{b} = \begin{pmatrix} 0 \\ b \end{pmatrix}, \hat{x} = \begin{pmatrix} x \\ x_0 \end{pmatrix}.$$

Here, zeros are of appropriate sizes and $\hat{x}_j := x_j, j = 1, \ldots, n,\ \hat{x}_{n+1} := x_0$. The dual of this problem is

$$D_t(\triangle \hat{A}) \qquad \max \Big\{ \hat{b}^T \hat{y} : (\hat{A} + t\triangle \hat{A})^T \hat{y} + \hat{s} = \hat{c}, \hat{y}^T = (v_1^T, v_2^T), \hat{s}^T = (s^T, 0), s \geq 0 \Big\},$$

with $v_1 \in \mathbb{R}$, and $v_2 \in \mathbb{R}^m$ where $s \in \mathbb{R}^n$ and zero is a number. Let $\mathcal{P}_t^*(\triangle \hat{A})$ and $\mathcal{D}_t^*(\triangle \hat{A})$ denote optimal solution sets of $P_t(\triangle \hat{A})$ and $D_t(\triangle \hat{A})$, respectively. Let $\pi_{t_0} = (B_{t_0}, N_{t_0})$ be the known optimal partition of Problems $P_{t_0}(\triangle A, \triangle c)$ and $D_{t_0}(\triangle A, \triangle c)$. We define the partition $\hat{\pi}_{t_0} = (\hat{B}_{t_0}, \hat{N}_{t_0})$, as $\hat{B}_{t_0} := B_{t_0} \cup B_{t_0}^- \cup B_{t_0}^+$ where

$$\begin{aligned} B_{t_0}^- &:= \{n+1 : \hat{x}_{n+1}^*(t_0) < 0,\ \text{for some } x^*(t_0) \in \mathcal{P}_{t_0}^*(\triangle \hat{A})\}, \\ B_{t_0}^+ &:= \{n+1 : \hat{x}_{n+1}^*(t_0) > 0,\ \text{for some } x^*(t_0) \in \mathcal{P}_{t_0}^*(\triangle \hat{A})\}, \end{aligned}$$

and $\hat{N}_{t_0} = N_{t_0} \cup N^{\circ}_{t_0}$ where

$$N^{\circ}_{t_0} := \{n+1 : \hat{x}^*_{n+1}(t_0) = 0, \text{ for all } x^*(t_0) \in \mathcal{P}^*_{t_0}(\triangle \hat{A})\}.$$

Recall that for all $i \in \{1,\ldots,n\}$, $\hat{x}^*_j + \hat{s}^*_j > 0$, and for $j = n+1$, $\hat{x}^*_j(t)$ and $\hat{s}^*_j(t)$ are not simultaneously zero. Thus, an induced optimal solution will be an induced strictly complementary optimal solution.

Now, let $\pi_{t_0} = (B_{t_0}, N_{t_0})$ denote the optimal partition of Problems $P_{t_0}(\triangle A, \triangle c)$ and $D_{t_0}(\triangle A, \triangle c)$, and $\hat{\pi}_{t_0} = (\hat{B}_{t_0}, \hat{N}_{t_0})$ stands for the induced optimal partition of Problems $P_{t_0}(\triangle \hat{A})$ and $D_{t_0}(\triangle \hat{A})$. Further, let $l = |B_{t_0}| \le n$ and $l \le \hat{l} = |\hat{B}_{t_0}| \le n+1$. We adapt the notations and concepts in Section 3 to the Problem $P_t(\triangle \hat{A})$. Let us consider

$$\begin{aligned} \hat{\tau} &: \{1,\ldots,l,\ldots,\hat{l}\} \to \{1,\ldots,n,n+1\}, \\ \hat{\tau}' &: \{1,\ldots,n+1-\hat{l}\} \to \{1,\ldots,n,n+1\}, \end{aligned}$$

are injective and increasing functions, $\text{Range}(\hat{\tau}) \cup \text{Range}(\hat{\tau}') = \{1,\ldots,n+1\}$, and $\text{Range}(\hat{\tau}) \cap \text{Range}(\hat{\tau}') = \varnothing$. For an induced optimal solution with induced optimal partition $\hat{\pi}_{t_0}$, one can show that

$$\hat{B}_{t_0} = \{\hat{\tau}(1),\ldots,\hat{\tau}(\hat{l})\}, \hat{N}_{t_0} = \{\hat{\tau}'(1),\ldots,\hat{\tau}'(n+1-\hat{l})\}.$$

If $n+1 \in \hat{N}_{t_0}$, then $l = \hat{l}$, and $\hat{B}_{t_0} = \{\tau(1),\ldots,\tau(l)\}$. Note that, for a fixed $t_0 \in \Lambda$, the induced optimal primal-dual solution $(\hat{x}^*_j(t_0), \hat{y}^*_j(t_0), \hat{s}^*_j(t_0))$ is defined as ones in general case with except that here, $m+n+1$ reduces to $n+1$, and some notations must be adapted.

Let us mention the necessary and sufficient conditions for an optimal solution of $P_0(\triangle \hat{A})$ being a strictly induced optimal solution. The proof is similar to the proof of Theorem 1, with the exception that functions τ and τ' are defined as (3) for Problem $P_t(\triangle A, \triangle c)$, where $\tau'(i) \subseteq \{i \in \{1,\ldots,n-l\}\}$ corresponds to the positive variables s_i in Problem $D_t(\triangle A, \triangle c)$.

Theorem 2. *Let for $t \in \Lambda$, $\pi_t = (B_t, N_t)$ be the optimal partition of Problems $P_t(\triangle A, \triangle c)$ and $D_t(\triangle A, \triangle c)$. Further, let $\hat{\pi}_t = (\hat{B}_t, \hat{N}_t)$ be the known induced optimal partition of problems $P_t(\triangle \hat{A})$ and $D_t(\triangle \hat{A})$. Then, $(\hat{x}^*(t), \hat{y}^*(t), \hat{s}^*(t))$ is a strictly induced complementary optimal solution of these problems if and only if*

Cond. 1 *For $1 \le q \le \hat{l}$, $\hat{x}^*_{\hat{\tau}(q)}(t) = e_q^T \hat{A}^{\dagger}_{\hat{\tau}}(t)\hat{b}$ is positive when $\hat{\tau}(q) \in B_t \cup B_t^+$, negative when $\hat{\tau}(q) \in B_t^-$, and zero otherwise,*

Cond. 2 *For $p \in \text{Range}(\tau')$, $\hat{s}^*_p(t) = (c + t\triangle c)_p - (c+t\triangle c)^T_{\tau} A^{\dagger}_{\tau}(t) A_p(t)$, is positive, and it is zero otherwise.*

5 Identifying an induced invariancy interval

Let $\tilde{\pi}_{t_0} = (\tilde{B}_{t_0}, \tilde{N}_{t_0})$ be the induced optimal partition of Problem $P_0(\triangle \tilde{A})$. First recall the fact that for two arbitrary matrices $U \in \mathbb{R}^{m\times n}$ and $W \in \mathbb{R}^{n\times n}$, where W is invertible, $(UW)^{\dagger} = W^{-1}U^{\dagger}$. Analogously, when $U \in \mathbb{R}^{m\times m}$, $W \in \mathbb{R}^{m\times n}$, and U is invertible, it holds $(UW)^{\dagger} = W^{\dagger}U^{-1}$ [25].

