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A linear theory of beams with deformable cross section

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Abstract. We present a direct model of beam which takes into consideration the deformation of the section by effect of orthogonal actions. The variation of size and the distortion of the transversal sections are taken into account as well as the usual rigid rotation-torsion-warping. We deduce the equations of motion in terms of the kinematic descriptors. A simple numerical example is also presented to show the consistence of the proposed model.

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1 Historical framework

The development of the *direct* (or *directed*) models of beams derives from the ideas of E. and F. Cosserat [6]. The motion of a single point of a Cosserat's continuum is composed by the usual translation, as in the Cauchy continuum model, as well as a rigid rotation; each element is so not a "simple" point but a "structured" point, as a small sphere. The Cosserat model of beam consists of a line of structured points: the beam is reduced to a representative line, i.e. its axis, moreover a triad of vectors (named *directors*) is attached to each point of this line. The vectors rigid rotation does not depend on the motion of this line. The deformation of a Cosserat beam is described by a map of a segment (usually the beam length) into the manifold $\mathcal{M} = \mathbb{R}^3 \times SO(3)$, $\mathbb{R}^3$ is the Euclidean space with the canonical metric and $SO(3)$ is the *Special Orthogonal Group*. If other behaviors of the directors are allowed the manifold $\mathcal{M}$ has a different structure. See the works of Ericksen and Truesdell [8], Antman [2–4], Green and Naghdi [11], Reissner [17], Simo [18], Simo and Vu-Quok [19] and Villaggio [20] in order to understand the evolution of direct models of beams. A non linear model for concrete beam with deformable cross section has been presented by Palacios and Cesnick [16]. Davì [7] obtained the equations of motion of a linear, homogeneous, anisotropic beam subject to axial extension, shear, bending,

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torsion and warping using a direct model of beam. All coefficients of the deduced equations are the geometric and inertial terms of the beam. The central idea of Davì's paper is the absolute kinematic description of the straight beam: the motion of the structural element is represented by two vectorial functions: the first one describes the motion in the direction of the axis, the second one belongs to the plane normal to it. Each function includes both the motion of the axis and of the cross section. The strain tensor is de-coupled in three terms in this frame: one part belongs to the axis, a second one is normal to it and a third mixed part has a component in the direction of the axis and a component in the section plane. A similar decomposition is used for the stress tensor. Davì shows also that the assumption of *thinness* of the beam is congruent and allows the computation of the beam motion. The displacement is not given "a priori" but reckoned "a posteriori". It is clear that the deduced motion equations are an adaptation of the Navier Stokes equations to direct beam models; this adaptation depends both on the practical shape of the element and on the chosen kinematic model.

The models of continua with structures are also used in the description of micro-fractured materials and on the propagation of fractures. Khodadadian, Noii and and others, with a probabilistic Bayesian approach, used the theory of continuum with structure studying the propagation of fractures: in their model the fracture surface is modeled through a shape function, [12]. In a similar framework, Noii, Aldakheel and others did model the phase-field of anisotropic brittle fracture via a shape function, [15]. The method of the orthogonal decomposition of the motion has been used by Abbaszadeh, Dehghan and others [1] in the numerical study of Navier Stokes equations coupled with heat transfer equation in the two dimensional Bussinesq model of continuum.

2 Presentation of the model

The above cited model proposed by Davì and all models with axial rigid directors are not suitable for describing some behaviors that are common in beam dynamics. If the the axis is subject to a compression (respectively extension) the sections reply with an extension (respectively contraction). An eventual rectangular mesh drawn in the section remains rectangular after the deformation, but possibly with a different aspect ratio; we talk about *re-sizing* of the section. A part of the section is normally compressed and a second part is normally stretched in the case of bending; the section replies as in the previous case, but a hypothetic rectangular mesh drawn in the section does not remain the same after the deformation because the edges assume curvilinear shape; each rectangle takes a pseudo-trapezoidal shape. We call this kind of deformation *re-shaping* of the section. We present a first approach to the description of the modifications of size and shape of transversal section of a straight linear uniform anisotropic beam. A system of three unit and mutually orthogonal vectors is attached to each point of the straight line joining the centers of mass of transversal sections, the *beam axis*, as in the classical above mentioned beam theories. One director belongs to the axis, the other two lay in the plane of the section. The directors rotate rigidly around their origin, this rotation is usually represented by a second order orthogonal matrix with determinant equal to one. This matrix is approximated with its first order part in the linearized kinematics; this approximation is done via a skew-symmetric

The part of the strain tensor belonging to the plane of section is identically zero, see Vlasov [21].

matrix of order three. It is usual, to make easiest the computations, the use of its unique real eigenvector rather than the whole matrix: this is the *rotation vector*. Two new directors are attached to the same point, they are able to describe both the re-sizing and re-shaping of the section; in this way we dismiss the assumption of thinness of the rod. It is clear that the uniform re-sizing of the transversal section may be described with an homogeneous strain tensor, on the contrary the modification of shape cannot be assigned with this kind of tensor, it is therefore necessary to use a strain tensor field at least linear in the coordinates of the section plane. It is important to note that the axis of bending is a fixed line of the plane deformation, i.e. no point belonging to it suffers a strain in the plane of the section but only a deformation in its normal direction. Unfortunately the direction of bending generally is an unknown of the problem and it is hard to describe this deformation using a frame with an axis coincident with the bending direction. Another description which uses two selected axes is possible but approximate, it will be used in this paper: the selected axes are the main inertia axes of the section. The motion of the transversal sections in their own plane is assumed to be the same for all sections, but the deformation size of each section is undertaken as an unknown of the problem; two scalar functions describe the re-sizing of the section and four scalar functions are able to sketch its re-shaping. The proposed model holds six degrees of freedom more than usual, thirteen functions scan the kinematics of the beam: three degrees of freedom are assigned to the axis motion and three are used to the section rigid rotation, one is assigned to the warping, six are able to sketch the re-sizing and the re-shaping of the section. An inertial term, a component of internal action and a component of external loads are associated to each degree of freedom. We circumscribe our study to linear dynamics, assuming that the balance occurs in the undeformed and reference configuration. The strain and the stress fields are consequently described by the *Lagrange tensor of small deformations* and the *Cauchy stress tensor* respectively. The motion equations are deduced using the weak formulation via the Principle of Virtual Power: we obtain the balance equations in term of resultant over the cross section of internal and external forces rather than in term of components of stress tensor, as it is usual in the beam theories. All achieved equations depend on geometric and inertial characteristics of the section; as well as the usual area, moments of inertia, Vlasov constant and some parameters deduced by warping shape functions, it is also necessary to introduce the moment of third and fourth order of the section.

3 Kinematic description and deformability of the beam element

A beam is a three-dimensional body which occupies the cylindrical region $\mathcal{R} = \mathcal{S} \times \mathcal{I} \subset \mathbb{E}^3$ of the Euclidean three-dimensional linear space $\mathbb{E}^3$ equipped with canonical metric, induced by the standard inner product. $\mathcal{S}$ is the *cross section*, an open, regular planar two dimensional domain with boundary ∂S and belonging to the plane $\mathcal{W}$ whose unit normal vector is $\mathbf{e}$. $\mathcal{I} = [\,0, L\,]$ is a closed interval of the real axis $\mathbb{R}$. The decomposition $\mathcal{R} = \mathcal{S} \times \mathcal{I}$ also allows to write the position of a generic point $\mathbf{p}$ of the beam as the sum of a vector $\mathbf{x}$ belonging to the section and a vector $\zeta \cdot \mathbf{e}$ normal to it, both vectors start from a selected point $\mathbf{o}$:

$$\mathbf{p}(\mathbf{x}, \zeta) = \mathbf{x} + \mathbf{e}\,\zeta, \qquad \mathbf{x} \cdot \mathbf{e} = 0, \qquad \zeta \in \mathcal{I}.$$

We shall keep $\mathbf{o}$ coincident with the *center of mass* of the section corresponding to $\zeta = 0$. The boundary $\partial\mathcal{R}$ of the beam is the disjointed union of three subsets:

$$\partial\mathcal{R} = \partial\mathcal{M} \amalg (\partial S \times \{0\}) \amalg (\partial S \times \{L\}) \ ,$$

$\partial M = \partial S \times \mathcal{I}$ is the *mantle*, $\partial S \times \{0\}$ is the *initial base* and $\partial S \times \{L\}$ is the *terminal base*.

A deformation of the beam is the vectorial field $\mathbf{p}' \colon \mathcal{R} \to \mathbb{E}^3$, $\mathbf{p}' = \mathbf{p}'(\mathbf{p}) = \mathbf{p}'(\mathbf{x}, \zeta)$; a monoparametric family of deformations $\mathbf{u} : \mathcal{R} \times [\,0,\, t_f\,] \to \mathbb{E}^3$ is a motion of the beam, the parameter t is the time, defined in the interval $[\,0,\, t_f\,]$. We decompose the motion $\mathbf{u}$ as a direct sum of two orthogonal sub-motions: $\mathbf{u}(\mathbf{x}, \zeta; t) = u(\mathbf{x}, \zeta; t)\,\mathbf{e} + \widehat{\mathbf{u}}(\mathbf{x}, \zeta; t)$, this assumption is consistent with the Davi's work. The motion $\mathbf{u_1}(\mathbf{x}, \zeta; t) = u_1(\mathbf{x}, \zeta; t) \cdot \mathbf{e}$ is parallel to the direction $\mathbf{e}$ and the motion $\widehat{\mathbf{u}}(\mathbf{x}, \zeta; t)$ belongs to the plane $\mathcal{W}$ normal to $\mathbf{e}$.

