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Existence and optimal harvesting of two competing species in a polluted environment with pollution reduction effect

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Abstract. This paper deals with the existence and optimal harvesting of two competing species in a polluted environment under the influence of pollution reduction effort. We propose and analyze a nonlinear system wherein harvesting and pollution reduction activities, respectively, are incorporated into the resource and pollution dynamic equations. We investigate the coexistence, competitive exclusion, and extinction of both species in the system. Further, considering pollution-dependent revenues, we study an optimal harvest problem on an infinite horizon. The results indicate that the extinction of both species is inevitable when pollutant inflow is sufficiently large. Otherwise, the proper effort allocation towards pollution reduction guarantees not only species coexistence but also improves the revenue. The significant outcomes of the study are verified by considering practical examples.

Keywords: Competition, Pollutants, Harvesting, Pollution reduction, Effort Allocation.
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1 Introduction

The ever-increasing effect of environmental pollution and exploitation is a major threat to the existence of ecologically-interdependent populations. Interspecific competition, on the other hand, is well known to affect the growth rate of the species as it reduces the amount of the available resources to each species when that resource is in short supply. Hence, studying the simultaneous effect of competition, pollution, and exploitation on the dynamics of the system is necessary for sustainable resource utilization.

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The effect of pollutants on the survival of a single-species population has been studied intensively in the literature [2, 10, 11, 13–15, 19, 29]. Considerable investigations were made to study the effect of pollution on the dynamics of the interacting populations [6, 9, 20–22, 25–28, 31]. In particular, Misra and Saxena [21] presented a mathematical model of the two species' competitive system by considering the growth rates of the species and the carrying capacity are directly affected by pollution. Dubey and Hussain [12] proposed and analyzed a mathematical model for studying the survival of two interacting species in a polluted environment by considering both constant and instantaneous introduction of pollutants into the environment. Shukla et al. [28] studied the existence and survival of two competing species in a polluted environment, and underlined that the amount of emission of toxicants into the environment plays a major role in the species coexistence.

To the best of our knowledge, studies with the emphasis on the dynamics of competing species (in a polluted environment) in the presence of exploitation and pollution reduction seen rarely in the literature. The works presented in [16, 17, 23, 30, 33, 34] mainly focus on the dynamics of exploited single-species populations in the presence of pollution. In this study, we consider the dynamics of harvested two-species competitive system in a polluted environment under the influence of pollution reduction. We investigate the stable coexistence, competitive exclusion, and extinction of both species in the system. Further, by considering pollution-dependent revenue function, we study an optimal harvest problem.

We organize the work as follows. In Section 2, we formulate the model, and analysis of steady-state equilibria is given in Section 3, wherein we investigate the existence and stability of the steady-state equilibria and the effect of effort allocation on the stock as well as the yield. In Section 4, we present the optimal harvest problem. We give numerical simulations in Section 5, which is followed by discussion and conclusions in Section 6.

2 Model formulation

Consider the ecosystem consisting of two competing species (subjected to harvesting) and pollutants. The presence of pollution affects both the growth rate and carrying capacity of the species, resulting in a decline in the yield. Consequently, it becomes necessary to allocate a part of the total effort capacity towards pollution reduction through environmental treatment (instead of utilizing the entire effort for harvesting) so that the yield gets improved. Since a higher resource level gives more catch per unit effort, and the resource stock gets increased by reducing pollution, it is a reasonable action to allocate a part of the available effort capacity towards pollution reduction.

Suppose the harvester has a total effort capacity of E^{total} (measured in terms of money) per unit time to put in harvesting, and this effort must be allocated between harvesting and pollution reduction. Let E_1 and E_2 be the efforts allocated towards harvesting species1 and species2, respectively, and the remaining effort $E^{total} - E_1 - E_2$ is used for pollution reduction. Further, let the maximum allowable effort for harvesting species i is E_i^{max} (i.e., $0 \le E_i \le E_i^{max}$ for $i = 1, 2$) and that

$$E_1^{max} + E_2^{max} = E^{total}. \quad (1)$$

Now, following [17, 21, 33, 34], the dynamics of two competing species in a polluted environment in the presence of harvesting and pollution reduction is given by

$$\frac{dx_1}{dt} = r_1 x_1 (1 - \frac{x_1}{K_1}) - c_{12} x_1 x_2 - d_1 x_1 z - \alpha_1 q_1 E_1 x_1, \tag{2a}$$

$$\frac{dx_2}{dt} = r_2 x_2 (1 - \frac{x_2}{K_2}) - c_{21} x_2 x_1 - d_2 x_2 z - \alpha_2 q_2 E_2 x_2, \tag{2b}$$

$$\frac{dz}{dt} = v - \left[\gamma_1 x_1 + \gamma_2 x_2 + \beta(E^{total} - E_1 - E_2) + \eta\right] z, \tag{2c}$$

$$x_1(0) > 0, \; x_2(0) > 0, \; z(0) > 0. \tag{2d}$$

Description of the parameters and constants involved in the model is presented in Table 1. The

Table 1: Description of the associated parameters and constants in the model.

Parameters	Symbol
Intrinsic growth rate for species i	r_i
Carrying capacity of species i	K_i
The action of species j upon the growth rate of species i	c_{ij}
The action of pollution upon the growth rate of species i	d_i
Uptake of pollutant by species i	γ_i
Pollutants inflow rate	v
Total effort capacity	E^{total}
Harvest effort associated with species i	E_i
Conversion factor associated with effort E_i	α_i
Conversion factor associated with de-pollution effort	β
Natural degradation of pollutants	η
Catchability coefficient for species i	q_i

quantity $x_i(t)$ represents the population size of species i (for $i = 1, 2$) and $z(t)$ stands for the stock of pollutants at time t. The expression $\alpha_i q_i E_i x_i$ represents the harvest rate associated with species i, where $\alpha_i E_i$ denotes the harvest effort in physical terms (such as the standard fishing vessels). The expression $\beta(E^{total} - E_1 - E_2)z$ in (2c) stands for the removal of pollutants in the environment by the de-pollution effort $E^{total} - E_1 - E_2$. In the model, the regeneration function for species i (in the absence of competition and harvesting) is given by

$$F_i(x_i, z) = R_i(z) x_i \big(1 - \frac{x_i}{k_i(z)}\big), \tag{3}$$

where $R_i(z) = r_i - d_i z$ and $k_i(z) = \frac{K_i(r_i - d_i z)}{r_i}$ are the pollution dependent intrinsic growth rate and the environmental carrying capacity of Species i, respectively. Clearly, r_i and K_i are the intrinsic growth rate and environmental carrying capacity of species i in the absence of pollution, respectively. In the model (2a)-(2c), the stock of pollutants (in the environment) directly affects the growth rate and carrying capacity of the species. Furthermore, the environment is assumed to be so large that the change of toxicant (in the environment) that comes from uptake and egestion by the species is neglected.

Before proceeding to study the analysis of steady-state equilibria, it is important to state the following lemmas which can easily established.

Proposition 1. *Consider system* (2a)-(2c) *with the initial condition in* (2d)*. Then*

(I) The system admits a unique solution $(x_1(t), x_2(t), z(t))$ *for all* $t \geq 0$.