Let us first mention a fact. Considering the original optimal partition $\pi = (B, N)$, the submatrix A_B, corresponding with the columns in B, has full row or column rank. Recall that $\tilde{A}_{\tilde{\tau}}(t)$ has some additional rows than A_B. If A_B has full column rank ($|B| < m$), then adding some rows to this matrix does not result in losing this property for $\tilde{A}_{\tilde{\tau}}(t)$. On the other hand, when A_B has full row rank ($|B| > m$), adding some rows to this submatrix may lead it to lose this property. In this case, one can remove redundant rows without affecting the in-hand optimal partition.

Now, to determine the matrix $\tilde{A}^{\dagger}_{\tilde{\tau}}(t)$, three possibilities would be considered. When $2m+1 < \tilde{l}$ and $\tilde{A}_{\tilde{\tau}}(t)$ has full-row rank, it holds $\tilde{A}_{\tilde{\tau}}(t_0)\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0) = I_{2m+1}$. Thus,

$$\tilde{A}^{\dagger}_{\tilde{\tau}}(t) = (\tilde{A}_{\tilde{\tau}}(t_0) + (t-t_0)\triangle\tilde{A}_{\tilde{\tau}})^{\dagger} = (\tilde{A}_{\tilde{\tau}}(t_0)(I_{\tilde{l}} + (t-t_0)\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)\triangle\tilde{A}_{\tilde{\tau}}))^{\dagger}. \tag{8}$$

Respecting the above-mentioned fact, when $(I_{\tilde{l}} + (t-t_0)\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)\triangle\tilde{A}_{\tilde{\tau}})$ is invertible, the right-hand-side of (8) is equivalent to

$$(I_{\tilde{l}} + (t-t_0)\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)\triangle\tilde{A}_{\tilde{\tau}})^{-1}\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0). \tag{9}$$

For $2m+1 > \tilde{l}$, when $\tilde{A}_{\tilde{\tau}}(t)$ has full-column rank, it holds $\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)\tilde{A}_{\tilde{\tau}}(t_0) = I_{\tilde{l}}$, and consequently

$$\tilde{A}^{\dagger}_{\tilde{\tau}}(t) = (\tilde{A}_{\tilde{\tau}}(t_0) + (t-t_0)\triangle\tilde{A}_{\tilde{\tau}})^{\dagger} = ((I_{2m+1} + (t-t_0)\triangle\tilde{A}_{\tau}\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0))\tilde{A}_{\tilde{\tau}}(t_0))^{\dagger}. \tag{10}$$

When $(I_{2m+1} + (t-t_0)\triangle\tilde{A}_{\tau}\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0))$ is invertible, then the right-hand-side of (10) is equivalent to

$$\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)(I_{2m+1} + (t-t_0)\triangle\tilde{A}_{\tau}\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0))^{-1}. \tag{11}$$

Finally for $2m+1 = \tilde{l}$, when $\tilde{A}_{\tilde{\tau}}(t)$ has full-row rank and full-column rank, since Moore-Penrose inverse of $\tilde{A}^{\dagger}_{\tilde{\tau}}(t)$ is identical with the standard inverse, i.e., $\tilde{A}^{-1}_{\tilde{\tau}}(t)$, thus

$$\tilde{A}^{\dagger}_{\tilde{\tau}}(t) = \tilde{A}^{-1}_{\tilde{\tau}}(t) = (I_{2m+1} + (t-t_0)\tilde{A}^{-1}_{\tilde{\tau}}(t_0)\triangle\tilde{A}_{\tilde{\tau}})^{-1}\tilde{A}^{-1}_{\tilde{\tau}}(t_0). \tag{12}$$

In all of these cases, the SVD of $\tilde{A}_{\tilde{\tau}}(t)$ is a general computational method to calculate the $\tilde{A}^{\dagger}_{\tilde{\tau}}(t)$ (See Page 152). In this study, our methodology is based on nonzero eigenvalues of $\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)\triangle\tilde{A}_{\tilde{\tau}}$, $\triangle\tilde{A}_{\tau}\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)$, or $\tilde{A}^{-1}_{\tilde{\tau}}(t_0)\triangle\tilde{A}_{\tilde{\tau}}$, depending on the magnitude of $\tilde{l}$. Respecting the fact that for two matrices Q and T with appropriate sizes, nonzero eigenvalues of TQ and QT are identical, and the extra ones are zero [16, Theorem 1.3.20], and depending on the size of TQ and QT, it is more-cost-effective to use the one with less dimension. Here, without loss of generality, we assume that $2m+1 \leq \tilde{l}$ and construct our methodology based on (9).

Let for t_0, we know $\mathcal{H}(\tilde{A}_{\tilde{\tau}}(t_0))$. The following theorem presents a tool that is capable to determine some intervals, where for $t \in \Lambda$ the associated inverse in (9) exists. Note that at these intervals, the corresponding index set $\mathcal{H}(\tilde{A}_{\tilde{\tau}}(t))$ of pseudo-inverse of $\tilde{A}_{\tilde{\tau}}(t)$ is identical to $\mathcal{H}(\tilde{A}_{\tilde{\tau}}(t_0))$. The proof is similar to the proof of Theorem 2 in [18], and omitted.

Theorem 3. *For a given t_0, let $\tilde{\tau}$ correspond to the induced optimal partition $\tilde{\pi}_{t_0}$ of Problem $P_{t_0}(\triangle\tilde{A})$. Then, for all $t \in \Lambda_{\tilde{\pi}_{t_0}}$ with $t \neq t_0$, the inverse of $(I_{\tilde{l}} + (t-t_0)\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)\triangle\tilde{A}_{\tilde{\tau}})$ in (9) exists if and only if*

$$1 + \alpha_j(t-t_0) \neq 0, \; j = 1, \ldots, \tilde{l}, \tag{13}$$

where $\alpha_1, \ldots, \alpha_{\tilde{l}}$ are nonzero eigenvalues of $\triangle\tilde{A}_{\tilde{\tau}}\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)$.