The symmetric part of the deformation gradient $\nabla_{\mathbf{x}}\mathbf{u}$ furnishes the infinitesimal strain field $\mathbf{E}(\mathbf{x}, \zeta; t)$, a second order symmetric tensor, defined in $\mathcal{R} \times [0, t_f]$; $\nabla_{\mathbf{x}}$ denotes the gradient operator with respect to $\mathbf{x}$. We introduce the following decomposition of the strain tensor

$$\mathbf{E}(\mathbf{x}, \zeta, t) = \widehat{\mathbf{E}}(\mathbf{x}, \zeta, t) + sym[\,\gamma(\mathbf{x}, \zeta, t) \otimes \mathbf{e}\,] + \varepsilon(\mathbf{x}, \zeta, t)\,\mathbf{e} \otimes \mathbf{e}\,, \tag{1}$$

- $\widehat{\mathbf{E}}(\mathbf{x}, \zeta, t) = \nabla_{\mathbf{x}}\widehat{\mathbf{u}}$ is the *plain strain field*, a second order symmetric tensor with kernel $\mathbf{e}$: $\widehat{\mathbf{E}}\,\mathbf{e} = \mathbf{0}$;
- $sym[\,\gamma(\mathbf{x}, \zeta, t) \otimes \mathbf{e}\,]$ denotes the symmetric part of the second order tensor $\gamma(\mathbf{x}, \zeta, t) \otimes \mathbf{e}$, $\gamma \in \mathcal{W}$ is the *shear deformation field* defined as $\gamma = \nabla_{\mathbf{x}} u + \partial_\zeta \widehat{\mathbf{u}}$;
- $\varepsilon = \partial_\zeta u$ is the *axial deformation*.

We assume that the plain strain field is the sum of two second order tensors $\widehat{\mathbf{E}} = \widehat{\mathbf{E}}_e + \widehat{\mathbf{E}}_d$. The first one

$$\widehat{\mathbf{E}}_e(\mathbf{x}, \zeta; t) = \alpha(\zeta; t)\mathbf{a} \otimes \mathbf{a} + \beta(\zeta; t)\mathbf{b} \otimes \mathbf{b} \tag{2}$$

represents the *plane extension strain tensor*; $\alpha(\zeta; t) : [\,0, L\,] \times [\,0, t_f\,[\to \mathbb{R}$ and $\beta(\zeta; t) : [\,0, L\,] \times [\,0, t_f\,[\to \mathbb{R}$ are the parameters of re-size of the area in the direction of two unit vectors $\mathbf{a}$ and $\mathbf{b}$. These vectors belong to the plane of the section, start from the center of mass and are mutually orthogonal and normal to $\mathbf{e}$. This deformation is thus defined as a product of two scalings centered in the center of mass of each section: the first $\Lambda_{\mathbf{a}}$ in direction $\mathbf{a}$ and ratio α, the second one $\Lambda_{\mathbf{b}}$ in direction $\mathbf{b}$ and ratio β. It is clear that if $\alpha < 0$ it has a contraction of the section in $\mathbf{a}$-direction, if $\alpha > 0$ it has an expansion; the same meaning is for β in $\mathbf{b}$-direction. A representation of a possible deformation of a rectangular section is reported in Figure 1, left side; we shall write in general terms of re-sizing of transversal section, without specifying the sign of α and β. The index "e" points out this kind of deformation: *the extension of the section without re-shaping*. A deeper kinematic description is necessary to sketch the section re-shaping. If $\mathbf{a}$ is the direction of the bending, i.e. the axis folds in direction $\mathbf{a}$, a possible displacement field able to describe the re-shaping is given by the quadratic form:

$$\widehat{\mathbf{u}}_{d\mathbf{a}}(\mathbf{x}, \zeta; t) = -\frac{1}{2}[\gamma_{\mathbf{a}}(\zeta; t)(\mathbf{a} \otimes \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{a}) - \delta_{\mathbf{a}}(\zeta; t)(\mathbf{b} \otimes \mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{b})][\mathbf{x} \otimes \mathbf{x}] \tag{3}$$

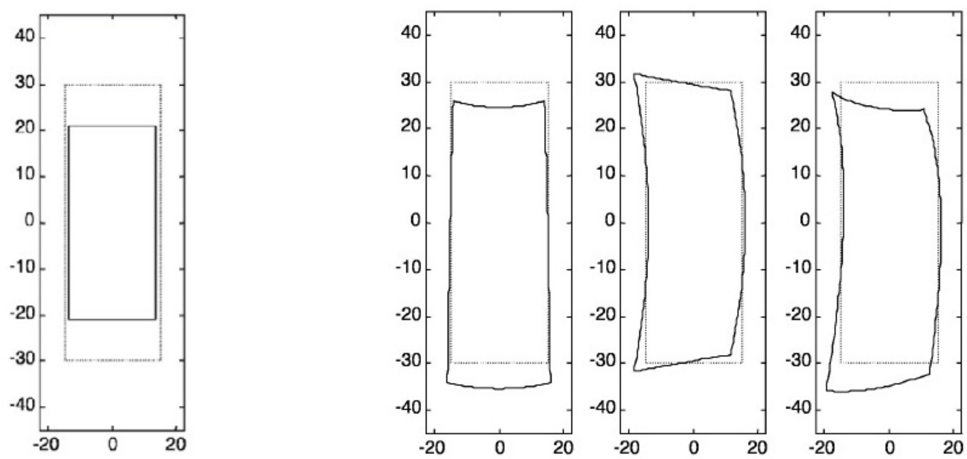

Figure 1: Left side: the transversal section subject to a contraction only. Right side The transversal section subject to a re-shaping: *left side* vertical axis of bending; *central side* horizontal axis of bending; *right side* the combination of the previous ones; (dotted line, undeformed shape; continuous line, deformed shape).

the index "$\mathbf{a}$" emphasizes that the re-shaping comes from the flexure in $\mathbf{a}$-direction (see Figure 1, right side). It is easy to verify that the gradient of (3) with respect to $\mathbf{x}$ is a second order symmetric tensor. The terms $\gamma_{\mathbf{a}}(\zeta;t) : [0,L] \times [0,t_f[\to \mathbb{R}$ and $\delta_{\mathbf{a}}(\zeta;t) : [0,L] \times [0,t_f[\to \mathbb{R}$ furnish the sizes of re-shaping, variable in space and in time. The term $\gamma_{\mathbf{a}}$ expresses the amplitude of the section re-shaping in $\mathbf{a}$-direction, conversely the term $\delta_{\mathbf{a}}$ furnishes the amplitude of the section re-shaping in $\mathbf{b}$-direction. A distorted rectangular section for a displacement field expressed by relation (3) is represented in the right side of Figure 1. Since the direction of bending is not known a priori, we suppose that the bending has a component due to a bending in $\mathbf{b}$ direction as well, this component of motion causes a field of re-shaping represented in the central side of Figure 1, right side. The quadratic form able to describe this re-shaping is:

$$\widehat{\mathbf{u}}_{d\mathbf{b}}(\mathbf{x},\zeta;t) = -\frac{1}{2}[\gamma_{\mathbf{b}}(\zeta;t)(\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{a}+\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{a}) - \delta(\zeta;t)(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{b}-\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a})][\mathbf{x}\otimes\mathbf{x}] \quad (4)$$

with an obvious meaning of symbols; $\gamma_{\mathbf{b}}(\zeta;t) : [0,L] \times [0,t_t[\to \mathbb{R}$ and $\delta_{\mathbf{b}}(\zeta;t) : [0,L] \times [0,t_t[\to \mathbb{R}$. The combination of (3) and (4) gives a re-shaping field on the cross section, represented in Figure 1, extreme right side.