(II) The solutions $x_1(t), x_2(t)$ *and* $z(t)$ *of the differential equations* (2a), (2b) *and* (2c)*, respectively, are nonnegative and bounded for all* $t \geq 0$*. In particular, if* $D_c \subset R^3_+$ *is a set given by*

$$D_c = \{(x_1, x_2, z) : 0 < x_1 \leq K_1, 0 < x_2 \leq K_2, 0 < z \leq \frac{v}{\eta}\},$$

then any solution that starts in D_c *stays in* D_c *for all* $t \geq 0$.

3 Analysis of the steady-state equilibria

Since the resource and pollution stocks are non negative in nature, we focus on the equilibria of system (2a)-(2c) that belong to the first octant of the x_1x_2z-space. With the help of the following transformations

$$b_i = \frac{r_i}{K_i}, \quad a_{ij} = \frac{c_{ij}}{b_i}, \quad \beta_i = \frac{d_i}{b_i}, \quad \omega_i = \frac{r_i - \alpha_i q_i E_i}{b_i}, \quad d = \eta + \beta(E^{total} - E_1 - E_2),$$

for $i, j = 1, 2$, the system under consideration can be rewritten as

$$\frac{dx_1}{dt} = b_1 x_1 \big(\omega_1 - x_1 - a_{12}x_2 - \beta_1 z\big), \tag{4a}$$

$$\frac{dx_2}{dt} = b_2 x_2 \big(\omega_2 - x_2 - a_{21}x_1 - \beta_2 z\big), \tag{4b}$$

$$\frac{dz}{dt} = v - z(\gamma_1 x_1 + \gamma_2 x_2 + d). \tag{4c}$$

And, the possible nonnegative equilibria are

$$\mathcal{Q}^0 = (x_1^0, x_2^0, z^0) = \big(0, 0, \frac{v}{d}\big), \tag{5a}$$

$$\mathcal{Q}^1 = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{z}) = \left(\widetilde{x}_1, 0, \frac{v}{d + \gamma_1 \widetilde{x}_1}\right), \tag{5b}$$

$$\mathcal{Q}^2 = (\overline{x}_1, \overline{x}_2, \overline{z}) = \left(0, \overline{x}_2, \frac{v}{d + \gamma_2 \overline{x}_2}\right), \tag{5c}$$

$$\mathcal{Q}^* = (\widehat{x}_1, \widehat{x}_2, \widehat{z}). \tag{5d}$$

Throughout the paper, the equilibria Q^0, Q^i and Q^* are referred to as the axial, boundary, and interior equilibria of system (2a)-(2c), respectively.

The axial equilibrium (Q^0) stands for the state where no species exists in the environment except pollution, and it always exist in the system. The boundary equilibrium (Q^i for $i = 1, 2$) represents the state where only species i exists in the environment. It can be easily verified from

(5b) & (5c) that for the given pair of harvest efforts (E_1, E_2), the existence of Q^i depends on the coefficients of the following quadratic equation

$$\gamma_i {x_i}^2 - (\gamma_i \omega_i - d) x_i + \beta_i v - \omega_i d = 0. \tag{6}$$

Clearly, (6) admits only one positive real root whenever

$$\beta_i v - \omega_i d < 0,$$

and it has two positive real roots for

$$\gamma_i \omega_i - d > 0,\ \beta_i v - \omega_i d > 0,\ (\gamma_i \omega_i - d)^2 - 4\gamma_i(\beta_i v - \omega_i d) > 0.$$

In the former case, the system under consideration admits two additional boundary equilibria:

$$Q^1_+ = (\widetilde{x}_1^+, 0, \widetilde{z}) \ \ \&\ \ Q^2_+ = (0, \overline{x}_2^+, \overline{z}),$$

whereas in the latter case, it admits four additional boundary equilibria:

$$Q^1_+ = (\widetilde{x}_1^+, 0, \widetilde{z}),\ Q^1_- = (\widetilde{x}_1^-, 0, \widetilde{z}),\ Q^2_+ = (0, \overline{x}_2^+, \overline{z})\ \& Q^2_- = (0, \overline{x}_2^-, \overline{z}),$$

where

$$ {x_i}^{\pm} = \frac{\gamma_i \omega_i - d \pm \sqrt{(\gamma_i \omega_i - d)^2 - 4\gamma_i(\beta_i v - \omega_i d)}}{2\gamma_i}. \tag{7}$$

The interior equilibrium (Q^*) corresponds to the coexistence of the species. Here, for each pair of the harvest efforts (E_1, E_2), the components of the interior equilibrium (if it exists) are

$$\begin{aligned} \widehat{x}_1 &= \frac{(\omega_1 - \omega_2 a_{12}) - (\beta_1 - a_{12}\beta_2)\widehat{z}}{1 - a_{12}a_{21}}, \\ \widehat{x}_2 &= \frac{(\omega_2 - \omega_1 a_{21}) - (\beta_2 - a_{21}\beta_1)\widehat{z}}{1 - a_{12}a_{21}}, \end{aligned} \tag{8}$$

and $\widehat{z}$ is the positive solution of a quadratic equation

$$Az^2 - Bz + C = 0, \tag{9}$$

where A, B and C are given by

$$\begin{aligned} A &= \gamma_1(\beta_1 - a_{12}\beta_2) + \gamma_2(\beta_2 - a_{21}\beta_1), \\ B &= \gamma_1(\omega_1 - \omega_2 a_{12}) + \gamma_2(\omega_2 - \omega_1 a_{21}) + (1 - a_{12}a_{21})d, \\ C &= (1 - a_{12}a_{21})v. \end{aligned} \tag{10}$$

Note that the existence of the interior equilibrium depends on the sign of coefficients of the quadratic equation in (9). Clearly, (9) has two positive real roots whenever

$$A > 0,\ B > 0,\ C > 0,\ B^2 - 4AC > 0 \tag{11}$$

and it has only one positive real root whenever

$$A > 0,\ C < 0.$$

In the former case, the system has two interior equilibria:

$$Q^*_+ = (\widehat{x}_1, \widehat{x}_2, \widehat{z}_+) \ \& \ Q^*_- = (\widehat{x}_1, \widehat{x}_2, \widehat{z}_-),$$

whereas in the latter case it has only one interior equilibrium ($Q^*_+ = (\widehat{x}_1, \widehat{x}_2, \widehat{z}_+)$), where

$$\widehat{z}_\pm = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{12}$$

Let's see the biological implications of the signs of the coefficients A, B and C. Note that $C > 0$ whenever

$$1 - a_{12}a_{21} > 0. \tag{13}$$

This situation represents the case where the interspecific competition is weak in the system, which allows the coexistence [18]. On the other hand, condition $1 - a_{12}a_{21} < 0$ depicts the case where competition is strong (at least for one of the species) in the system, and hence it leads to the competitive exclusion.