Remark 1. *Recall that for* $2m+1 < \tilde{l}$*, the size of* $\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)\triangle\tilde{A}_{\tilde{\tau}}$ *is greater than the size of* $\triangle\tilde{A}_{\tilde{\tau}}\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)$ *. Thus, it is more cost-effective to consider* $\alpha_1,\dots,\alpha_{\tilde{l}}$ *as the non zero eigenvalues of* $\triangle\tilde{A}_{\tilde{\tau}}\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)$*.*

Corollary 1. *Let* $P_{t_0}(\triangle\tilde{A})$ *have a unique basic optimal solution, i.e.,* $\tilde{l} = 2m+1$*. In this case,* $\tilde{A}_{\tilde{\tau}}(t)$ *is invertible for all* $t \in \Lambda_{\tilde{\pi}_{t_0}}$ *if and only if* (13) *holds, where* $\alpha_1,\dots,\alpha_{2m+1}$ *are eigenvalues of* $\tilde{A}^{-1}_{\tilde{\tau}}(t_0)\triangle\tilde{A}_{\tilde{\tau}}$*.*

The following theorem relates Cond. 1, the feasibility of $P_{t_0}(\triangle\tilde{A})$, to the eigenvalues of some other matrices. The proof is similar to the proof of Theorem 3 in [18] and omitted.

Theorem 4. *Let* $\tilde{\tau}$ *correspond to the induced optimal partition* $\tilde{\pi}_{t_0}$ *of Problems* $P_{t_0}(\triangle\tilde{A})$ *and* $D_{t_0}(\triangle\tilde{A})$*. Further, let* $\beta_{q,1},\dots,\beta_{q,\tilde{l}}$ *be nonzero eigenvalues of* $(\triangle\tilde{A}_{\tilde{\tau}}+\tilde{b}e_q^T)\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)$*, and* $\alpha_1,\dots,\alpha_{\tilde{l}}$ *be nonzero eigenvalues of* $\triangle\tilde{A}_{\tilde{\tau}}\tilde{A}^{\dagger}_{\tilde{\tau}}(t_0)$*. Then, for* $1\le q\le\tilde{l}$*, and* $\tilde{\tau}(q)\in B\cup B^+$*,* $\tilde{x}^*_{\tilde{\tau}(q)}(t) > 0$ *is identical with*

$$\prod_{j=1}^{\tilde{l}}\frac{1+(t-t_0)\beta_{q,j}}{1+(t-t_0)\alpha_j}\begin{cases}\ge 1, & \text{if } (t-t_0)\ge 0,\\ \le 1, & \text{if } (t-t_0)\le 0.\end{cases}$$

When $\tilde{\tau}(q)\in B^-$*,* $\tilde{x}^*_{\tilde{\tau}(q)}(t) < 0$ *is identical with*

$$\prod_{j=1}^{\tilde{l}}\frac{1+(t-t_0)\beta_{q,j}}{1+(t-t_0)\alpha_j}\begin{cases}\le 1, & \text{if } (t-t_0)\ge 0,\\ \ge 1, & \text{if } (t-t_0)\le 0.\end{cases}$$

Next theorem relates Cond. 2, the feasibility of $D_{t_0}(\triangle\tilde{A})$, to the eigenvalues of another matrices.

Theorem 5. *Let* $\tilde{\pi}_{t_0}$ *be the known induced optimal partition of Problems* $P_{t_0}(\triangle\tilde{A})$ *and* $D_{t_0}(\triangle\tilde{A})$*. Moreover, let* $\alpha_1,\dots,\alpha_l$*;* $\gamma_{p,1},\dots,\gamma_{p,l}$*;* $\bar{\gamma}_{p,1},\dots,\bar{\gamma}_{p,l}$*;* $\delta_{p,1},\dots,\delta_{p,l}$*; and* $\bar{\delta}_{p,1},\dots,\bar{\delta}_{p,l}$ *be nonzero eigenvalues of* $\triangle A_\tau A^{\dagger}_\tau(t_0)$*;* $(\triangle A_\tau+\triangle A_p c_\tau^T)A^{\dagger}_\tau(t_0)$*;* $(\triangle A_\tau+\triangle A_p\triangle c_\tau^T)A^{\dagger}_\tau(t_0)$*;* $(\triangle A_\tau+(A_p+t_0\triangle A_p)c_\tau^T)A^{\dagger}_\tau(t_0)$*; and* $(\triangle A_\tau+(A_p+t_0\triangle A_p)\triangle c_\tau^T)A^{\dagger}_\tau(t_0)$*, respectively. Then, for* $p\in\text{Range}(\tau')$*, inequality*

$$(c_\tau+t\triangle c_\tau)^T A^{\dagger}_\tau(t)A_p(t)-(c_p+t\triangle c_p) < 0, \tag{14}$$

reduces to

$$\begin{aligned}\frac{1}{t-t_0}\Big(t\prod_{j=1}^{l}\frac{1+(t-t_0)\bar{\delta}_{p,j}}{1+(t-t_0)\alpha_{p,j}}+\prod_{j=1}^{l}\frac{1+(t-t_0)\delta_{p,j}}{1+(t-t_0)\alpha_{p,j}}\Big)+t\prod_{j=1}^{l}\frac{1+(t-t_0)\bar{\gamma}_{p,j}}{1+(t-t_0)\alpha_{p,j}}\\ +\prod_{j=1}^{l}\frac{1+(t-t_0)\gamma_{p,j}}{1+(t-t_0)\alpha_{p,j}} < \frac{t^2+(2-t_0)t+(1-t_0)}{t-t_0}+c_p+t\triangle c_p.\end{aligned} \tag{15}$$