The part of strain tensor painting the section re-shaping is given by the *plane re-shaping strain tensor*:

$$\begin{aligned}\widehat{\mathbf{E}}_d(\mathbf{x},\zeta;t) = & -[\gamma_{\mathbf{a}}(\zeta;t)(\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{b}+\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{a}) - \delta_{\mathbf{a}}(\zeta;t)(\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{a}-\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b})]\mathbf{x} \\ & -[\gamma_{\mathbf{b}}(\zeta;t)(\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{a}+\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{a}) - \delta_{\mathbf{b}}(\zeta;t)(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{b}+\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a})]\mathbf{x}. \quad (5)\end{aligned}$$

We emphasize here that $\nabla_x \widehat{\mathbf{u}}$ is defined at less of a skew-symmetric part Ψ, representing a rigid rotation, i.e.,

$$\nabla_x \widehat{\mathbf{u}} = \widehat{\mathbf{E}} + \Psi. \tag{6}$$

Supposing that the beam is restrained so as not to allow rigid motions, the explicit expression of $\widehat{\mathbf{u}}(\mathbf{x}, \zeta; t)$ is deduced by direct integration of (6):

$$\begin{aligned}
\widehat{\mathbf{u}}(\mathbf{x}, \zeta; t) \; = \; & \widehat{\mathbf{v}}(\mathbf{x}, \zeta; t) + [\alpha(\zeta; t)\mathbf{a} \otimes \mathbf{a} + \beta(\zeta; t)\mathbf{b} \otimes \mathbf{b}]\mathbf{x} \\
& -\frac{1}{2}[\gamma_{\mathbf{a}}(\zeta; t)(\mathbf{a} \otimes \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{a}) - \delta_{\mathbf{a}}(\zeta; t)(\mathbf{b} \otimes \mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{b})]\mathbf{x} \otimes \mathbf{x} \\
& -\frac{1}{2}[\gamma_{\mathbf{b}}(\zeta; t)(\mathbf{b} \otimes \mathbf{b} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{a}) - \delta_{\mathbf{b}}(\zeta; t)(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{b} - \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a})]\mathbf{x} \otimes \mathbf{x} \\
& +\vartheta(\zeta; t)\mathbf{e} \times \mathbf{x} + [\mathbf{e} \times \varphi(\zeta; t) - \varphi(\zeta; t) \times \mathbf{e}]\mathbf{x} \,.
\end{aligned}$$

where $\widehat{\mathbf{v}} \in \mathcal{W}$ is the displacement of a point of the beam axis; $\vartheta \in \mathbb{R}$ describes the torsion of the section $\mathcal{S}$, an infinitesimal rotation of $\mathcal{S}$ around $\mathbf{e}$, $\varphi \in \mathcal{W}$ denotes the angle of shear and bending as an infinitesimal rotation of $\mathcal{S}$ around one axis normal to $\mathbf{e}$, it is the unique real eigenvector of the matrix Ψ.

The motion of the beam along the $\mathbf{e}$-axis is given by a sum of two terms, the first one $v(\zeta; t) \in \mathbb{R}$ furnishes the displacement of a point of the beam axis, it does not depend on the shape of the transversal section; the second one represents the warping of the section parallel to the $\mathbf{e}$-axis and is expressed by the product $\omega(\zeta; t)\Phi(\mathbf{x})$:

$$u(\mathbf{x}, \zeta; t) = v(\zeta; t) + \omega(\zeta; t)\Phi(\mathbf{x}) \,.$$

$\Phi(\mathbf{x}) : \mathcal{S} \to \mathbb{R}$ is the warping shape function and $\omega(\zeta; t) : \; [\, 0\, L\,] \times [\, 0, t_t\, [\to \mathbb{R}$ is its amplitude, variable in $\mathbf{e}$-axis and in time. The components of the infinitesimal strain tensor are given by:

$$\begin{aligned}
\widehat{\mathbf{E}}(\mathbf{x}, \zeta; t) \; = \; & \alpha(\zeta; t)\mathbf{a} \otimes \mathbf{a} + \beta(\zeta; t)\mathbf{b} \otimes \mathbf{b} \\
& -[\gamma_{\mathbf{a}}(\zeta; t)(\mathbf{a} \otimes \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{a}) - \delta_{\mathbf{a}}(\zeta; t)(\mathbf{b} \otimes \mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{b})]\mathbf{x} \\
& -[\gamma_{\mathbf{b}}(\zeta; t)(\mathbf{b} \otimes \mathbf{b} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{a}) - \delta_{\mathbf{b}}(\zeta; t)(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{b} - \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a})]\mathbf{x}, \qquad (7) \\
\gamma(\mathbf{x}, \zeta; t) \; = \; & \partial_\zeta \widehat{\mathbf{v}}(\mathbf{x}, \zeta; t) + \varphi(\zeta; t) + [\partial_\zeta \alpha(\zeta; t)\mathbf{a} \otimes \mathbf{a} + \partial_\zeta \beta(\zeta; t)\mathbf{b} \otimes \mathbf{b}]\mathbf{x} \\
& -\frac{1}{2}[\partial_\zeta \gamma_{\mathbf{a}}(\zeta; t)(\mathbf{a} \otimes \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{a}) - \partial_\zeta \delta_{\mathbf{a}}(\zeta; t)(\mathbf{b} \otimes \mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{b})]\mathbf{x} \otimes \mathbf{x} \\
& -\frac{1}{2}[\partial_\zeta \gamma_{\mathbf{b}}(\zeta; t)(\mathbf{b} \otimes \mathbf{b} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{b}) - \partial_\zeta \delta_{\mathbf{b}}(\zeta; t)(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{b} - \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{a})]\mathbf{x} \otimes \mathbf{x} \\
& +\partial_\zeta \vartheta(\zeta; t)\mathbf{e} \times x + \omega(\zeta; t)\nabla_{\mathbf{x}}\Phi(\mathbf{x}). \qquad (8) \\
\varepsilon(\mathbf{x}, \zeta; t) \; = \; & \partial_\zeta v(\zeta; t) + \partial_\zeta \varphi(\zeta; t) \cdot \mathbf{x} + \partial_\zeta \omega(\zeta; t)\Phi(\mathbf{x}) \,. \qquad (9)
\end{aligned}$$

It is important to emphasize that, in order to avoid a reversal of the section orientation after the deformation, it is necessary that the product $\alpha\beta > 0$ is strictly positive; this inequality excludes pathological behaviors for which a contraction in one direction is coupled with an extension in the other one and therefore ensures the regularity of the motion. Conversely no particular conditions have to be satisfied by $\gamma_{\mathbf{a}}$, $\gamma_{\mathbf{b}}$, $\delta_{\mathbf{a}}$ and $\delta_{\mathbf{a}}$.

4 Dynamics and balance equations

Let $\mathbf{S}(\mathbf{x},\zeta;t) \in Sym$ be *the Cauchy stress tensor* defined in each point of the beam; we assume an additive decomposition of the stress tensor likewise for the deformation state:

$$\mathbf{S}(\mathbf{x},\zeta;t) = \mathbf{S}(\mathbf{x},\zeta;t) + sym[\tau(\mathbf{x},\zeta;t) \otimes \mathbf{e}] + \sigma(\mathbf{x},\zeta;t)\mathbf{e} \otimes \mathbf{e}\,. \tag{10}$$

$\mathbf{S}(\mathbf{x},\zeta;t)$ is the *plane stress field*, a second order symmetric tensor with kernel $\mathbf{e}$, $\tau(\mathbf{x},\zeta;t) \in \mathcal{W}$ is the *shear tension vector* and $\sigma(\mathbf{x},\zeta;t) \in \mathbb{R}$ is the *axial tension field.* We suppose that the stress tensor verifies the Cauchy equations of motion, at each point of the beam and in every time instant

$$\begin{cases} div\,\mathbf{S} + \mathbf{q} = \rho\,\partial^2_{tt}\mathbf{u}, & \forall(\mathbf{x};t) \in (\mathcal{R} \setminus \partial\mathcal{R}) \times [\,0, t_f\,[, \qquad (a) \\ \mathbf{S}\mathbf{n} = \mathbf{s}, & \forall(\mathbf{x};t) \in \partial\mathcal{R} \times [\,0, t_f\,[. \qquad (b) \end{cases} \tag{11}$$

ρ is the mass density, $\mathbf{q}$ and $\mathbf{s}$ are respectively the external mass and surface body forces and $\mathbf{n}$ is the unit normal field pointing out the beam. We express the equation of motion in terms of resultants over the cross section rather than in terms of components of stress, as it is usual in beam theories. To make so, we write the equations (11) in the integral form using the Principle of Virtual Power. If $(\mathbf{u}^*, \mathbf{E}^*)$ is an admissible displacement-strain field, the equation (11.a) is equivalent to

$$\int_{\mathcal{R}} \mathbf{S} \cdot \mathbf{E}^* - \int_{\mathcal{R}} \mathbf{q} \cdot \mathbf{u}^* - \int_{\partial\mathcal{R}} \mathbf{s} \cdot \mathbf{u}^* - \int_{\mathcal{R}} \partial^2_{tt}\mathbf{u} \cdot \mathbf{u}^* = 0\,.$$

The use of both decompositions (1) and (10) and of the representation $\mathcal{R} = \mathcal{S} \times [\,0, L\,]$ turns the previous equality in the following equation

$$\int_0^L \left(\int_{\mathcal{S}} \left(\widehat{\mathbf{S}} \cdot \widehat{\mathbf{E}}^* + \tau \cdot \gamma^* + \sigma\varepsilon^* \right) - \int_{\mathcal{S}} \mathbf{q} \cdot \mathbf{u}^* - \int_{\partial\mathcal{S}} \mathbf{s} \cdot \mathbf{u}^* - \int_{\mathcal{S}} \rho\partial^2_{tt}\mathbf{u} \cdot \mathbf{u} \right) = 0\,. \tag{12}$$