We know that the intrinsic growth rate r_i has a direct relationship with the stock $\widehat{x}_i$ and it has an inverse relationship with that of $\widehat{x}_j$ for $i \neq j,\ i, j = 1, 2$. Thus, expressions $\omega_1 - \omega_2 a_{12}$ and $\omega_2 - \omega_2 a_{21} > 0$ in (8) are positive provided that $C > 0$. Consequently, we have the following important restrictions on the harvest efforts E_1 and E_2 associated with species1 and species2, respectively:

$$\begin{aligned} E_1 &< \frac{1}{\alpha_1 q_1}\big[r_1 - \omega_2 c_{12}\big], \\ E_2 &< \frac{1}{\alpha_2 q_2}\big[r_2 - \omega_1 c_{21}\big]. \end{aligned} \tag{14}$$

This implies $B > 0$. Naturally the resource stock has an inverse relationship with that of pollution. Hence, it follows (together with $C > 0$) that coefficients $\beta_1 - a_{12}\beta_2$ and $\beta_2 - a_{21}\beta_1$ in (8) are positive i.e.,

$$a_{12} < \frac{\beta_1}{\beta_2} \text{ and } a_{21} < \frac{\beta_2}{\beta_1}. \tag{15}$$

As a result $A > 0$. Finally, since the resource stock is a positive quantity, the numerators in (8) are positive (provided that $C > 0$). This gives the following important restriction on the stock of pollution:

$$\widehat{z}_\pm < \min\left\{\frac{\omega_1 - \omega_2 a_{12}}{\beta_1 - \beta_2 a_{12}},\ \frac{\omega_2 - \omega_1 a_{21}}{\beta_2 - \beta_1 a_{21}}\right\}. \tag{16}$$

Clearly, the system admits at most one interior equilibrium provided that

$$\widehat{z}_- < \min\left\{\frac{\omega_1 - \omega_2 a_{12}}{\beta_1 - \beta_2 a_{12}},\ \frac{\omega_2 - \omega_1 a_{21}}{\beta_2 - \beta_1 a_{21}}\right\} < \widehat{z}_+. \tag{17}$$

We observe that the system under consideration has at most seven equilibria: an axial equilibrium (Q^0), four boundary equilibria ($Q^1_\pm, Q^2_\pm$), and two interior equilibria ($Q^*_\pm$). However, we give more attention to the equilibria: Q^0, Q^1_+, Q^2_+ and Q^*_-, and also we assume that these are the only equilibria of the system under consideration, i.e., Q^1_+, Q^2_+ and Q^*_- are unique in the interior of the x_1z-space, x_2z-space and x_1x_2z-space, respectively. Note that these equilibria give the higher resource stocks and lower stocks of pollution. In the rest of the paper, we denote Q^1_+, Q^2_+ and Q^*_- by Q^1, Q^2, and Q^*, respectively, unless mentioned otherwise. The local and global stabilities of these equilibria are given below.

Theorem 1 (**Local stability**). *Suppose Q^0, Q^1, Q^2, and Q^* are the only nonnegative equilibria of system* (2a)-(2c). *Then*

(a) Q^0 is locally asymptotically stable whenever

$$z^0 > max\left\{\frac{r_1 - \alpha_1 q_1 E_1}{d_1}, \frac{r_2 - \alpha_2 q_2 E_2}{d_2}\right\}, \tag{18}$$

and unstable otherwise.

(b) Q^1 is locally asymptotically stable whenever

$$\omega_2 - a_{21}\widetilde{x}_1 - \beta_2\widetilde{z} < 0, \quad \widetilde{z} < \sqrt{\frac{b_1 v}{\gamma_1 d_1}}, \tag{19}$$

and unstable otherwise.

(c) Q^2 is locally asymptotically stable whenever

$$\omega_1 - a_{12}\overline{x}_2 - \beta_1\overline{z} < 0, \quad \overline{z} < \sqrt{\frac{b_2 v}{\gamma_2 d_2}}, \tag{20}$$

and unstable otherwise.

(d) Q^* is locally asymptotically stable whenever

$$a_2 > 0, \; a_3 > 0, \; a_1 a_2 - a_3 > 0. \tag{21}$$

and unstable otherwise, where

$$\begin{aligned}
a_1 &= b_1\widehat{x}_1 + b_2\widehat{x}_2 + \frac{v}{\widehat{z}}, \\
a_2 &= (b_1\widehat{x}_1(\frac{v}{\widehat{z}}) - d_1\gamma_1\widehat{x}_1\widehat{z}) + (b_2\widehat{x}_2(\frac{v}{\widehat{z}}) - d_2\gamma_2\widehat{x}_2\widehat{z}) + (b_1b_2 - c_{12}c_{21})\widehat{x}_1\widehat{x}_2, \\
a_3 &= (b_1b_2 - c_{12}c_{21})\widehat{x}_1\widehat{x}_2(\frac{v}{\widehat{z}}) + (c_{12}\gamma_1 d_2 + c_{21}\gamma_2 d_1 - b_1\gamma_2 d_2 - b_2\gamma_1 d_1)\widehat{x}_1\widehat{x}_2\widehat{z}.
\end{aligned}$$

Theorem 2 (**Global stability**). *Suppose Q^0, Q^1, Q^2, and Q^* are the only nonnegative equilibria of system* (2a)-(2c). *Then* *(a) Q^0 is globally asymptotically stable whenever the following conditions are satisfied:*

$$\begin{aligned}&r_1-\alpha_1q_1E_1-d_1z^0<0,\ r_2-\alpha_2q_2E_2-d_2z^0<0,\\&8(r_1-\alpha_1q_1E_1-d_1z^0)(r_2-\alpha_2q_2E_2-d_2z^0)\frac{v}{(z^0)^2}+\gamma_1^2(r_2-\alpha_2q_2E_2-d_2z^0)\\&+\gamma_2^2(r_1-\alpha_1q_1E_1-d_1z^0)>0.\end{aligned}\tag{22}$$

(b) Q^1 is globally asymptotically stable whenever the following conditions are satisfied:

$$4b_1b_2-(c_{12})^2>0,\ (4b_1b_2-(c_{12})^2)(\frac{v}{\tilde{z}^2})+c_{12}\gamma_2(\gamma_1+d_1)-b_1(\gamma_2)^2-b_2(\gamma_1+d_1)^2>0.\tag{23}$$

(c) Q^2 is globally asymptotically stable whenever the following conditions are satisfied:

$$4b_1b_2-(c_{21})^2>0,\ (4b_1b_2-(c_{21})^2)(\frac{v}{\bar{z}^2})+c_{21}\gamma_1(\gamma_2+d_2)-b_2(\gamma_1)^2-b_1(\gamma_2+d_2)^2>0.\tag{24}$$

(d) Q^* is globally asymptotically stable whenever the following conditions are satisfied:

$$\begin{aligned}&4b_1b_2-(c_{12}+c_{21})^2>0,\\&(4b_1b_2-(c_{12}+c_{21})^2)(\tfrac{v}{\hat{z}^2})+(c_{12}+c_{21})(\gamma_2+d_2)(\gamma_1+d_1)-b_1(\gamma_2+d_2)^2-b_2(\gamma_1+d_1)^2>0.\end{aligned}\tag{25}$$

3.1 Influence of the effort allocation

Here, we wish to see the influence of efforts E_1, E_2 on the stocks $\widehat{x}_1, \widehat{x}_2, \widehat{z}$ and yield. For each pair of efforts (E_1, E_2), the resource and pollution stocks ($\widehat{x}_1, \widehat{x}_2$ and $\widehat{z}$) can be computed using (8) and (12). Clearly, these components can be considered as functions of only two variables E_1 and E_2. Now, by applying partial differentiation on $\widehat{z}$ with respect to E_1,E_2 and using (10), we obtain

$$\frac{\partial\widehat{z}}{\partial E_1}=\frac{1}{2A}\big[1-\frac{B}{\sqrt{B^2-4AC}}\big]\big[\frac{\alpha_1q_1}{b_1}(a_{21}\gamma_2-\gamma_1)-\beta(1-a_{12}a_{21})\big],\tag{26a}$$

$$\frac{\partial\widehat{z}}{\partial E_2}=\frac{1}{2A}\big[1-\frac{B}{\sqrt{B^2-4AC}}\big]\big[\frac{\alpha_2q_2}{b_2}(a_{12}\gamma_1-\gamma_2)-\beta(1-a_{12}a_{21})\big].\tag{26b}$$