Proof. First, recall that the feasibility of $D_t(\triangle\tilde{A})$, i.e. $\tilde{s}\ge 0$, is identical with the feasibility of $D_t(\triangle)$ (See the proof of Theorem 1). Consider the sets of indices corresponding to positive and zero elements of x in $\tilde{x}$ are respectively denoted by τ and τ' (See Section 3). For $p \in \text{Range}(\tau')$, by substituting of

$$A_p(t) = A_p + t\triangle A_p = A_p + (t-t_0)\triangle A_p + t_0\triangle A_p,$$

in (14), and adding $1 + \frac{1}{t-t_0} + t(1+\frac{1}{t-t_0})$ to its both sides, it simply reduces to

$$\begin{aligned}
&1 + \frac{1}{t-t_0} + c_\tau^T A_\tau^\dagger(t)(A_p + (t-t_0)\triangle A_p + t_0\triangle A_p)\\
&+t\Big(1 + \frac{1}{t-t_0} + \triangle c_\tau^T A_\tau^\dagger(t)(A_p + (t-t_0)\triangle A_p + t_0\triangle A_p)\Big)\\
&< \frac{t^2 + (2-t_0)t + (1-t_0)}{t-t_0} + c_p + t\triangle c_p,
\end{aligned} \tag{16}$$

or equivalently to

$$\begin{aligned}
&1 + (t-t_0)c_\tau^T A_\tau^\dagger(t)\triangle A_p + \frac{1}{t-t_0}\Big(1 + (t-t_0)c_\tau^T A_\tau^\dagger(t)(A_p + t_0\triangle A_p)\Big)\\
&+t\Big(1 + (t-t_0)\triangle c_\tau^T A_\tau^\dagger(t)\triangle A_p + \frac{1}{t-t_0}(1 + (t-t_0)\triangle c_\tau^T A_\tau^\dagger(t)(A_p + t_0\triangle A_p))\Big)\\
&< \frac{t^2 + (2-t_0)t + (1-t_0)}{t-t_0} + c_p + t\triangle c_p.
\end{aligned} \tag{17}$$

By considering (9), (17) can be translated to (15). The proof is complete. □

6 Closed form of the optimal value function

Let $\tilde{Z}(t)$ and $\hat{Z}(t)$ denote the optimal value functions of Problems $P_t(\triangle)$ and $P_t(\triangle A, \triangle c)$, respectively. Without loss of generality, we determine their representation when $l > m$. Other cases go similarly.

Theorem 6. *Let Conds 1 and 2 satisfy for a fixed parameter $t \in \Lambda$. Further, let $\alpha_1, \dots, \alpha_l$, $\alpha_1^\times, \dots, \alpha_l^\times$, $\bar{\alpha}_1^\times, \dots, \bar{\alpha}_l^\times$, $\breve{\alpha}_1^\times, \dots, \breve{\alpha}_l^\times$ and $\check{\alpha}_1^\times, \dots, \check{\alpha}_l^\times$ be nonzero eigenvalues of $\triangle A_\tau A_\tau^\dagger(t_0)$, $(\triangle A_\tau + \triangle b\triangle c_\tau^T)A_\tau^\dagger(t_0)$, $(\triangle A_\tau + \triangle b c_\tau^T)A_\tau^\dagger(t_0)$, $(\triangle A_\tau + b\triangle c_\tau^T)A_\tau^\dagger(t_0)$ and $(\triangle A_\tau + bc_\tau^T)A_\tau^\dagger(t_0)$, respectively. Then*

$$\begin{aligned}
\tilde{Z}(t) = \ & \frac{1}{t-t_0}\Bigg(t^2\prod_{j=1}^{l}\frac{1+(t-t_0)\alpha_j^\times}{1+(t-t_0)\alpha_j} + t\Big(\prod_{j=1}^{l}\frac{1+(t-t_0)\bar{\alpha}_j^\times}{1+(t-t_0)\alpha_j} + \prod_{j=1}^{l}\frac{1+(t-t_0)\breve{\alpha}_j^\times}{1+(t-t_0)\alpha_j}\Big)\\
&+\prod_{j=1}^{l}\frac{1+(t-t_0)\check{\alpha}_j^\times}{1+(t-t_0)\alpha_j} - (t+1)^2\Bigg).
\end{aligned}$$

Proof. Note that in this case

$$\begin{aligned}
\tilde{Z}(t) &= \tilde{c}_{\tilde{\tau}}^T\tilde{x}_{\tilde{\tau}}(t) = x_0 = (c+t\triangle c)_\tau^T x_\tau\\
&= (c+t\triangle c)_\tau^T A_\tau^\dagger(t)(b+t\triangle b)\\
&= t^2\triangle c_\tau^T A_\tau^\dagger(t)\triangle b + t(c_\tau^T A_\tau^\dagger(t)\triangle b + \triangle c_\tau^T A_\tau^\dagger(t)b) + c_\tau^T A_\tau^\dagger(t)b.
\end{aligned}$$

By (9), we have

$$\begin{aligned}(t^2+2t+1)+(t-t_0)\tilde{Z}(t) = \; & t^2\Big(1+(t-t_0)\triangle c_\tau^T A_\tau^\dagger(t)\triangle b\Big)\\ & +t\Big(1+(t-t_0)c_\tau^T A_\tau^\dagger(t)\triangle b+1+(t-t_0)\triangle c_\tau^T A_\tau^\dagger(t)b\Big)\\ & +1+(t-t_0)c_\tau^T A_\tau^\dagger(t)b.\end{aligned} \tag{18}$$

Using the tools from realization theory (See Page 152 and Eq. (5)) on each terms in the right-hand-side of (18), we have

$$\begin{aligned}(t^2+2t+1)+(t-t_0)\tilde{Z}(t) = \; & t^2\prod_{j=1}^{l}\frac{1+(t-t_0)\alpha_j^\times}{1+(t-t_0)\alpha_j}\\ & +t\Big(\prod_{j=1}^{l}\frac{1+(t-t_0)\bar{\alpha}_j^\times}{1+(t-t_0)\alpha_j}+\prod_{j=1}^{l}\frac{1+(t-t_0)\breve{\alpha}_j^\times}{1+(t-t_0)\alpha_j}\Big)\\ & +\prod_{j=1}^{l}\frac{1+(t-t_0)\breve{\alpha}_j^\times}{1+(t-t_0)\alpha_j}.\end{aligned}$$

The proof is complete. □

Corollary 2. *In the spacial case $l=m$, it holds*

$$\begin{aligned}\tilde{Z}(t) = \; & \frac{1}{t-t_0}\Bigg(t^2\prod_{j=1}^{m}\frac{1+(t-t_0)\alpha_j^\times}{1+(t-t_0)\alpha_j}+t\Big(\prod_{j=1}^{m}\frac{1+(t-t_0)\bar{\alpha}_j^\times}{1+(t-t_0)\alpha_j}+\prod_{j=1}^{m}\frac{1+(t-t_0)\breve{\alpha}_j^\times}{1+(t-t_0)\alpha_j}\Big)\\ & +\prod_{j=1}^{m}\frac{1+(t-t_0)\breve{\alpha}_j^\times}{1+(t-t_0)\alpha_j}-t^2-2t-1\Bigg),\end{aligned}$$

where $\alpha_1,\dots,\alpha_m$, $\alpha_1^\times,\dots,\alpha_m^\times$, $\bar{\alpha}_1^\times,\dots,\bar{\alpha}_m^\times$, $\breve{\alpha}_1^\times,\dots,\breve{\alpha}_m^\times$ and $\breve{\alpha}_1^\times,\dots,\breve{\alpha}_m^\times$ are the eigenvalues of $A_\tau^{-1}(t_0)\triangle A_\tau$, $A_\tau^{-1}(t_0)(\triangle A_\tau+\triangle b\triangle c_\tau^T)$, $A_\tau^{-1}(t_0)(\triangle A_\tau+\triangle bc_\tau^T)$, $A_\tau^{-1}(t_0)(\triangle A_\tau+b\triangle c_\tau^T)$ and $A_\tau^{-1}(t_0)(\triangle A_\tau+bc_\tau^T)$, respectively.