It is convenient to decompose body and surface forces as the sum of a part belonging to the section plane $\mathcal{W}$, indicated with a superimposed $(\,\widehat{\cdot}\,)$, and a part orthogonal to it: $\mathbf{q} = \widehat{\mathbf{q}} + \mathbf{e}\,q$ and $\mathbf{s} = \widehat{\mathbf{s}} + \mathbf{e}\,s$. Usual developments applied to equation (12) lead to the following balance equations (see also [7])

$$\begin{cases} \partial_\zeta r + p & = & \rho A \partial^2_{tt}\, v, \\ \partial_\zeta \mathbf{r} + \widehat{\mathbf{p}} & = & \rho A\, \partial^2_{tt}\, \mathbf{v}, \\ \partial_\zeta m + c & = & \rho\, tr\mathbf{J}^*\, \partial^2_{tt}\, \vartheta, \\ \mathbf{e} \times \partial_\zeta \mathbf{m} - \widehat{\mathbf{r}} + \widehat{\mathbf{c}} & = & \rho\, \mathbf{J}\, \partial^2_{tt}\, \varphi, \\ \partial_\zeta m_\Phi - r_\Phi + c_\Phi & = & \rho\, D_2\, \partial^2_{tt}\, \omega. \end{cases} \tag{13}$$

$A > 0$ is the area of the cross section; the positive definite *inertia tensor* $\mathbf{J}$, the *Euler tensor* $\mathbf{J}^*$ and the *Vlasov constant* D_2 (the inertial term related to the section warping) are respectively defined by:

$$\mathbf{J} = \int_{\mathcal{S}} \mathbf{x} \otimes \mathbf{x}, \qquad \mathbf{J}^* = \int_{\mathcal{S}} (\mathbf{e} \times \mathbf{x}) \otimes (\mathbf{e} \times \mathbf{x}), \qquad D_2 = \int_{\mathcal{S}} \Phi^2.$$

The external load terms are defined by

$$\widehat{\mathbf{p}} = \int_{\mathcal{S}} \widehat{\mathbf{q}} + \int_{\partial\mathcal{S}} \widehat{\mathbf{s}}, \qquad p = \int_{\mathcal{S}} q + \int_{\partial\mathcal{S}} s, \qquad c = \mathbf{e} \cdot \left(\int_{\mathcal{S}} \mathbf{x} \times \widehat{\mathbf{q}} + \int_{\partial\mathcal{S}} \mathbf{x} \times \widehat{\mathbf{s}} \right),$$
$$c_{\Phi} = \int_{\mathcal{S}} \Phi\, q + \int_{\partial\mathcal{S}} \Phi\, s, \qquad \widehat{\mathbf{c}} = \int_{\mathcal{S}} \mathbf{x}\, q + \int_{\partial\mathcal{S}} \mathbf{x}\, s.$$

Numerous forces concur, however, to the balance in the plane of the section: the balance of re-sizing forces is expressed by the following equations

$$\begin{cases} \partial_\zeta \mathtt{A} + \mathrm{A} - \mathrm{a} = \rho J \cdot (\mathbf{a} \otimes \mathbf{a})\, \partial^2_{tt} \alpha, \\ \partial_\zeta \mathtt{B} + \mathrm{B} - \mathrm{b} = \rho J \cdot (\mathbf{b} \otimes \mathbf{b})\, \partial^2_{tt} \beta, \end{cases} \tag{14}$$

where

$$\mathtt{A} = \int_{\mathcal{S}} (\tau(\mathbf{x}, \zeta; t) \cdot \mathbf{a})(\mathbf{x} \cdot \mathbf{a}) = \int_{\mathcal{S}} (\tau(\mathbf{x}, \zeta; t) \otimes \mathbf{x})(\mathbf{a} \otimes \mathbf{a}),$$
$$\mathtt{B} = \int_{\mathcal{S}} (\tau(\mathbf{x}, \zeta; t) \cdot \mathbf{b})(\mathbf{x} \cdot \mathbf{b}) = \int_{\mathcal{S}} (\tau(\mathbf{x}, \zeta; t) \otimes \mathbf{x})(\mathbf{b} \otimes \mathbf{b}).$$

$\mathtt{A}$ and $\mathtt{B}$ are the *re-sizing*-**a**-force and the *re-sizing*-**b**-force as the resultant over the area of the "moment" of the **a**-component of the shear vector τ to respect **b** axis and the resultant over the area of the "moment" of the **b**-component of the shear vector τ to respect **a** axis respectively. The *zero*-**a**-force and the *zero*- **b**-force are the projection of the resultant of the plane stress field in **a** and **b** directions and are defined respectively by

$$\mathrm{a} = \int_{\mathcal{S}} \widehat{\mathbf{S}}(\mathbf{x}, \zeta; t) \cdot \mathbf{a} \otimes \mathbf{a}, \qquad \mathrm{b} = \int_{\mathcal{S}} \widehat{\mathbf{S}}(\mathbf{x}, \zeta; t) \cdot \mathbf{b} \otimes \mathbf{b}.$$

The external load terms are

$$\mathrm{A} = \int_{\mathcal{S}} \widehat{\mathbf{q}} \cdot [(\mathbf{a} \otimes \mathbf{a})\mathbf{x}] + \int_{\partial\mathcal{S}} \widehat{\mathbf{s}} \cdot [(\mathbf{a} \otimes \mathbf{a})\mathbf{x}], \qquad \mathrm{B} = \int_{\mathcal{S}} \widehat{\mathbf{q}} \cdot [(\mathbf{b} \otimes \mathbf{b})\mathbf{x}] + \int_{\partial\mathcal{S}} \widehat{\mathbf{s}} \cdot [(\mathbf{b} \otimes \mathbf{b})\mathbf{x}].$$

The balance equations of internal actions distorting the section in its own plane are

$$\begin{cases} \partial_\zeta \mathtt{G}_{\mathbf{a}} + \mathrm{G}_{\mathbf{a}} - \mathrm{g}_{\mathbf{a}} = \rho \left(\mathbb{A}_4 + \mathbb{A}_2 + \frac{1}{2}\mathbb{A}_{3r} + \frac{1}{2}\mathbb{B}_{3l} \right) \cdot \mathbb{J}\, \partial^2_{tt} \gamma_{\mathbf{a}}, \\ \partial_\zeta \mathtt{G}_{\mathbf{b}} + \mathrm{G}_{\mathbf{b}} - \mathrm{g}_{\mathbf{b}} = \rho \left(\mathbb{B}_4 + \mathbb{B}_2 + \frac{1}{2}\mathbb{A}_{3r} + \frac{1}{2}\mathbb{B}_{3l} \right) \cdot \mathbb{J}\, \partial^2_{tt} \gamma_{\mathbf{b}}, \\ \partial_\zeta \mathtt{D}_{\mathbf{a}} + \mathrm{D}_{\mathbf{a}} - \mathrm{d}_{\mathbf{a}} = \rho \left(\mathbb{A}_2 - \mathbb{B}_4 + \frac{1}{2}\mathbb{A}_{3r} - \frac{1}{2}\mathbb{B}_{3l} \right) \cdot \mathbb{J}\, \partial^2_{tt} \delta_{\mathbf{a}}, \\ \partial_\zeta \mathtt{D}_{\mathbf{b}} + \mathrm{D}_{\mathbf{b}} - \mathrm{d}_{\mathbf{b}} = \rho \left(\mathbb{B}_2 - \mathbb{A}_4 + \frac{1}{2}\mathbb{A}_{3r} - \frac{1}{2}\mathbb{B}_{3l} \right) \cdot \mathbb{J}\, \partial^2_{tt} \delta_{\mathbf{b}}, \end{cases} \tag{15}$$

with

$$\mathtt{G}_{\mathbf{a}} = \int_{\mathcal{S}} (\tau(\mathbf{x}, \zeta; t) \otimes \mathbf{x} \otimes \mathbf{x}) \cdot (\mathbf{a} \otimes \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{a}),$$
$$\mathtt{G}_{\mathbf{b}} = \int_{\mathcal{S}} (\tau(\mathbf{x}, \zeta; t) \otimes \mathbf{x} \otimes \mathbf{x}) \cdot (\mathbf{b} \otimes \mathbf{b} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{b}),$$
$$\mathtt{D}_{\mathbf{a}} = \int_{\mathcal{S}} (\tau(\mathbf{x}, \zeta; t) \otimes \mathbf{x} \otimes \mathbf{x}) \cdot (\mathbf{b} \otimes \mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{a}),$$
$$\mathtt{D}_{\mathbf{b}} = \int_{\mathcal{S}} (\tau(\mathbf{x}, \zeta; t) \otimes \mathbf{x} \otimes \mathbf{x}) \cdot (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{b} - \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{b}),$$

being the *section re-shaping forces* and

$$\mathtt{g}_{\mathbf{a}} = \int_{\mathcal{S}} (\widehat{\mathbf{S}}(\mathbf{x}, \zeta; t) \otimes \mathbf{x} \otimes \mathbf{x}) \cdot (\mathbf{a} \otimes \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{a}),$$
$$\mathtt{g}_{\mathbf{b}} = \int_{\mathcal{S}} (\widehat{\mathbf{S}}(\mathbf{x}, \zeta; t) \otimes \mathbf{x} \otimes \mathbf{x}) \cdot (\mathbf{b} \otimes \mathbf{b} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{b}),$$
$$\mathtt{d}_{\mathbf{a}} = \int_{\mathcal{S}} (\widehat{\mathbf{S}}(\mathbf{x}, \zeta; t) \otimes \mathbf{x} \otimes \mathbf{x}) \cdot (\mathbf{b} \otimes \mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b} \otimes \mathbf{a}),$$
$$\mathtt{d}_{\mathbf{b}} = \int_{\mathcal{S}} (\widehat{\mathbf{S}}(\mathbf{x}, \zeta; t) \otimes \mathbf{x} \otimes \mathbf{x}) \cdot (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{b} - \mathbf{a} \otimes \mathbf{a} \otimes \mathbf{b}),$$