From (26a) and (26b), it can be observed that the stock of pollution increases with efforts E_1 and E_2 if

$$a_{21}<\frac{\gamma_1}{\gamma_2}\ \text{ and }\ a_{12}<\frac{\gamma_2}{\gamma_1},\tag{27}$$

respectively. The partial derivatives of $\widehat{x}_1$ with respect to E_1 and E_2 are

$$\frac{\partial\widehat{x}_1}{\partial E_1}=-\frac{1}{1-a_{12}a_{21}}\big[\frac{\alpha_1q_1}{b_1}+(\beta_1-a_{12}\beta_2)\frac{\partial\widehat{z}}{\partial E_1}\big],\tag{28a}$$

$$\frac{\partial\widehat{x}_1}{\partial E_2}=\frac{1}{1-a_{12}a_{21}}\big[\frac{\alpha_2q_2}{b_2}a_{12}-(\beta_1-a_{12}\beta_2)\frac{\partial\widehat{z}}{\partial E_2}\big],\tag{28b}$$

respectively. From (28a) we observe that the stock x_1 decreases as E_1 increases. On the other hand, an increase in E_2 has two different effects on x_1: a positive effect (due to a reduction in the interspecific competition) and a negative effect (due to a rise in the stock of pollution). If the right side of (28b) is negative i.e.,

$$\frac{\alpha_2 q_2}{b_2}a_{12} - (\frac{\beta_1 - a_{12}\beta_2}{2A})(1 - \frac{B}{\sqrt{B^2-4AC}})[\frac{\alpha_2 q_2}{b_2}(a_{12}\gamma_1 - \gamma_2) - \beta(1 - a_{12}a_{21})] < 0, \tag{29}$$

then the negative impact is dominant and hence x_1 decreases. If the right side of (28b) is positive i.e.,

$$\frac{\alpha_2 q_2}{b_2}a_{12} - (\frac{\beta_1 - a_{12}\beta_2}{2A})(1 - \frac{B}{\sqrt{B^2-4AC}})[\frac{\alpha_2 q_2}{b_2}(a_{12}\gamma_1 - \gamma_2) - \beta(1 - a_{12}a_{21})] > 0, \tag{30}$$

then the positive effect is dominant and hence x_1 increases. Similarly, applying partial differentiation on $\widehat{x}_2$ we obtain

$$\frac{\partial \widehat{x}_2}{\partial E_2} = -\frac{1}{1 - a_{12}a_{21}}[\frac{\alpha_2 q_2}{b_2} + (\beta_2 - a_{21}\beta_1)\frac{\partial \widehat{z}}{\partial E_2}], \tag{31a}$$

$$\frac{\partial \widehat{x}_2}{\partial E_1} = \frac{1}{1 - a_{12}a_{21}}[\frac{\alpha_1 q_1}{b_1}a_{21} - (\beta_2 - a_{21}\beta_1)\frac{\partial \widehat{z}}{\partial E_1}]. \tag{31b}$$

From (31a) we observe that the stock x_2 decreases as E_2 increases (provided that (27) holds). On the other hand, an increase in effort E_1 has two different effects on x_2: positive and negative effects. If the right side of (31b) is negative i.e.,

$$\frac{\alpha_1 q_1}{b_1}a_{21} - (\frac{\beta_2 - a_{21}\beta_1}{2A})(1 - \frac{B}{\sqrt{B^2-4AC}})[\frac{\alpha_1 q_1}{b_1}(a_{21}\gamma_2 - \gamma_1) - \beta(1 - a_{21}a_{12})] < 0, \tag{32}$$

then the negative effect is dominant and hence x_2 decreases. If the positive effect is dominant (the right side of (31b) is positive), i.e.,

$$\frac{\alpha_1 q_1}{b_1}a_{21} - (\frac{\beta_2 - a_{21}\beta_1}{2A})(1 - \frac{B}{\sqrt{B^2-4AC}})[\frac{\alpha_2 q_1}{b_1}(a_{21}\gamma_2 - \gamma_1) - \beta(1 - a_{21}a_{12})] > 0, \tag{33}$$

then x_2 increases. We have the following proposition.

Proposition 2. *Let $\widehat{x}_1, \widehat{x}_2, \widehat{z}$ be the components of the unique interior equilibrium Q^* of system* (2a)-(2c)*, and let* (27) *holds. Then the following statements are true:*

(a) The stock $\widehat{z}$ has a direct relationship with effort E_i for $i = 1, 2$.

(b) The stock $\widehat{x}_i$ has an inverse relationship with effort E_i for $i = 1, 2$.

(c) The stock $\widehat{x}_1$ has a direct relationship with with E_2 if (30) *holds and it has an inverse relationship with E_2 if* (29) *holds. Similarly, the stock $\widehat{x}_2$ has a direct relationship with E_1 if* (33) *holds and it has an inverse relationship with E_1 if* (32) *holds.*

3.2 Maximum sustainable yield

In the discussion above, we have seen the influence of effort allocation on the stock levels. Here, we are interested to evaluate the maximum sustainable yield. We observed that an increase in the harvest effort results in a rise in the stock of pollution (under the condition in (27)). This will affect the resource stock and hence the yield. Thus, increasing the harvest effort may not guarantee an improvement of the yield. Therefore, it is important to identify the proper effort allocation (between harvesting and pollution reduction) to maximize the yield. For each pair of efforts (E_1, E_2), the yield (which we denote by h) is given by

$$h(t) = \alpha_1 q_1 E_1 x_1 + \alpha_2 q_2 E_2 x_2. \tag{34}$$

Clearly, h is a function of two variables $E_1 \& E_2$, and it is continuously differentiable. Hence, the maximum sustainable yield (which we denote by h_{MSY}) occurs either at the boundary or in the interior of rectangular region $\mathcal{D}$, which is given by

$$\mathcal{D} = \{(E_1, E_2) : 0 \le E_1 \le E_1^{max},\ 0 \le E_2 \le E_2^{max},\ E_1^{max} + E_2^{max} = E^{total}\}. \tag{35}$$

Applying partial differentiation on the yield function, we obtain

$$\frac{\partial h}{\partial E_1} = \alpha_1 q_1 x_1 + \alpha_1 q_1 E_1 \frac{\partial x_1}{\partial E_1} + \alpha_2 q_2 E_2 \frac{\partial x_2}{\partial E_1}, \tag{36a}$$

$$\frac{\partial h}{\partial E_2} = \alpha_2 q_2 \widehat{x}_2 + \alpha_2 q_2 E_2 \frac{\partial x_2}{\partial E_2} + \alpha_1 q_1 E_1 \frac{\partial x_1}{\partial E_2}. \tag{36b}$$

We observe that the yield increases with E_1 and E_2 as long as the expressions on the right of (36) are positive and attains it's maximum (local) at (E_1^{MSY}, E_2^{MSY}) where the equations are equal to zero. Beyond this level, the yield starts to decline due to the higher level of pollution.