Corollary 3. *For $\triangle b=0$, we have*

$$\hat{Z}(t)=\frac{1}{t-t_0}(t\prod_{j=1}^{l}\frac{1+(t-t_0)\breve{\alpha}_j^\times}{1+(t-t_0)\alpha_j}+\prod_{j=1}^{l}\frac{1+(t-t_0)\breve{\alpha}_j^\times}{1+(t-t_0)\alpha_j}-t-1),$$

where the nonzero values of $\alpha_1,\dots,\alpha_l$, $\breve{\alpha}_1^\times,\dots,\breve{\alpha}_l^\times$ and $\breve{\alpha}_1^\times,\dots,\breve{\alpha}_l^\times$ are nonzero eigenvalues of $\triangle A_\tau A_\tau^\dagger(t_0)$, $(\triangle A_\tau+b\triangle c_\tau^T)A_\tau^\dagger(t_0)$ and $(\triangle A_\tau+bc_\tau^T)A_\tau^\dagger(t_0)$, respectively. When $l=m$, the closed form of $\hat{Z}(t)$ is as

$$\hat{Z}(t)=\frac{1}{t-t_0}(t\prod_{j=1}^{m}\frac{1+(t-t_0)\breve{\alpha}_j^\times}{1+(t-t_0)\alpha_j}+\prod_{j=1}^{m}\frac{1+(t-t_0)\breve{\alpha}_j^\times}{1+(t-t_0)\alpha_j}-t-1),$$

where $\alpha_1,\dots,\alpha_m$, $\breve{\alpha}_1^\times,\dots,\breve{\alpha}_m^\times$ and $\breve{\alpha}_1^\times,\dots,\breve{\alpha}_m^\times$ are as stated in Corollary 2.

7 Finding all transition and change points

It should be noted that the algorithm presented in [18] can be generalized to determine the corresponding invariancy intervals, change points and transition points to the problem $P_t(\triangle\tilde{A})$ and its specific case $P_t(\triangle\hat{A})$. In this algorithm, we first consider a sufficiently small precision value $\epsilon > 0$ and identify induced optimal partition of Problem $P_t(\triangle\tilde{A})$ for $t = t_0 = 0$ and $t = t_0 + \epsilon$ (if the latter exists). Recall that $P_{t_0}(\triangle\tilde{A})$ always has optimal solution. If $P_{t_0+\epsilon}(\triangle\tilde{A})$ is unbounded or infeasible, then t_0 is a transition or change point, the algorithmic process terminates. Otherwise, by using Theorems 3-5 for Problem $P_{t_0+\epsilon}(\triangle\tilde{A})$, one can find the corresponding induced optimal partition invariancy interval by an almost identical process in [18] with some differences which are related to the converted form of Cond. 2. If the upper bound of this interval is infinite, we stop. Otherwise, one could continue the process, after adding ϵ to this end value, until to the point where problem $P_t(\triangle\tilde{A})$ is unbounded or infeasible.

To find the induced optimal invariancy intervals, and transition or change points to the left of $t_0 = 0$, one replaces $\epsilon < 0$ and adjust the process accordingly. We refer to [18] for more details.

8 Illustrative examples

In this section, we first present two concrete examples to clarify the approach. We also present a prototype instance for a financial application. In these examples, we set $\epsilon = 0.015$. The first example is an instance in general case which includes some points that are both transition and change point or one of them. The second example is an instance in special case which only include a transition point. The third example is a prototype instance of the application in finance.

Example 1. Consider the following problem

$$\begin{aligned} \min \quad & (-1+\frac{1}{2}t)x_1 + (-1+\frac{1}{2}t)x_2 & \\ s.t. \quad & tx_1 + (1+t)x_2 + x_3 &= 1+2t, \\ & (1-t)x_1 + (1-2t)x_2 + x_4 &= 1-t, \\ & x_1, x_2, x_3, x_4 &\geq 0, \end{aligned}$$

where $t \in \mathbb{R}$ is the parameter. This problem can be rewritten as

$$\begin{aligned} \min \quad & & x_0 & \\ s.t. \quad & (-1+\frac{1}{2}t)x_1 + (-1+\frac{1}{2}t)x_2 - x_0 &= 0, \\ & x_2 + x_3 + t(x_1 + x_2 - 2) &= 1, \\ & x_1 + x_2 + x_4 + t(-x_1 - 2x_2 + 1) &= 1, \\ & x_1, x_2, x_3, x_4 &\geq 0. \end{aligned} \tag{19}$$

By substituting $x_5 = x_1 + x_2 - 2$ and $x_6 = -x_1 - 2x_2 + 1$, Problem (19) converts to the following

Table 1: Invariancy intervals, induced optimal partitions and the optimal value function in Example 1.

Inv. Int.	B	B^+	B^-	N	N^0	$Z(t)$
-1	$\{1\}$	$\varnothing$	$\{5,7\}$	$\{2,3,4\}$	$\{6\}$	-1.5
$(-1,0)$	$\{1,3\}$	$\varnothing$	$\{5,7\}$	$\{2,4\}$	$\{6\}$	$\frac{t-2}{2}$
0	$\{1,2,3\}$	$\varnothing$	$\{5,6,7\}$	$\{4\}$	$\varnothing$	-1
$(0,0.5)$	$\{1,2\}$	$\varnothing$	$\{5,6,7\}$	$\{3,4\}$	$\varnothing$	$\frac{t^3-2t^2+0.5t-1}{t^2-t+1}$
0.5	$\{1,2\}$	$\varnothing$	$\{6,7\}$	$\{3,4\}$	$\{5\}$	-1.5
$(0.5,1)$	$\{1,2\}$	$\{5\}$	$\{6,7\}$	$\{3,4\}$	$\varnothing$	$\frac{t^3-2t^2+0.5t-1}{t^2-t+1}$
1	$\{1\}$	$\{5\}$	$\{6,7\}$	$\{2,3,4\}$	$\varnothing$	-1.5
$(1,2)$	$\{1,4\}$	$\{5\}$	$\{6,7\}$	$\{2,3\}$	$\varnothing$	$\frac{t^2+1.5t+1}{t}$
2	$\{1,2,3,4\}$	$\varnothing$	$\{5,6\}$	$\varnothing$	$\{7\}$	0
$(2,\infty)$	$\{2,3\}$	$\{6,7\}$	$\{5\}$	$\{1,4\}$	$\varnothing$	$\frac{t^2-3t+2}{4t-2}$

equivalent one with only perturbation in the coefficient matrix.