the *zero-re-shaping forces*. See the analogy with the balance equations (13). The external load terms are given by

$$\begin{aligned}
\mathrm{G}_{\mathbf{a}} &= \textstyle\int_{\partial\mathbf{S}} \widehat{\mathbf{s}} \cdot [(\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{b}+\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{a})(\mathbf{x}\otimes\mathbf{x})] - \int_{\mathcal{S}} \widehat{\mathbf{q}} \cdot [(\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{b}+\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{a})(\mathbf{x}\otimes\mathbf{x})],\\
\mathrm{G}_{\mathbf{b}} &= \textstyle\int_{\partial\mathbf{S}} \widehat{\mathbf{s}} \cdot [(\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{a}+\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{b})(\mathbf{x}\otimes\mathbf{x})] - \int_{\mathcal{S}} \widehat{\mathbf{q}} \cdot [(\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{a}+\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{b})(\mathbf{x}\otimes\mathbf{x})],\\
\mathrm{D}_{\mathbf{a}} &= \textstyle\int_{\partial\mathbf{S}} \widehat{\mathbf{s}} \cdot [(\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{a}-\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b})(\mathbf{x}\otimes\mathbf{x})] - \int_{\mathcal{S}} \widehat{\mathbf{q}} \cdot [(\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{a}-\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b})(\mathbf{x}\otimes\mathbf{x})],\\
\mathrm{D}_{\mathbf{b}} &= \textstyle\int_{\partial\mathbf{S}} \widehat{\mathbf{s}} \cdot [(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{b}-\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a})(\mathbf{x}\otimes\mathbf{x})] - \int_{\mathcal{S}} \widehat{\mathbf{q}} \cdot [(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{b}-\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a})(\mathbf{x}\otimes\mathbf{x})].
\end{aligned}$$

The fourth order inertial tensor $\mathbb{J}$ is defined as

$$\mathbb{J} = \int_{\mathcal{S}} \mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x},$$

and the fourth order tensors related to description of section motion in its own plane are

$$\mathbb{A}_4 = \mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}, \qquad \mathbb{A}_2 = \mathbf{a}\otimes\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{b}, \qquad \mathbb{A}_{3r} = \mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{b}, \tag{16}$$

and

$$\mathbb{B}_2 = \mathbf{b}\otimes\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{a}, \qquad \mathbb{B}_4 = \mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b}, \qquad \mathbb{B}_{3l} = \mathbf{a}\otimes\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b}.$$

The balance equations (13), (14) and (15) are accompanied by the boundary conditions on $\{0, L\} \times [\,0, t_f\,[$:

$$\begin{gathered}
\mathbf{r}\cdot\mathbf{u} = 0, \qquad \mathbf{m}\cdot(\varphi\times\mathbf{e}+\vartheta\,\mathbf{e}) = 0, \qquad m_\varphi\omega = 0,\\
\mathtt{A}\alpha = 0, \qquad \mathtt{B}\beta = 0, \qquad \mathtt{G}_{\mathbf{a}}\gamma_{\mathbf{a}} = 0, \qquad \mathtt{G}_{\mathbf{b}}\gamma_{\mathbf{b}} = 0, \qquad \mathtt{D}_{\mathbf{a}}\delta_{\mathbf{a}} = 0, \qquad \mathtt{D}_{\mathbf{b}}\delta_{\mathbf{b}} = 0.
\end{gathered} \tag{17}$$

The balance equations (14), (15) and the boundary conditions (17) are characteristic of the proposed model and arise from the kinematic behavior of the transversal section.

4.1 Note on the warping shape function

The warping shape function is usually obtained solving the uniform de Saint-Venant torsion problem for the prismatic body $\mathcal{R} = \mathcal{S}\times\mathcal{I}$. The balance equations and the boundary conditions on the lateral surface of the beam $\partial S \times [\,0, L\,]$ bring to the classical Neumann problem for the function Φ on the section $\mathcal{S}$:

$$\left\{\begin{array}{ll}
\Delta_{\mathbf{x}}\Phi = 0, & \mathbf{x}\in\mathcal{S}\setminus\partial\mathcal{S},\\
(\Delta_{\mathbf{x}}\Phi + \mathbf{e}\times\mathbf{x})\cdot n = 0, & \mathbf{x}\in\partial\mathcal{S},
\end{array}\right. \tag{18}$$

where $\Delta_{\mathbf{x}}$ is the Laplace-Beltrami operator on the section and $\mathbf{n}$ the unit normal to boundary of $\mathcal{S}$ pointing out of it. Standard theorems of calculus assure the existence of the solution Φ of the problem (18) and its uniqueness up to a constant which defines a rigid translation of the section $\mathcal{S}$ along the $\mathbf{e}$-axis; see, for example, Michajlov [14]. We imposed also that the function Φ satisfies the following orthogonality conditions (see [19] and [7]):

$$\int_{\mathcal{S}} \Phi = 0, \qquad \int_{\mathcal{S}} \Phi\,\mathbf{x} = \mathbf{0}, \qquad \int_{\mathcal{S}} \nabla_{\mathbf{x}}\Phi = \mathbf{0}.$$

5 Constitutive relations and equations of motion

The equations (13), (14) and (15) are very general and represent the balance conditions for a beam without specification of its material constituent, they allow the determination of the internal actions if the dynamical state is known. The motion under the external loads is however an unknown of the problem and the equations (13), (14) and (15) are not enough to solve the beam motion problem. Furthermore, no material specifications have been provided so far. The relation between the strain and the stress tensors is the *constitutive relation* of the continuum, it specifies the type of material and its behavior under external loads. We restrict our attention to *linear homogeneous anisotropic materials* for which the stress-strain relation is expressed by the linear form

$$\mathbf{S}(\mathbf{x},\zeta;t) = \mathbb{A}[\mathbf{E}(\mathbf{x},\zeta;t)]\,. \tag{19}$$

$\mathbb{A}$ is the elasticity tensor, a fourth order tensor mapping *Sym* into *Sym*. The assumption of homogeneity implies that each term of $\mathbb{A}$ is constant and it does not depend either on spatial coordinates or on time variable. We suppose, beside, that the tensor $\mathbb{A}$ is positive definite and symmetric, i.e. $\forall\ \mathbf{T}_1, \mathbf{T}_2 \in Sym \setminus \{\mathbf{0}\}:$

$$\mathbf{T}_1 \cdot \mathbb{A}[\mathbf{T}_1] > 0, \tag{20}$$

$$\mathbf{T}_1 \cdot \mathbb{A}[\mathbf{T}_2] = \mathbf{T}_2 \cdot \mathbb{A}[\mathbf{T}_1]. \tag{21}$$

The assumption (20) enables one to eliminate degenerations of material, the assumption (21) and the decompositions (1) and (10) allow to write the constitutive law in terms of components of stress and strain as

$$\begin{aligned}
\widehat{\mathbf{S}}(\mathbf{x},\zeta;t) &= \mathbb{C}[\widehat{\mathbf{E}}(\mathbf{x},\zeta;t)] + \mathbb{L}[\gamma(\mathbf{x},\zeta;t)] + \mathbf{D}\varepsilon(\mathbf{x},\zeta;t)\\
\tau(\mathbf{x},\zeta;t) &= \mathbb{L}^T[\widehat{\mathbf{E}}(\mathbf{x},\zeta;t)] + \mathbf{G}[\gamma(\mathbf{x},\zeta;t)] + \mathbf{h}\varepsilon(\mathbf{x},\zeta;t),\\
\sigma(\mathbf{x},\zeta;t) &= \mathbf{D}\cdot[\widehat{\mathbf{E}}(\mathbf{x},\zeta;t)] + \mathbf{h}[\cdot\gamma(\mathbf{x},\zeta;t)] + E\varepsilon(\mathbf{x},\zeta;t),
\end{aligned} \tag{22}$$

where $\mathbb{C}$ is a fourth order symmetric tensor mapping *Sym* into *Sym*, $\mathbf{D}$ and $\mathbf{G}$ are second order symmetric tensors with kernel $\mathbf{e}$, E is a scalar, $\mathbb{L}$ is a third order tensor mapping $\mathcal{W}$ into *Sym*, and $h \in \mathcal{W}$ is a vector belonging to the plane of the section. The positive definitiveness of the tensor $\mathbb{A}$ implies in turn the positive definitiveness of $\mathbb{C}$, $\mathbf{G}$ and E.
The symmetry of the tensor $\mathbb{A}$ also implies that the linear form (19) is exact and can be derived from a potential, the *stored energy*. If external mass and surface body forces constitute a conservative system, with potential energy $\int_{\mathcal{R}} \mathbf{q}\cdot\mathbf{u} + \int_{\partial\mathcal{S}} \mathbf{s}\cdot\mathbf{u}$, writing the kinetic energy as usual as $\frac{1}{2}\int_{\mathcal{R}} \rho\,\partial_t\mathbf{u}\cdot\partial_t\mathbf{u}$ we get the total energy (*Hamiltonian*) $\mathcal{H}:\ H^{2,2}(\mathcal{S}\times[0,L]\times[0,t_f[\,)\to\mathbb{R}$,