We have the following observations. Unlike the basic Lotka-Volterra competition system (ref. [18]), the extinction of both species is possible in the system under consideration if the inflow rate of pollutants is too large. Moreover, the inflow of pollutants, the harvest efforts, and the strength of interspecific competition play a crucial role in the existence of the interior equilibrium. If the inflow of pollutants is moderate and competition is too strong for both species, then the boundary equilibria are locally asymptotically stable (provided that (19) and (20) are satisfied). In such a case the survival of the species depends on the initial position. If the effect of competition on species2 (by species1) is relatively high and the effect on species1 (by species2) is relatively low, then species2 will go extinct leaving species1 alone. In such case, an increase in E_1 affects the dominance of species1 which may result in the coexistence. Similar statement can be given when species2 dominates species1. Finally, in the absence of harvesting and pollution reduction activities, condition (14) is replaced by

$$a_{12} < \frac{K_1}{K2}, \quad a_{21} < \frac{K_2}{K1}, \tag{37}$$

and (17) is replaced by

$$\widehat{z}_- < \min\left\{\frac{K_1 - K_2 a_{12}}{\beta_1 - \beta_2 a_{12}},\ \frac{K_2 - K_1 a_{21}}{\beta_2 - \beta_1 a_{21}}\right\} < \widehat{z}_+,$$

for the existence of the unique interior equilibrium. In this case, A and C are unchanged and the expression for B is replaced by

$$B' = \gamma_1(K_1 - a_{12}K_2) + \gamma_2(K_2 - a_{21}K_1) + \eta(1 - a_{12}a_{21}).$$

Here, the existence of the interior equilibrium depends only on the stock of pollution and the interspecific competition. Furthermore, if the environment is pollution free, then condition (37) alone is sufficient for the existence of interior equilibrium.

4 Optimal harvest problem

Consider the exploitation of two competing species in a polluted environment under the sole ownership. A decline in the revenue (due to the presence of pollution) drives the owner to invest a part of the available effort capacity on pollution reduction. The aim is to determine the optimal effort allocation that maximizes the revenue.

Let $B(h, z)$ be a pollution dependent gross benefit from resource harvesting. This benefit is assumed to increase with increasing yield, and it decreases as pollution increases. Moreover, the marginal (negative) impact of pollution increases with pollution. Thus, the function B may behave as follows (ref. [34]):

$$B_h > 0, B_z < 0, B_{zz} < 0. \tag{38}$$

Therefore, we consider the following explicit form for the function B:

$$B(h, z) = (1 - \theta_1 z^2)p_1\alpha_1 q_1 E_1 x_1 + (1 - \theta_2 z^2)p_2\alpha_2 q_2 E_2 x_2.$$

Here, $p_1(1 - \theta_1 z^2)$ and $p_2(1 - \theta_2 z^2)$ represent the pollution dependent prices per unit harvest associated with species1 and species2, respectively, where θ_i is positive constant such that $0 \leq \theta_i z^2 \leq 1$ for $i = 1, 2$. Clearly, p_i is the price per unit catch associated with species i in the absence of pollution. Note that the function B can also be expressed as

$$B(E_1, E_2, x_1, x_2, z) = (1 - \theta_1 z^2)p_1\alpha_1 q_1 E_1 x_1 + (1 - \theta_2 z^2)p_2\alpha_2 q_2 E_2 x_2. \tag{39}$$

Hence, the instantaneous net revenue (which we denote by R) is defined by

$$R(E_1, E_2, x_1, x_2, z) = B(E_1, E_2, x_1, x_2, z) - E^{total}, \tag{40}$$

and the present value of the total net revenues is

$$PV = \int_0^\infty e^{-\delta t} R(E_1, E_2, x_1, x_2, z)dt. \tag{41}$$

The objective is to maximize the integral in (41) by the proper allocation of the available effort capacity (E^{total}) between harvesting (E_1, E_2) and pollution reduction ($E^{total} - E_1 - E_2$). Formally expressed, the problem is as follows:

$$\begin{aligned} &\max_{\{E_1, E_2\}} \; PV \\ &\text{Subject to: } (2a) - (2d) \\ &0 \leq E_1 \leq E_1^{max},\; 0 \leq E_2 \leq E_2^{max}. \end{aligned} \tag{42}$$

The problem given in (42) is an optimal control problem where x_1, x_2, z are the state variables and E_1, E_2 are the control variables. The effort $E^{total} - E_1 - E_2$ is determined once E_1, E_2 are identified. Solving problem (42) is to find out the optimal allocation of the total effort capacity (E^{total}) between harvesting ($\widehat{E}_1, \widehat{E}_2$) and pollution reduction ($E^{total} - \widehat{E}_1 - \widehat{E}_2$) so that the total discounted revenue is as large as possible.

Let us consider the following notations:

$$\begin{aligned}
f_0(x_1, x_2, z, E_1, E_2, t) &= e^{-\delta t} R(x_1, x_2, z, E_1, E_2),\\
f_1(x_1, x_2, z, E_1, E_2, t) &= b_1 x_1(\omega_1 - x_1 - a_{12}x_2 - \beta_1 z),\\
f_2(x_1, x_2, z, E_1, E_2, t) &= b_2 x_2(\omega_2 - x_2 - a_{21}x_1 - \beta_2 z),\\
f_3(x_1, x_2, z, E_1, E_2, t) &= v - (\gamma_1 x_1 + \gamma_2 x_2 + d)z.
\end{aligned} \tag{43}$$

As per the maximum principle (ref. [4,8,24]), the Hamiltonian (H) associated with problem (42) is given by

$$H(x_1, x_2, z, E_1, E_2, \mu_1, \mu_2, \mu_3, t) = f_0 + \mu_1 f_1 + \mu_2 f_2 + \mu_3 f_3.$$

The associated adjoint differential equations are given as follows:

$$\begin{aligned}
\frac{d\mu_1}{dt} &= -e^{-\delta t}(1 - \theta_1 z^2)(p_1 q_1 \alpha_1 E_1) - \mu_1 b_1\big[\omega_1 - 2x_1 - a_{12}x_2 - \beta_1 z\big] + \mu_2 x_2 c_{21} + \mu_3 \gamma_1 z,\\
\frac{d\mu_2}{dt} &= -e^{-\delta t}(1 - \theta_2 z^2)(p_2 q_2 \alpha_2 E_2) - \mu_2 b_2\big[\omega_2 - 2x_2 - a_{21}x_1 - \beta_2 z\big] + \mu_1 x_1 c_{12} + \mu_3 \gamma_2 z,\\
\frac{d\mu_3}{dt} &= e^{-\delta t}[(2\theta_1 z)p_1 q_1 \alpha_1 E_1 x_1 + (2\theta_2 z)p_2 q_2 \alpha_2 E_2 x_2] + \mu_1 x_1 d_1 + \mu_2 x_2 d_2 + \mu_3(\gamma_1 x_1 + \gamma_2 x_2 + d),
\end{aligned}$$