$$\begin{array}{lrcl}
\min & x_0 & & \\
s.t. & (-1+\frac{1}{2}t)x_1+(-1+\frac{1}{2}t)x_2-x_0 & = & 0, \\
& x_2+x_3+tx_5 & = & 1, \\
& x_1+x_2+x_4+tx_6 & = & 1, \\
& x_1+x_2-x_5 & = & 2, \\
& -x_1-2x_2-x_6 & = & -1, \\
& x_1,x_2,x_3,x_4 & \geq & 0.
\end{array}$$

To be clear

$$\tilde{A}=\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \triangle\tilde{A}=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{c}^T=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{b}^T=\begin{pmatrix} 0 & 1 & 1 & 2 & -1 \end{pmatrix}.$$

Table 1 has a summary of the results and Fig. 1 denotes the corresponding optimal value function. The domain of the optimal value function is the closed unbounded interval $[-1,\infty)$ in this example.

As it is seen from this table, $t=0$ and $t=2$ are simultaneously transition and change points, while $t=-1$ and $t=1$ are mere transition points. The optimal value function is continuous on transition points but not differentiable. On the other hand, $t=0.5$ is merely a change point, and

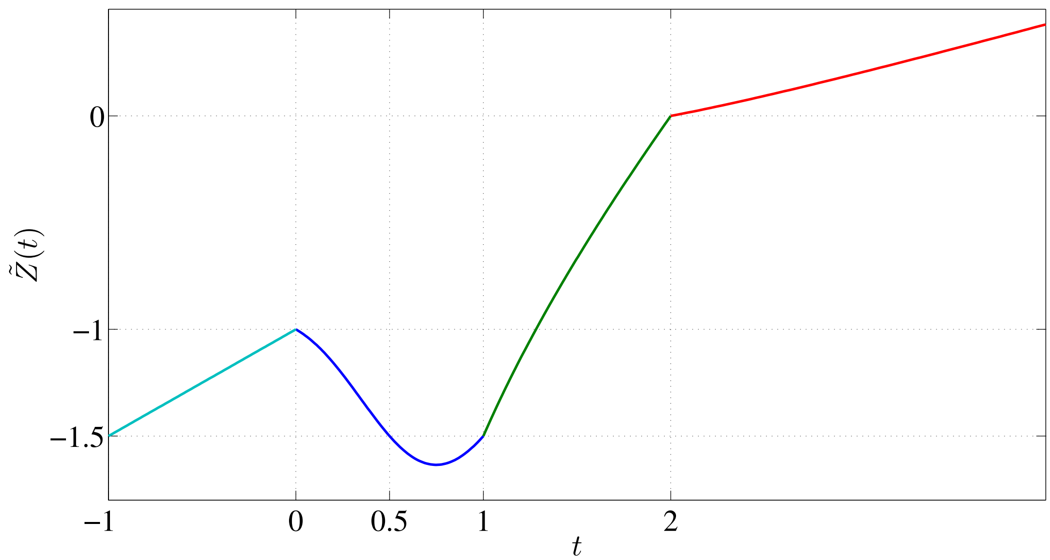

Figure 1: The optimal value function $\tilde{Z}(t)$ in Example 1.

representations of the optimal value function before and after this point are identical. Clearly, it is continuous and differentiable at this point. And as another observation, its slope does not alter at this change point.

Example 2. Consider the family of linear programs as

$$\begin{array}{lllll} \min & -(1-t)x_1 & -(2+t)x_2 & & \\ s.t. & (1+t)x_1 & +(1-t)x_2 & +x_3 & =1, \\ & x_1, & x_2, x_3 \geq 0, & & \end{array}$$

where $t \in \mathbb{R}$. This problem can be rewritten as

$$\begin{array}{llllll} \min & & & & x_0 & \\ s.t. & -(1-t)x_1 & -(2+t)x_2 & & -x_0 & =0, \\ & x_1 & +x_2+x_3 & +t(x_1-x_2) & & =1, \\ & x_1, & x_2, x_3 \geq 0, & & & \end{array} \tag{20}$$

Note that in problem (20) only the coefficient matrix is perturbed and

$$\hat{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} -1 & -2 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad \triangle\hat{A} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The results are summarized in Table 2 and the corresponding optimal value function is denoted in Fig. 2. As it is seen from Table 2, the point $t = -0.2$ is a transition point. The optimal

Table 2: Invariancy intervals, induced optimal partitions and the optimal value function of Example 2.

t	B	B^+	B^-	N	N^0	$Z_b(t)$
$(-1,-0.2)$	$\{1\}$	$\varnothing$	$\{4\}$	$\{2,3\}$	$\varnothing$	$\frac{t-1}{1+t}$
-0.2	$\{1,2\}$	$\varnothing$	$\{4\}$	$\{3\}$	$\varnothing$	-1.5
$(-0.2,1)$	$\{2\}$	$\varnothing$	$\{4\}$	$\{1,3\}$	$\varnothing$	$\frac{t+2}{t-1}$

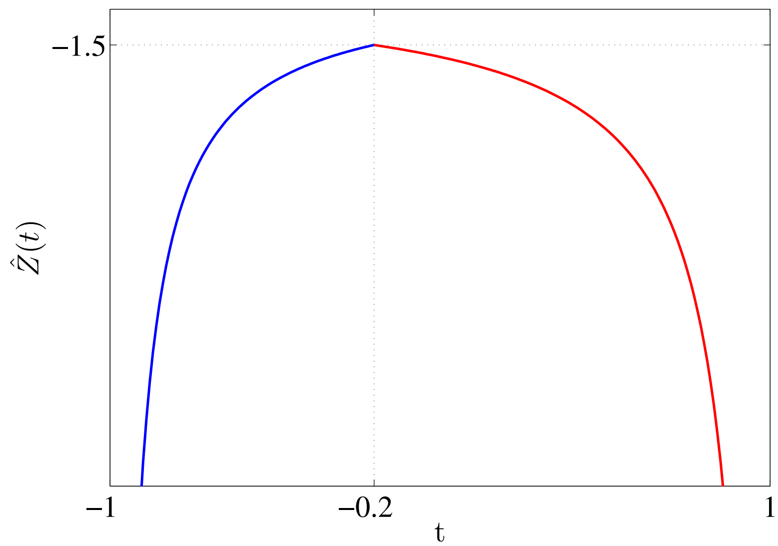

Figure 2: The optimal value function $\hat{Z}(t)$ in Example 2.

value function in this example is not differentiable at $t = -0.2$, as it is easily observed that $\tilde{Z}'_- = 3.125$ and $\tilde{Z}'_+ = -2.083$. Moreover, its domain is the open interval $(-1, +1)$ unlike the previous example.