$$\begin{aligned}
\mathcal{H}(\mathbf{u},\partial_\zeta\mathbf{u},\partial_t\mathbf{u},\zeta,t) &= \int_0^{t_f}\left(\frac{1}{2}\int_{\mathcal{R}}\mathbf{S}\cdot\mathbf{E} - \frac{1}{2}\int_{\mathcal{R}}\rho\partial_t\mathbf{u}\cdot\partial_t\mathbf{u} - \int_{\mathcal{R}}\mathbf{q}\cdot\mathbf{u} - \int_{\partial\mathcal{S}}\mathbf{s}\cdot\mathbf{u}\right)\\
&= \frac{1}{2}\int_0^{t_f}\int_0^L\left(\int_{\mathcal{S}}\widehat{\mathbf{E}}\cdot\mathbb{C}[\widehat{\mathbf{E}}] + \gamma\cdot\mathbf{G}\gamma + E\varepsilon^2 + 2\widehat{\mathbf{E}}\cdot\mathbf{D}\varepsilon + 2\gamma\cdot\mathbf{h}\varepsilon\right.\\
&\quad\left. -\frac{1}{2}\int_{\mathcal{R}}\rho\partial_t\mathbf{u}\cdot\partial_t\mathbf{u} - \int_{\mathcal{R}}\mathbf{q}\cdot\mathbf{u} - \int_{\partial\mathcal{S}}\mathbf{s}\cdot\mathbf{u}\right)
\end{aligned} \tag{23}$$

The Hamiltonian (23) is a convex functional because the stored energy is a positive definite quadratic form, as a consequence it owns a local minimum in the Sobolev space $H^{2,2}(\mathcal{S} \times [\,0, L\,] \times [\,0, t_f\,[\,)$, (Lions-Stampacchia theorem, see [13] and Ciarlet [5], p. 290). This minimum furnishes the dynamical balance positions of the beam. Standard techniques of calculus allow to get the motion equations of the beam as the Euler-Lagrange equations for the Hamiltonian (23). These equations are resumed in the linear differential system

$$\mathtt{A}\,\partial^2_{\zeta\zeta}\mathbf{v} + \mathtt{B}\,\partial_{\zeta}\mathbf{v} + \mathtt{C}\,\mathbf{v} = \mathtt{M}\,\partial^2_{tt}\mathbf{v}, \tag{24}$$

where the matrices $\mathtt{A}$, $\mathtt{B}$, $\mathtt{C}$ and $\mathtt{M}$ couple all unknown functions and their spatial and temporal derivatives. We do not report the explicit expressions of each terms of above mentioned matrices to avoid trivial repetitions; we go to particularize the system (24) to monocline and orhotropic-rhombic beams in the next developments. The equations (24) form a system of coupled partial differential equations of the second order, the convexity of the stored energy and the positivity of the kinetic energy assure the total hyperbolicity of the system and the possibility of waves propagation and the establishment of vibrations. It is possible to see that all deformations are coupled by the material anisotropy. The specification of particular anisotropy allows to decouple the problem of the beam dynamic in sub-problems, it will be shown in the next sections.

6 Monocline and orthotropic beams

The material symmetry allows to split the problem of the bean motion under external loads in uncoupled sub-problems. For this to happen it is still necessary that the *local* symmetry exhibited by the material is reflected to the *global* symmetry of the beam: i.e. an axis of the symmetry of the material coincides with the axis of the beam. The properties of the material crystalline lattice put off the macroscopic behavior of the element via the terms of elasticity tensor $\mathbb{A}$ and its global symmetries. To do this it is necessary to specify the symmetry group of isometries $\mathcal{G}$ that leave unaltered the crystalline lattice. For a detailed exposition on group theory and its use see e. g. Golubisky and others [10] and Forte and Vianello, [9] in special reference to elasticity.

6.1 Monocline beams

A material is called *monocline* with respect to an axis of unit vector $\mathbf{z}$ if its symmetry group $\mathcal{G}$ contains all rotations of amplitude π around $\mathbf{z}$ and a central reflection, these isometries are respectively represented by the second order tensors $-\mathbf{I}$ and $2\mathbf{z}\otimes\mathbf{z}-\mathbf{I}$ ($\mathbf{I}$ is the identity tensor). The representation of the symmetry group of monocline materials is given by:

$$\mathcal{G}_m = \langle\, 2\mathbf{z}\otimes\mathbf{z} - \mathbf{I}, -\mathbf{I}\,\rangle.$$

The axis of anisotropy coincides with the axis of the beam in a monocline beam, i.e. $\mathbf{z}$=$\mathbf{e}$. Standard computations of group theory get $\mathbb{L} = \mathbf{0}$ and $\mathbf{h} = \mathbf{0}$. The representations of $\mathbb{C}$, $\mathbf{D}$, $\mathbf{G}$ are more complex; without going into details (for which reference should be made to Xiao [22]), $\mathbb{C}$ is expressed by a linear combination of six fourth order tensors, as in what follows. Let $\mathbf{e}$, $\mathbf{a}$

and $\mathbf{b}$ be three unitary and mutually orthogonal vectors, we define the second order tensors Ω_i by

$$\Omega_i = c_{i1}\mathbf{a}\otimes\mathbf{a} + c_{i2}\mathbf{b}\otimes\mathbf{b} + c_{i3}\mathbf{e}\otimes\mathbf{e} + c_{i4}(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a}), \qquad i = 1,2,3,4$$

$$\Omega_5 = \frac{1}{\sqrt{2}}(\mathbf{e}\otimes\mathbf{a}+\mathbf{a}\otimes\mathbf{e}), \qquad \Omega_6 = \frac{1}{\sqrt{2}}(\mathbf{e}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{e}), \tag{25}$$

with c_{ij} being dimensionless constants such that $\sum_k^4 c_{ik}c_{jk} + c_{i4}c_{j4} = \delta_{ij}$; we have

$$\mathbb{C} = \sum_{k=1}^{6} C_k \Omega_k \otimes \Omega_k, \tag{26}$$

with C_k being the elastic constants. The tensors $\mathbf{D}$ and $\mathbf{G}$ are represented by a linear combination of second order tensors

$$\begin{aligned}\mathbf{D} &= D_a\mathbf{a}\otimes\mathbf{a} + D_b\mathbf{b}\otimes\mathbf{b} + D_{ab}(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a}),\\ \mathbf{G} &= G_a\mathbf{a}\otimes\mathbf{a} + G_b\mathbf{b}\otimes\mathbf{b} + G_{ab}(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a}).\end{aligned} \tag{27}$$

The coefficients of the linear combinations are the elastic constants. We obtain the following constitutive relations for monocline beams putting conditions (27) and (26) into equations (13), (14) and (15):

$$r = EA\partial_\zeta v + AD_a\alpha + AD_b\beta, \tag{28}$$

$$\begin{aligned}\widehat{\mathbf{r}} &= A[\,G_a\mathbf{a}\otimes\mathbf{a} + G_b\mathbf{b}\otimes\mathbf{b} + G_{ab}(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a})](\varphi + \partial_\zeta\widehat{\mathbf{v}})\\ &\quad -[G_a\mathbb{T}_1 + G_{ab}\mathbb{T}_2][\mathbf{J}]\partial_\zeta\gamma_{\mathbf{a}} - [G_{\mathbf{b}}\mathbb{T}_3 + G_{\mathbf{ab}}\mathbb{T}_4][J]\partial_\zeta\gamma_{\mathbf{b}}\\ &\quad +[G_b\mathbb{T}_3 + G_{ab}\mathbb{T}_4][\mathbf{J}]\partial_\zeta\delta_{\mathbf{a}} + [G_{\mathbf{a}}\mathbb{T}_4 + G_{\mathbf{ab}}\mathbb{T}_3][J]\partial_\zeta\delta\mathbf{b},\end{aligned} \tag{29}$$

$$\begin{aligned}m &= \mathbf{J}^*\cdot[G_{\mathbf{a}}\mathbf{a}\otimes\mathbf{a} + G_{\mathbf{b}}\mathbf{b}\otimes\mathbf{b} + G_{\mathbf{ab}}(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a})]\partial_\zeta\vartheta\\ &\quad +[\,G_{\mathbf{a}}\mathbf{a}\otimes\mathbf{a} + G_{\mathbf{b}}\mathbf{b}\otimes\mathbf{b} + G_{\mathbf{ab}}(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a})]\cdot\mathbf{L}_1\omega\\ &\quad +G_{\mathbf{a}}\mathbf{a}\cdot[\mathbf{e}\times\mathbf{Ja}]\partial_\zeta\alpha + G_{\mathbf{b}}\mathbf{b}\cdot[\mathbf{e}\times\mathbf{J}_{\mathbf{b}}]\partial_\zeta\beta,\end{aligned} \tag{30}$$