Here, μ_i is known as the adjoint variable for $i = 1, 2, 3$. Because of the presence of the term $e^{-\delta t}$ no steady state is possible for the above system, and hence the following transformation is needed;

$$\lambda_i(t) = \mu_i(t)e^{\delta t}, \quad \text{for} \quad i = 1, 2, 3 \quad \text{and} \quad \mathcal{H} = He^{\delta t}, \tag{44}$$

where $\mathcal{H}$ is known as the current value Hamiltonian, which is given by

$$\mathcal{H}\big(x_1, x_2, z, E_1, E_2, \lambda_1, \lambda_2, \lambda_3\big) = R + \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3, \tag{45}$$

and λ_1, λ_2, λ_3 are known as the current adjoint variables satisfying the differential equations:

$$\frac{d\lambda_1}{dt} = \delta\lambda_1 - (1 - \theta_1 z^2)(\alpha_1 p_1 q_1 E_1) - b_1[\omega_1 - 2x_1 - a_{12}x_2 - \beta_1 z]\lambda_1 + x_2 c_{21}\lambda_2 + \gamma_1 z \lambda_3, \tag{46a}$$

$$\frac{d\lambda_2}{dt} = \delta\lambda_2 - (1 - \theta_2 z^2)(p_2 \alpha_2 q_2 E_2) - b_2[\omega_2 - 2x_2 - a_{21}x_1 - \beta_2 z]\lambda_2 + x_1 c_{12}\lambda_1 + \gamma_2 z \lambda_3, \tag{46b}$$

$$\frac{d\lambda_3}{dt} = [(2\theta_1 z)p_1 \alpha_1 q_1 E_1 x_1 + (2\theta_2 z)p_2 \alpha_2 q_2 E_2 x_2] + x_1 d_1 \lambda_1 + x_2 d_2 \lambda_2 + [\delta + \gamma_1 x_1 + \gamma_2 x_2 + d]\lambda_3. \tag{46c}$$

Clearly the problem under consideration is a linear control problem, and hence the optimal control shall be a combination of bang-bang and singular controls (ref. [7]). First we investigate the optimal singular control and the associated optimal singular solution.

Differentiating the current value Hamiltonian (in (45)) with respect to E_1 and E_2 gives

$$\begin{aligned}\mathcal{H}_{E_1} &= (1-\theta_1 z^2)(p_1\alpha_1 q_1 x_1) - \lambda_1\alpha_1 q_1 x_1 + \lambda_3\beta z,\\ \mathcal{H}_{E_2} &= (1-\theta_2 z^2)(p_2\alpha_2 q_2 x_2) - \lambda_2\alpha_2 q_2 x_2 + \lambda_3\beta z,\end{aligned}$$

and the switching functions are

$$\begin{aligned}s_1(t) &= (1-\theta_1 z^2)(p_1\alpha_1 q_1 x_1) - \lambda_1\alpha_1 q_1 x_1 + \lambda_3\beta z,\\ s_2(t) &= (1-\theta_2 z^2)(p_2\alpha_2 q_2 x_2) - \lambda_2\alpha_2 q_2 x_2 + \lambda_3\beta z.\end{aligned} \tag{47}$$

It is known that in the case of singular solution we have $s_1(t) = 0,\ s_2(t) = 0$, i.e.,

$$\begin{aligned}&(1-\theta_1 z^2)(p_1\alpha_1 q_1 x_1) - \lambda_1\alpha_1 q_1 x_1 + \lambda_3\beta z = 0,\\ &(1-\theta_2 z^2)(p_2\alpha_2 q_2 x_2) - \lambda_2\alpha_2 q_2 x_2 + \lambda_3\beta z = 0.\end{aligned} \tag{48}$$

Now, the unique interior steady state of the six dimensional system ((2a)-(2c), (46a)-(46c)) is given by $(\widehat{x}_1, \widehat{x}_2, \widehat{z}, \widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3)$, where

$$\begin{aligned}\widehat{x}_1 &= \frac{(\omega_1-\omega_2 a_{12}) - (\beta_1 - a_{12}\beta_2)\widehat{z}}{1-a_{12}a_{21}},\\ \widehat{x}_2 &= \frac{(\omega_2-\omega_1 a_{21}) - (\beta_2 - a_{21}\beta_1)\widehat{z}}{1-a_{12}a_{21}},\\ \widehat{z} &= \frac{B-\sqrt{B^2-4AC}}{2A},\\ \widehat{\lambda}_1 &= \frac{(1-\theta_1\widehat{z}^2)(p_1\alpha_1 q_1 E_1) - c_{21}\widehat{x}_2\widehat{\lambda}_2 - \gamma_1\widehat{z}\widehat{\lambda}_3}{\delta + b_1 x_1},\\ \widehat{\lambda}_2 &= \frac{\zeta_0}{c_{12}c_{21}\widehat{x}_1\widehat{x}_2 - (\delta+b_1\widehat{x}_1)(\delta+b_2\widehat{x}_2)},\\ \widehat{\lambda}_3 &= \frac{\zeta_1-\zeta_2}{\zeta_3-\zeta_4},\end{aligned} \tag{49}$$

with A, B and C are given in (10), and

$$\begin{aligned}\zeta_0 &= [c_{12}\widehat{x}_1(1-\theta_1\widehat{z}^2)(p_1\alpha_1 q_1 E_1) - (\delta+b_1\widehat{x}_1)(1-\theta_2\widehat{z}^2)(p_2\alpha_2 q_2 E_2)]\\ &\quad -[(c_{12}\gamma_1\widehat{x}_1\widehat{z}) - (\delta+b_1\widehat{x}_1)\gamma_2\widehat{z}]\widehat{\lambda}_3,\\ \zeta_1 &= [d_1\widehat{x}_1(\delta+b_2\widehat{x}_2) - c_{12}d_2\widehat{x}_1\widehat{x}_2]\\ &\quad [c_{12}\widehat{x}_1(1-\theta_1\widehat{z}^2)p_1\alpha_1 q_1 E_1 - (\delta+b_1\widehat{x}_1)(1-\theta_2\widehat{z}^2)p_2\alpha_2 q_2 E_2],\\ \zeta_2 &= [c_{12}c_{21}\widehat{x}_1\widehat{x}_2 - (\delta+b_1\widehat{x}_1)(\delta+b_2\widehat{x}_2)]\\ &\quad [d_1\widehat{x}_1(1-\theta_2\widehat{z}^2)p_2\alpha_2 q_2 E_2 + c_{12}\widehat{x}_1(2\theta_1\widehat{z}p_1\alpha_1 q_1 E_1\widehat{x}_1 + 2\theta_2\widehat{z}p_2\alpha_2 q_2 E_2\widehat{x}_2)],\\ \zeta_3 &= [(c_{12}\gamma_1\widehat{x}_1\widehat{z}) - (\delta+b_1\widehat{x}_1)\gamma_2\widehat{z}][d_1\widehat{x}_1(\delta+b_2\widehat{x}_2) - c_{12}d_2\widehat{x}_1\widehat{x}_2],\\ \zeta_4 &= [d_1\gamma_2\widehat{x}_1\widehat{z} - c_{12}\widehat{x}_1(\delta+d+\gamma_1 x_1+\gamma_2 x_2)][c_{12}c_{21}\widehat{x}_1\widehat{x}_2 - (\delta+b_1\widehat{x}_1)(\delta+b_2\widehat{x}_2)].\end{aligned}$$

Substituting the solution $\left(\widehat{x}_1, \widehat{x}_2, \widehat{z}, \widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3\right)$ into (48) gives us the following system of equations involving only E_1, E_2:

$$\begin{aligned}&(1-\theta_1\widehat{z}^2)(p_1\alpha_1 q_1\widehat{x}_1) - \widehat{\lambda}_1\alpha_1 q_1\widehat{x}_1 + \widehat{\lambda}_3\beta\widehat{z} = 0,\\ &(1-\theta_2\widehat{z}^2)(p_2\alpha_2 q_2\widehat{x}_2) - \widehat{\lambda}_2\widehat{\alpha}_2 q_2\widehat{x}_2 + \widehat{\lambda}_3\beta\widehat{z} = 0.\end{aligned} \tag{50}$$

If the pair $(\widehat{E}_1, \widehat{E}_2)$ uniquely solves the system of equations (in (50)) and satisfies conditions $0 < \widehat{E}_1 < E_1^{max}$ and $0 < \widehat{E}_2 < E_2^{max}$, then it becomes the optimal singular control (otherwise the maximum among the appropriate solutions becomes optimal), and the associated optimal solution becomes $\big(\widehat{x}_1(\widehat{E}_1, \widehat{E}_2), \widehat{x}_2(\widehat{E}_1, \widehat{E}_2), \widehat{z}(\widehat{E}_1, \widehat{E}_2)\big)$ (ref. [3, 5]).