The following example is a simple description of the financial application proposed in Page 147.

Example 3. Consider the parametric problem (2). For a numerical experience, let $b = 0.04$, $n = 4$ and time unit is set as a "week". The ask prices, the bid prices and prespecified respective variation rates $\theta_1, \theta_2, \theta_3, \theta_4$ are shown in Table 3. We replace the following constraints

$$x_j^a + z_j^a = 1, \quad x_j^b + z_j^b = 1, \quad j = 1, \dots, 4,$$

instead of the upper bounds of variables where $z_j^a, z_j^b \geq 0, j = 1, \dots, 4$, are as slack variables.

Table 3: The data for Example 3.

Bond	1	2	3	4
Bid Price	1.335	0.7487	455.18	304.48
Ask Price	1.3354	0.7489	455.2	304.5
Coupon	0.03	0.027	0.004	0.007
Variation Rate	0.2	0.3	0.04	0.5

Table 4: Optimal solution at $t = 0$ and its corresponding induced optimal partition in Example 3.

B	$\{3, 4, 5, 6, 10, 11, 14, 16, 17\}$
$\tilde{x}_B^{*T}(0)$	$\big(1, 1, 0.8, 1, 1, 1, 0.2, 1, 1\big)$
B^-	$\{18, 27\}$
$\tilde{x}_{B^-}^{*T}(0)$	$\big(-0.0184, -757.8428\big)$
N	$\{1, 2, 7, 8, 9, 12, 13, 15\}$
$S_{1_N}^T$	$\big(0.0004, 0.4532, 455.02, 304.19, 44.51, 455.0019, 304.17, 0.4530\big)$
S_2	$\big(1\big)$
N^0	$\{19, 20, 21, 22, 23, 24, 25, 26\}$
y	$\big(-1\big)$
v_1^T	$\big(44.51, 0, 0, -455.0019, -304.17, 0, -0.4530, 0, 0\big)$

As a result, the index set is $\mathcal{I} = \{1, 2, \ldots, 17\}$, and the standard form of this instance is

$$\begin{array}{lrrr}
\max & 1.335(1+0.2t)x_1^b & +0.7487(1+0.2t)x_2^b & +455.18(1+0.2t)x_3^b \\
& +304.48(1+0.2t)x_4^b & -1.3354(1+0.3t)x_1^a & -0.7489(1+0.3t)x_2^a \\
& -455.2(1+0.3t)x_3^a & -304.5(1+0.3t)x_4^a & \\
s.t. & -0.03(1+0.4t)x_1^b & -0.027(1+0.4t)x_2^b & -0.004(1+0.4t)x_3^b \\
& -0.007(1+0.4t)x_4^b & +0.03(1+0.4t)x_1^a & +0.027(1+0.4t)x_2^a \\
& +0.004(1+0.4t)x_3^a & +0.007(1+0.4t)x_4^a & -x_5 = 0.04+0.5t, \\
& x_j^a + z_j^a = 1, & x_j^b + z_j^b = 1, & j = 1, \ldots, 4, \\
& x_1^a, x_2^a, x_3^a, x_4^a, x_1^b, & x_2^b, x_3^b, x_4^b, x_5 \geq 0. &
\end{array}$$

Observe that the assumption $p_j^a > p_j^b$ holds for $t \geq 0$. By some manipulations as described in Page 153, this problem can be transformed to a uni-parametric problem as $P_t(\triangle\tilde{A})$. For $t = 0$, the optimal solution and its corresponding induced optimal partition are reflected in Table 4. Table 5 includes all induced optimal partition invariancy intervals and corresponding induced optimal partitions, as well as the representation of the optimal value function. Fig. 3 depicts the optimal value function for $t \geq 0$.

As Table 5 shows, at transition points where some indices interchange between B and N, the optimal value function is continuous but not differentiable. While it is continuous from left at the end point $t = \frac{175}{108}$, the problem is infeasible for $t > \frac{175}{108}$. Moreover, the point $t = 1.1986$ is a change point where displacement of the index 27 occurs between B^+, B^- and N^0 before and after this point. Moreover, the representation of the optimal value function in the neighborhood

Table 5: Induced optimal partitions, invariancy intervals, and the representation of optimal value function in Example 3.

t	B	B^+	B^-	N	N^0	$\tilde{Z}(t)$
$(-1.2, \frac{75}{227})$	$\{3,4,5,6,10,11,14,16,17\}$	$\{\}$	$\{18,27\}$	$\mathcal{I}\backslash B$	$\{19,\ldots,26\}$	$\frac{145.24t^2+4522t+18946}{t+25}$
$\frac{75}{227}$	$\{3,4,5,6,10,11,16,17\}$	$\{\}$	$\{18,27\}$	$\mathcal{I}\backslash B$	$\{19,\ldots,26\}$	807.5669
$(\frac{75}{227}, \frac{325}{447})$	$\{3,4,5,6,10,11,13,16,17\}$	$\{\}$	$\{18,27\}$	$\mathcal{I}\backslash B$	$\{19,\ldots,26\}$	$\frac{-3798.2t^2-13903t+25464}{t+25}$
$\frac{325}{447}$	$\{3,5,6,10,11,13,16,17\}$	$\{\}$	$\{18,27\}$	$\mathcal{I}\backslash B$	$\{19,\ldots,26\}$	518.8306
$(\frac{325}{447}, \frac{25}{22})$	$\{3,5,6,8,10,11,13,16,17\}$	$\{\}$	$\{18,27\}$	$\mathcal{I}\backslash B$	$\{19,\ldots,26\}$	$\frac{-5742.9t^2-12490t+25465}{t+25}$
$\frac{25}{22}$	$\{3,5,6,8,10,11,13,16\}$	$\{\}$	$\{18,27\}$	$\mathcal{I}\backslash B$	$\{19,\ldots,26\}$	147.5283
$(\frac{25}{22}, 1.1986)$	$\{3,5,6,8,10,11,12,13,16\}$	$\{\}$	$\{18,27\}$	$\mathcal{I}\backslash B$	$\{19,\ldots,26\}$	$\frac{-10015t^2-38565t+60612}{t+25}$
1.1986	$\{3,5,6,8,10,11,12,13,16\}$	$\{\}$	$\{18\}$	$\mathcal{I}\backslash B$	$\{19,\ldots,27\}$	0
$(1.1986, \frac{150}{109})$	$\{3,5,6,8,10,11,12,13,16\}$	$\{27\}$	$\{18\}$	$\mathcal{I}\backslash B$	$\{19,\ldots,26\}$	$\frac{-10015t^2-38565t+60612}{t+25}$
$\frac{150}{109}$	$\{5,6,8,10,11,12,13,16\}$	$\{27\}$	$\{18\}$	$\mathcal{I}\backslash B$	$\{19,\ldots,26\}$	−433.1558
$(\frac{150}{109}, \frac{175}{108})$	$\{5,6,7,8,10,11,12,13,16\}$	$\{27\}$	$\{18\}$	$\mathcal{I}\backslash B$	$\{19,\ldots,26\}$	$\frac{-14977t^2-31739t+60615}{t+25}$
$\frac{175}{108}$	$\{5,6,7,8,10,11,12,13\}$	$\{27\}$	$\{18\}$	$\mathcal{I}\backslash B$	$\{19,\ldots,26\}$	−1132.1