$$\begin{aligned}\mathbf{e}\times\widehat{\mathbf{m}} &= E\,\mathbf{J}\,\partial_\zeta\varphi - D_{\mathbf{ab}}[(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a})\otimes\mathbf{a}]\mathbf{J}\gamma_{\mathbf{a}} - D_{\mathbf{ab}}[(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a})\otimes\mathbf{b}]\mathbf{J}\gamma_{\mathbf{b}}\\ &\quad +[(D_{\mathbf{b}}\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b} + D_{\mathbf{a}}\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{b})]\mathbf{J}\delta_{\mathbf{a}} + [(D_{\mathbf{a}}\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a} + D_{\mathbf{b}}\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{a})]\mathbf{J}\delta_{\mathbf{b}},\end{aligned} \tag{31}$$

$$m_\Phi = ED_2\partial_\zeta\omega, \tag{32}$$

$$r_\Phi = [\,G_{\mathbf{a}}\mathbf{a}\otimes\mathbf{a} + G_{\mathbf{b}}\mathbf{b}\otimes\mathbf{b} + G_{\mathbf{ab}}(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a})\,]\cdot[\mathbf{L}_1\partial_\zeta\vartheta + \mathbf{L}_2\partial_\zeta\omega\,], \tag{33}$$

$$\mathtt{A} = (G_{\mathbf{a}} + G_{\mathbf{ab}})\mathbf{a}\cdot(\mathbf{e}\times\mathbf{Ja})\partial_\zeta\vartheta + G_{\mathbf{a}}\mathbf{J}\cdot(\mathbf{a}\otimes\mathbf{a})\partial_\zeta\alpha + G_{\mathbf{ab}}\mathbf{J}\cdot(\mathbf{a}\otimes\mathbf{b})\partial_\zeta\beta, \tag{34}$$

$$a = AD_{\mathbf{a}}\partial_\zeta v + A\mathbb{C}\cdot\mathbb{A}_4\alpha + A\mathbb{C}\cdot(\mathbb{A}_2 + \mathbb{B}_2)\beta, \tag{35}$$

$$\mathtt{B} = (G_{\mathbf{b}} + G_{\mathbf{ab}})\mathbf{b}\cdot(\mathbf{e}\times\mathbf{Jb})\partial_\zeta\vartheta + G_{\mathbf{b}}a\mathbf{J}\cdot(\mathbf{b}\otimes\mathbf{a})\partial_\zeta\alpha + G_{\mathbf{b}}\mathbf{J}\cdot(\mathbf{b}\otimes\mathbf{b})\partial_\zeta\beta, \tag{36}$$

$$b = AD_{\mathbf{b}}\partial_\zeta v + A\mathbb{C}\cdot(\mathbb{A}_2 + \mathbb{B}_2)\alpha + A\mathbb{C}\cdot\mathbb{B}_4\beta, \tag{37}$$

$$\begin{aligned}\mathtt{G}_{\mathbf{a}} &= -[G_{\mathbf{a}}\mathbb{T}_1 + G_{\mathbf{ab}}\mathbb{T}_2][\mathbf{J}]\partial_\zeta\widehat{\mathbf{v}} + [G_{\mathbf{b}}\mathbb{A}_4 + G_{\mathbf{ab}}(\mathbb{A}_{3l} + \mathbb{A}_{3r}) + G_{\mathbf{a}}\mathbb{A}_2]\cdot\mathbb{J}\partial_\zeta\gamma_{\mathbf{a}}\\ &\quad +[\,G_{\mathbf{ab}}(\mathbb{A}_{3r} + \mathbb{A}_2) + G_{\mathbf{a}}(\mathbb{A}_{3r} + \mathbb{B}_{3r})]\cdot\mathbb{J}\partial_\zeta\gamma_{\mathbf{b}}\\ &\quad -[G_{\mathbf{ab}}(\mathbb{A}_{3r} - \mathbb{B}_{3r}) + (G_{\mathbf{a}} - G_{\mathbf{b}})\mathbb{A}_2\,]\cdot\mathbb{J}\partial_\zeta\delta_{\mathbf{a}}\\ &\quad -[(G_{\mathbf{b}} - G_{\mathbf{a}})\mathbb{A}_{3r} + G_{\mathbf{ab}}(\mathbb{A}_2 - \mathbb{A}_4)\,]\cdot\mathbb{J}\partial_\zeta\delta_{\mathbf{b}} - D_{\mathbf{ab}}[(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a})\otimes\mathbf{a}]\mathbf{J}\varphi,\end{aligned} \tag{38}$$