Having identified the steady state solution, now it remains to reach this solution optimally starting from the given initial state $(x_1(0), x_2(0), z(0))$. Since the problem under consideration is linear in the control variables, the optimal steady solution shall be reached by a bang-bang control (ref. [24]). If we denote it by a control vector $\big(\overline{E}_1, \overline{E}_2\big)$, then we have

$$\overline{E_1}(t) = \begin{cases} 0, & \text{if } s_1(t) < 0, \\ E_1^{max}, & \text{if } s_1(t) > 0, \end{cases} \tag{51}$$

$$\overline{E_2}(t) = \begin{cases} 0, & \text{if } s_2(t) < 0, \\ E_2^{max}, & \text{if } s_2(t) > 0. \end{cases} \tag{52}$$

If T^* represents the time taken to reach the optimal steady state optimally from the given initial state, then the optimal control $\big(E_1^*, E_2^*\big)$ to the given problem is

$$\big(E_1^*(t), E_2^*(t)\big) = \begin{cases} \big(\overline{E_1}(t), \overline{E_2}(t)\big), & \text{for } 0 \le t < T^* \\ \big(\widehat{E}_1, \widehat{E}_2\big), & \text{for } t \ge T^*. \end{cases} \tag{53}$$

Suppose $(\breve{x}_1(t), \breve{x}_2(t), \breve{z}(t))$ represents the trajectory from some initial state to the optimal steady state solution. Then the optimal stock path (denoted by $(x_1^*(t), x_2^*(t), z^*(t))$) is traced out by

$$\big(x_1^*(t), x_2^*(t), z^*(t)\big) = \begin{cases} \big(\breve{x}_1(t), \breve{x}_2(t), \breve{z}(t)\big), & \text{for } 0 \le t \le T^* \\ \big(\widehat{x}_1, \widehat{x}_2, \widehat{z}\big), & \text{for } t \ge T^*. \end{cases} \tag{54}$$

Note that if the singular control is employed right from the initial state, then the corresponding stock path approaches the optimal singular solution asymptotically (by the global asymptotic stability of the singular solution). The resulting stock path is known as suboptimal path.

5 Applications

This section presents numerical simulations to demonstrate the significant outcomes of the study. The examples represent the dynamics of two competing species in a polluted lake environment, where the biological and economic parameters are related to the actual values one might have in a fishery (ref. [1]). Table 2 presents the values assigned to the associated parameters and constants.

Consider the set of values given in Table 2. For each pair of efforts (E_1, E_2) conditions (13), (14), (15), (17) are satisfied, and hence system (2a)-(2c) admits a unique interior equilibrium Q^*. Moreover, (25) is also satisfied and hence the unique interior equilibrium is globally asymptotically stable. The relationships between efforts E_1, E_2 and the stocks x_1, x_2, z as well as the yield h are shown in Figure 1. For the given values of the parameters, conditions (27), (30) and (33) are satisfied. Consequently, the stock x_1 decreases with E_1 and it increases with E_2 (see Frame (a)). Similarly, x_2 decreases with E_2 and it increases with E_1 (see Frame (b)). The

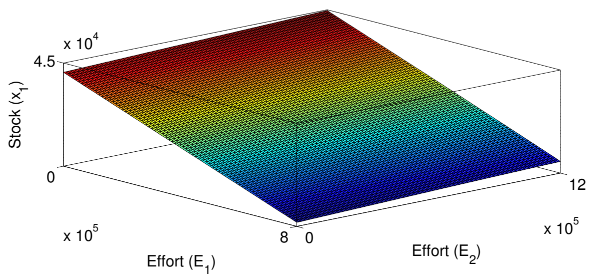
Table 2: The values assigned to the associated parameters and constants in the model.

Symbols	Values	Units
r_1 (r_2)	0.42 (0.36)	1/year
K_1 (K_2)	4.2×10^4 (3.6×10^4)	tonnes
c_{12} (c_{21})	1×10^{-6} (1.2×10^{-6})	1/tonne/year
d_1 (d_2)	1×10^{-5} (1×10^{-5})	1/tonne/year
γ_1 (γ_2)	0.8×10^{-4} (0.6×10^{-4})	1/tonne/year
v	600	tonne/year
E^{total}	2.0×10^6	US$/year
E_1^{max} (E_2^{max})	8×10^5 (1.2×10^6)	US$/year
α_1 (α_2)	1.0×10^{-3} (0.8×10^{-3})	vessel/US$/year
β	1×10^{-6}	1/US$
η	5×10^{-3}	1/year
q_1 (q_2)	0.0004 (0.0003)	1/vessel/year
δ	0.05	1/year
p_1 (p_2)	3000 (4000)	US$/tonne
θ_1 (θ_2)	1×10^{-5} (1×10^{-5})	1/tonne^2

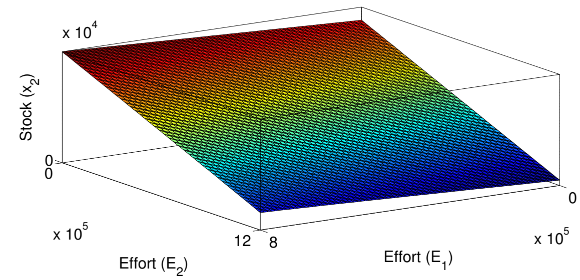
stock of pollution z increases with both E_1 and E_2 (see Frame (c)). Here, the highest level of pollution occurs at the steady-state where the entire effort is utilized for harvesting, and the lowest level of pollution occurs where the entire effort goes for pollution reduction. The figure further highlights the maximum sustainable yield $h_{MSY} = 7.238 \times 10^3$ (in tonnes) and the corresponding critical effort level $(E_1^{MSY}, E_2^{MSY}) = (5.24 \times 10^5, 7.34 \times 10^5)$ (in US$) (see Frame (d)). Note that this yield is larger than the one we would obtain by utilizing the entire effort for harvesting alone. The associated globally asymptotically stable interior equilibrium is $(2.05 \times 10^4, 1.66 \times 10^4, 177)$ (in tonnes).

Now let us consider the set of parameter values as presented in Table 2 except for the inflow of pollutants which is replaced by $v = 3 \times 10^3$ (tonnes per year). In this case, the maximum sustainable yield h_{MSY} is 6.98×10^3 (in tonnes) and the corresponding effort allocation is $(5.06 \times 10^5, 7.04 \times 10^5)$ (in US$). The associated globally asymptotically stable interior equilibrium is $(2.06 \times 10^4, 1.67 \times 10^4, 871)$ (in tonnes). The coexistence of the species (in this case) is shown in Frame (a) of Figure 2. Now, if the effort allocation is taken as (E_1^{max}, E_2^{max}), i.e., the entire effort is utilized for harvesting alone, then no species will survive in the environment (see Frame (b) of Figure 2)). This underlines the crucial role played by the effort allocation towards pollution reduction on the coexistence.