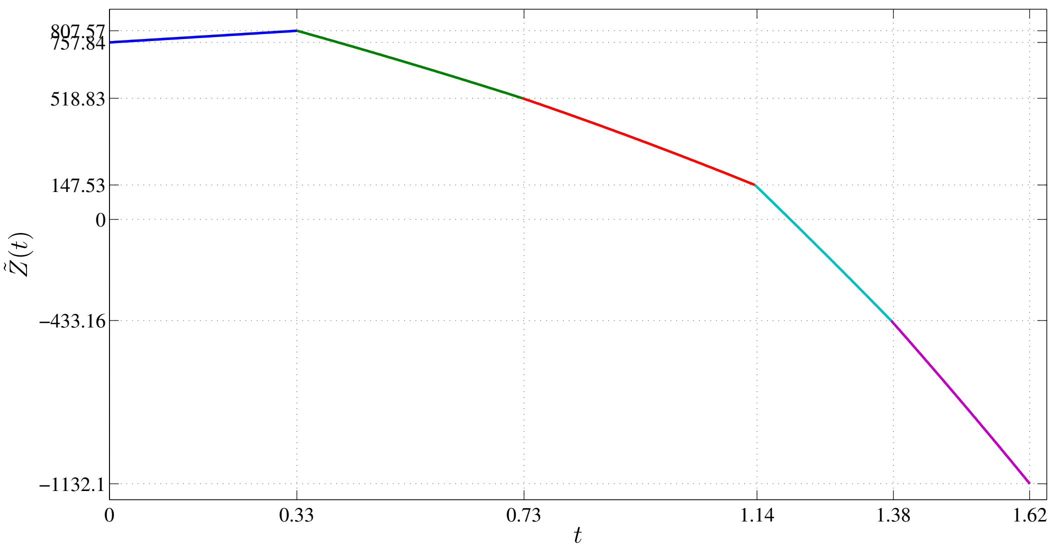

Figure 3: Optimal value function $\tilde{Z}(t)$ in Example 3.

of this point does not change.
As Fig. 3 depicts, for $0 \leq t < \frac{75}{227}$, simultaneously selling some amount of bonds 3 and 4 and buying some amount of bonds 1 and 2 increase the total profit, while for $\frac{75}{227} < t < \frac{325}{447}$ and $\frac{325}{227} < t < \frac{25}{22}$ the optimal value function decreases. Note that for $\frac{25}{22} < t < 1.1986$ the profit goes down while for $1.1986 < t < \frac{150}{109}$ the loss goes up. Analogous descriptions can be provided for the results in other intervals.

8.1 Computational results

To observe the result of the model in large scale problems, the computational results are reported on some Netlib test problems. The characteristics of these problems are reflected in Table 6.

Name	Rows	Columns
Afiro	27	51
Blend	74	114
Stocfor1	117	165
Scagr7	129	185

Table 6: Characteristics of test problems.

The hardware consisted of an Intel Corei7 @ 1.80 GHz Processor, 12 GB of RAM running Windows 10 Enterprise. The algorithm has been implemented in MATLAB R2019b using the `linprog` solver. In implementing the proposed algorithm, it is necessary to determine the intersection of some intervals where Conds 1 and 2 satisfy. INTLAB is a toolbox for Matlab, supporting real and complex intervals operations. Here, we used some standard commands of the interval arithmetic toolbox INTLAB Ver. 11 for interval computations.

In each parametric Problem $P_t(\triangle)$, for $\triangle A$, $\lfloor \frac{n}{2} \rfloor$ of elements are randomly selected and their values are produced by the pseduorandom normal codes of the Matlab. The elements of $\triangle b$ and $\triangle c$ are produced by uniform distributions from intervals $[0, 3]$ and $[-1, 2]$, respectively. Computational results corresponding to $\epsilon = 0.015$ are depicted in Table 7.

Table 7: Computational results for $\epsilon = 0.015$.

Problem	Convex section of $\Lambda; 0 \in \Lambda$	No. of detected Ind.				Total CPU time (Sec.)
		Inv. Int.	Trans. Points	Chang. Points	Both	
Afiro	(-3.136,22.8]	66	32	18	15	6831.8
Blend	[-0.024,2.089]	48	44	3	0	37273
Scagr7	[-0.4659,0.6677]	36	31	0	4	76934

In Table 7, the first column denotes the name of problems; the second column is the corresponding convex section of the domain of the optimal value function containing $t_0 = 0$. The next four consequent columns are the number of detected invariancy intervals, transition points, change points and those that are both transition and change points, respectively. The last column shows the total time (in CPU time) for determining all invariancy intervals.

As Tables 6 and 7 show, the computation time seems to increase as the size of the problem increases. This impression cannot be regarded as a general rule. For example, the problem `Stocfor1` only revealed one invariancy interval $(-0.011, 0.012)$ for several random selections of $\triangle A$, $\triangle b$ and $\triangle c$; and the total CPU time was almost less than 850 Sec. Our experiments revealed that the time required to determine an interval satisfying Cond. 1, is as least as the two-thirds of the total time.

9 Conclusion

In this paper, we considered a uni-parametric linear program when left and right-hand-side of constraints, in addition to the objective coefficients, were linearly perturbed by an identical parameter. Based on the induced optimal partition concept, a methodology for identifying the corresponding invariancy interval was provided. The concept of change point was introduced. It was observed that the optimal value function is fractional on each interval. Using a computational algorithm, one could found all invariancy intervals. Provided examples indicated the validity of the findings. This study could be continued for more than one parameter and the case when the perturbation is not linear with respect to the parameter.

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