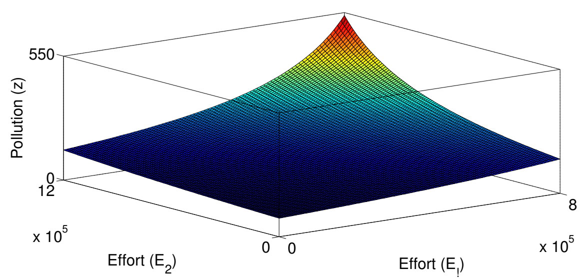
Next, let's consider the set of parameter values as presented in Table 2 except for the inflow of pollutants (v) and the interspecific coefficient c_{21} which are replaced by 3×10^3 and 2.2×10^{-5}, respectively. Here, the effect of species1 on the growth rate of species2 is high, and that of species2 on the growth rate of species1 is relatively low. Consequently, species1 excludes species2 (see Frame (a) of Figure 3). Now, if we put some pressure on species1 by applying additional fishing effort (e.g., $E_1 = E_1^{max}$), then its dominance will be limited, resulting in the coexistence



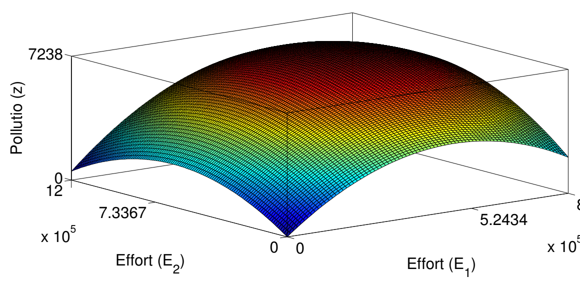
(a) Species1



(b) Species2



(c) Pollution



(d) Yield

Figure 1: This figure presents the relationship between efforts (E_1, E_2) and the stocks x_1, x_2, z, and the yield (h).

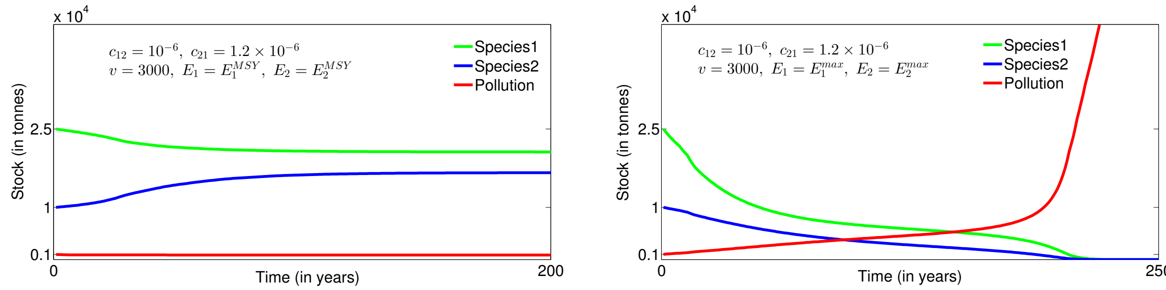
in the system (see Frame (b) of Figure 3). Similar is true when species2 is dominant.

Finally, consider the optimal harvest problem (42). With the set of parameter values given in Table 2, the unique interior steady state of the six dimensional dynamical systems ((2a), (2b), (2c), (46a), (46b), (46c)) is $(2.18 \times 10^4, 1.74 \times 10^4, 166.5, 2.34 \times 10^3, 2.99 \times 10^3, -1.15 \times 10^4)$ where the unique solution $(\widehat{E}_1, \widehat{E}_2)$ of (50) is $(4.93 \times 10^5, 6.98 \times 10^5)$ (in US\$). Clearly, $0 < \widehat{E}_1 < E_1^{max}$, $0 < \widehat{E}_2 < E_2^{max}$ and hence $(\widehat{E}_1, \widehat{E}_2)$ is optimal. The associated optimal singular solution is $(2.18 \times 10^4, 1.74 \times 10^4, 166.5)$ (in tonnes). The yield corresponding to the optimal steady state is $h(\widehat{E}_1, \widehat{E}_2) = 7.22 \times 10^3$ (in tones). Observe that this yield doesn't exceed the maximum sustainable yield $h_{MSY} = 7.238 \times 10^3$ (in tonnes) as it was expected.

6 Concluding remarks

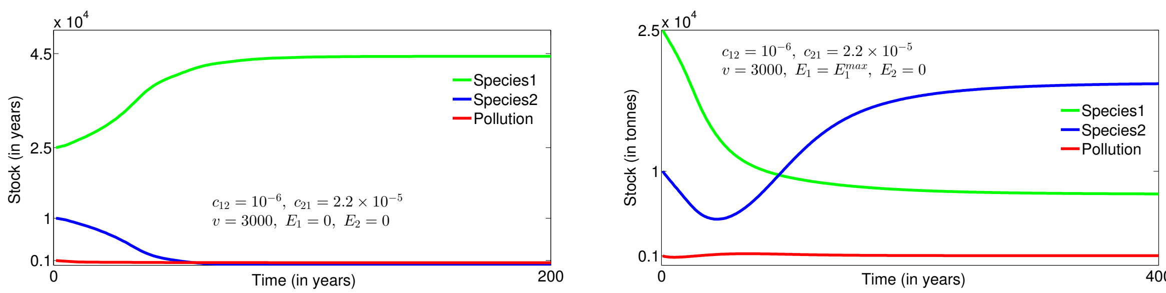
In this paper, we have presented the dynamics of two competing species within a polluted environment in the presence of harvesting and pollution reduction. Fall in the revenue (due to pollutants) is the driving force for investing a part of the effort capacity on pollution reduction. Environmental pollution is assumed to affect both the growth rate and the quality of biomass. We have captured the effect of pollution on resource growth through the intrinsic growth rate and the saturation level (in the regeneration function), and the effect on the quality of biomass through the revenue function.

We have considered environmental treatment (by the depollution effort) as a feasible alter-



(a) Coexistence of the species (b) Extinction of the species

Figure 2: This figure highlights the influence of effort allocation towards pollution reduction on the survival of the species.



(a) Dominance of species1 (b) Coexistence

Figure 3: This figure highlights the crucial role played by harvesting the dominant species on the coexistence.

native to reduce the stock of pollutants in the environment. We have investigated the stable coexistence, competitive exclusion and the extinction of both species. We observed that the competition, pollutants and exploitation play a crucial role on the coexistence. In particular, if the inflow of pollutants is too large, then the extinction of both species is inevitable. Otherwise, it is possible to ensure the permanence of species through the depollution effort. Further, we have provided some criteria for the extinction of both species, competitive exclusion, and stable coexistence in the system. In the case where one species dominates the other, it is possible to ensure the stable coexistence in the system by increasing the harvesting effort associated with the dominant species.

By considering the revenue function that is pollution dependent, we have studied the optimal harvest problem. We observed that when the inflow of pollutants increases, then the optimal harvest strategy recommends increasing the efforts towards pollution reduction. Moreover, the proper allocation of the available effort capacity between harvesting and pollution reduction, not only improves the revenues but also the survival rate of the species.

Finally, in the current work, we have considered a system consisting of two competing species that are surviving in a polluted environment. One can establish similar results for the system consisting of more than two species. Moreover, we have considered a competition type of ecological-interdependence between two species. Similar results can be established for the other types of ecological-interdependence such as predation, mutualism, etc.

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