$$
\begin{aligned}
\mathtt{G}_{\mathbf{b}} &= -[G_{\mathbf{b}}\mathbb{T}_2 + G_{\mathbf{ab}}\mathbb{T}_1][\mathbf{J}]\partial_\zeta\widehat{\mathbf{v}} + [\,G_{\mathbf{ab}}(\mathbb{A}_{3r}+\mathbb{A}_2) + G_{\mathbf{a}}(\mathbb{A}_{3r}+\mathbb{B}_{3r})\,]\cdot\mathbb{J}\partial_\zeta\gamma_{\mathbf{a}} \\
&\quad +[\,G_{\mathbf{a}}\mathbb{B}_4 + G_{\mathbf{ab}}(\mathbb{B}_{3l}+\mathbb{A}_{3l}) + G_{\mathbf{b}}\mathbb{A}_2\,]\cdot\mathbb{J}\partial_\zeta\gamma_{\mathbf{b}} \qquad (39)\\
&\quad -[\,G_{\mathbf{a}}(\mathbb{A}_{3r}+\mathbb{B}_{3r}) - G_{\mathbf{ab}}(\mathbb{B}_4-\mathbb{A}_2)\,]\cdot\mathbb{J}\partial_\zeta\delta_{\mathbf{a}} \\
&\quad -[G_{\mathbf{ab}}(\mathbb{B}_{3l}-\mathbb{A}_{3r}) + (G_{\mathbf{b}}-G_{\mathbf{a}})\mathbb{A}_2\,]\cdot\mathbb{J}\partial_\zeta\delta_{\mathbf{b}} - D_{\mathbf{ab}}[(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a})\otimes\mathbf{b}]\mathbf{J}\varphi, \\
g_{\mathbf{a}} &= 2\mathbb{C}[(\mathbb{A}_2+\mathbb{B}_2)(\mathbf{a}\otimes\mathbf{a})\cdot\mathbf{J}\gamma_{\mathbf{a}} + 2(\mathbb{A}_2+\mathbb{B}_2)(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{J}\gamma_{\mathbf{b}}] \\
&\quad +\frac{1}{2}\mathbb{C}\cdot(\mathbb{A}_{3r}-\mathbb{B}_{2r}+\mathbb{A}_2-\mathbb{B}_{3r})(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{J}\delta_{\mathbf{a}} - \frac{1}{2}\mathbb{C}\cdot(\mathbb{A}_{3r}-\mathbb{B}_2+\mathbb{A}_2-\mathbb{B}_{3r})(\mathbf{a}\otimes\mathbf{a})\cdot\mathbf{J}\delta_{\mathbf{b}} \\
&\quad +\frac{1}{2}[D_{\mathbf{ab}}(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a})]\mathbf{J}\mathbf{a}\cdot\partial_\zeta\varphi, \qquad (40)\\
g_{\mathbf{b}} &= 2\mathbb{C}[(\mathbb{A}_2+\mathbb{B}_2)(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{J}\gamma_{\mathbf{a}} + 2(\mathbb{A}_2+\mathbb{B}_2)(\mathbf{a}\otimes\mathbf{a})\cdot\mathbf{J}\gamma_{\mathbf{b}}] \\
&\quad +\frac{1}{2}\mathbb{C}\cdot(\mathbb{A}_{3r}-\mathbb{B}_{2r}+\mathbb{A}_2-\mathbb{B}_{3r})(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{J}\delta_{\mathbf{a}} \\
&\quad -\frac{1}{2}\mathbb{C}\cdot(\mathbb{A}_{3r}-\mathbb{B}_2+\mathbb{A}_2-\mathbb{B}_{3r})(\mathbf{a}\otimes\mathbf{a})\cdot\mathbf{J}\delta_{\mathbf{b}} \qquad (41)\\
&\quad +\frac{1}{2}[D_{\mathbf{ab}}(\mathbf{b}\otimes\mathbf{a}+\mathbf{a}\otimes\mathbf{b})]\mathbf{J}\mathbf{b}\cdot\partial_\zeta\varphi, \\
4\,\mathtt{D}_{\mathbf{a}} &= 2\,[\,G_{\mathbf{b}}(\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{a}-\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b}) + G_{\mathbf{ab}}(\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}-\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{b})\,][\,\mathbf{J}\,]\partial_\zeta\widehat{\mathbf{v}} \qquad (42)\\
&\quad -2\,[\,G_{\mathbf{ab}}(\mathbb{A}_{3r}-\mathbb{B}_{3r}) + (G_{\mathbf{a}}-G_{\mathbf{b}})\mathbb{A}_2\,]\cdot[\,\mathbb{J}\,]\partial_\zeta\gamma_{\mathbf{a}} \\
&\quad -2[\,(G_{\mathbf{a}}-G_{\mathbf{b}})\mathbb{B}_{3l} - G_{\mathbf{ab}}(\mathbb{B}_4-\mathbb{A}_2)\,]\cdot[\,\mathbb{J}\,]\partial_\zeta\gamma_{\mathbf{b}} \\
&\quad +2[\,G_{\mathbf{a}}\mathbb{A}_2 - 2G_{\mathbf{ab}}\mathbb{B}_{3l} + G_{\mathbf{b}}\mathbb{B}4\,]\cdot[\,\mathbb{J}\,]\partial_\zeta\delta_{\mathbf{a}} + 2[\,2G_{\mathbf{ab}}\mathbb{A}_2 - G_{\mathbf{a}}\mathbb{A}_{3r} - G_{\mathbf{b}}\mathbb{B}_{3l}\,]\cdot[\,\mathbb{J}\,]\partial_\zeta\delta_{\mathbf{a}} \\
&\quad +[\,(G_{\mathbf{a}}+G_{\mathbf{ab}})\mathbf{J}\cdot(\mathbf{a}\otimes\mathbf{a}) - (G_{\mathbf{b}}+G_{\mathbf{ab}})\mathbf{J}\cdot(\mathbf{b}\otimes\mathbf{b})\,]\cdot\varphi, \\
4\,\mathtt{D}_{\mathbf{b}} &= 2\,[\,G_{\mathbf{a}}(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{b}-\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}) + G_{\mathbf{ab}}(\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b}-\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{a})\,][\,\mathbf{J}\,]\partial_\zeta\widehat{\mathbf{v}} \qquad (43)\\
&\quad -2[(\,G_{\mathbf{b}}-G_{\mathbf{a}})\mathbb{A}_{3r} + G_{\mathbf{ab}}(\mathbb{A}_2-\mathbb{A}_4)\,]\cdot\,[\,\mathbb{J}\,]\partial_\zeta\gamma_{\mathbf{a}} \\
&\quad -2[\,G_{\mathbf{ab}}(\mathbb{B}_{3l}-\mathbb{A}_{3r}) - (G_{\mathbf{a}}-G_{\mathbf{b}})\mathbb{A}_2]\cdot[\,\mathbb{J}\,]\partial_\zeta\gamma_{\mathbf{b}} \\
&\quad +2[\,2G_{\mathbf{ab}}\mathbb{A}_2 - G_{\mathbf{a}}\mathbb{A}_{3r} - G_{\mathbf{b}}\mathbb{B}_{3l}\,]\cdot[\,\mathbb{J}\,]\partial_\zeta\delta_{\mathbf{a}} + 2[\,G_{\mathbf{b}}\mathbb{A}_2 + G_{\mathbf{a}}\mathbb{A}_4 - 2G_{\mathbf{ab}}\mathbb{A}_{3r}\,]\cdot[\,\mathbb{J}\,]\partial_\zeta\delta_{\mathbf{b}} \\
&\quad -[\,(G_{\mathbf{b}}+G_{\mathbf{ab}})\mathbf{J}\cdot\mathbf{b}\otimes\mathbf{b} - (G_{\mathbf{a}}+G_{\mathbf{ab}})\mathbf{J}\cdot(\mathbf{a}\otimes\mathbf{a})]\cdot\varphi, \\
\mathtt{d}_{\mathbf{a}} &= \frac{1}{2}\mathbb{C}\cdot(\mathbb{A}_{3r}-\mathbb{B}_2+\mathbb{A}_2-\mathbb{B}_{3r})[\,(\mathbf{a}\otimes\mathbf{a})\cdot\mathbf{J}\gamma_{\mathbf{a}} + (\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{J}\gamma_{\mathbf{b}}\,] \qquad (44)\\
&\quad -\mathbb{C}\cdot(\mathbb{A}_4-\mathbb{B}_2-\mathbb{A}_2+\mathbb{B}_4)[\,(\mathbf{b}\otimes\mathbf{b})\cdot\mathbf{J}\delta_{\mathbf{a}} - (\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{J}\delta_{\mathbf{b}}\,] \\
&\quad +\frac{1}{2}[(D_{\mathbf{b}}\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b} + D_{\mathbf{a}}\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{b})]\mathbf{J}\cdot\partial_\zeta\varphi, \\
\mathtt{d}_{\mathbf{3}} &= \frac{1}{2}\mathbb{C}\cdot(\mathbb{B}_{3r}-\mathbb{A}_2+\mathbb{B}_2-\mathbb{A}_{3r})[\,(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{J}\gamma_{\mathbf{a}} + (\mathbf{b}\otimes\mathbf{b})\cdot\mathbf{J}\gamma_{\mathbf{b}}\,] \qquad (45)\\
&\quad -\mathbb{C}\cdot(\mathbb{B}_4-\mathbb{A}_2-\mathbb{B}_2+\mathbb{A}_4)[\,(\mathbf{a}\otimes\mathbf{a})\cdot\mathbf{J}\delta_{\mathbf{a}} - (\mathbf{b}\otimes\mathbf{a})\cdot\mathbf{J}\delta_{\mathbf{b}}\,] \\
&\quad +\frac{1}{2}[(D_{\mathbf{a}}\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{b} + D_{\mathbf{b}}\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{a})]\mathbf{J}\cdot\partial_\zeta\varphi.
\end{aligned}
$$

For sake of simplicity, we did mean with $\mathbb{C}$ the tensor expressed by relation (26) conditionally to (25) and (26). The symbology is consistent with the papers of [17] and [7]. The fourth order tensors related to description of the motion are defined above (see (16)) while the tensors indicated with $\mathbb{B}$ can be easily obtained swapping $\mathbf{b}$ with $\mathbf{a}$ in the definition of $\mathbb{A}$. The third order tensor are

$$\begin{array}{ll} \mathbb{T}_1 = \frac{1}{2}(\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{b}+\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{a}), & \mathbb{T}_2 = \frac{1}{2}(\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{a}+\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{b}), \\ \mathbb{T}_3 = \frac{1}{2}(\mathbf{b}\otimes\mathbf{a}\otimes\mathbf{a}+\mathbf{b}\otimes\mathbf{b}\otimes\mathbf{b}), & \mathbb{T}_4 = \frac{1}{2}(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{b}+\mathbf{a}\otimes\mathbf{a}\otimes\mathbf{a}). \end{array} \tag{46}$$

We obtain the balance equations, expressed in term of kinematic descriptors of the beam, substituting the previous relations (46) in the expressions (13), (14) and (15). It is possible to note that the equations (28)-(45) exhibit the splitting of the general problem in two uncoupled sub-problems, as previously announced. The axial extension–re-sizing–torsion–warping is the first sub-problem, the second one is the connexion between bending-shear and section's re-shaping. The re-sizing of transversal section depends both on the axial extension and the twist while its re-shaping just hinges on the bending. The normal force depends on the re-sizing of the transversal section as well as on the expected axial deformation. The re-sizing forces $\mathtt{A}$ and $\mathtt{B}$ are functions of the torsional deformation $\partial_\zeta\vartheta$ as well as of the components of the plane deformation tensor, the zero **a** and **b** forces depend on the axial deformation. The torque m depends on the components of the plane deformation tensor. The bi-shear and the bi-moment are functions of the warping shape function, as usual. The structure of the bending moment, of the shear and of the section re-shaping forces does explicit the interaction between bending and re-shaping of transversal section. We obtain the equations of Davì's cited paper neglecting the parameters of re-sizing and re-shaping of section from equations (28)-(45).

6.2 Orthotropic-rhombic beams

A material is said *orthotropic-rhombic* if its symmetry group contains the reflections on three mutually orthogonal directions; in analogy with monocline beams, we define an orthotropic rhombic beam if its axis coincides with one axis of symmetry of the material. Assuming that **e**, **a** and **b** are the unit vectors defining the axes of anisotropy, the symmetry group of a orthotropic material is

$$\mathcal{G}_h = \langle\, 2\mathbf{e}\otimes\mathbf{e}-\mathbf{I}, 2\mathbf{a}\otimes\mathbf{a}-\mathbf{I}, 2\mathbf{b}\otimes\mathbf{b}-\mathbf{I}, -\mathbf{I}\,\rangle\,. \tag{47}$$

The tensors **D** and **G** assume the following representation

$$\mathbf{D} = D_{\mathbf{a}}\mathbf{a}\otimes\mathbf{a}+D_{\mathbf{b}}\mathbf{b}\otimes\mathbf{b}, \qquad \mathbf{G} = G_{\mathbf{a}}\mathbf{a}\otimes\mathbf{a}+G_{\mathbf{b}}\mathbf{b}\otimes\mathbf{b}. \tag{48}$$

with $D_{\mathbf{a}}$, $D_{\mathbf{b}}$, $G_{\mathbf{a}}$ and $G_{\mathbf{b}}$ being the elastic constants. The tensor $\mathbb{C}$ is still represented by the linear combination

$$\mathbb{C} = \sum_{k=1}^{6} C_k\,\Omega_h\otimes\Omega_k, \tag{49}$$

but the tensors Ω_k are given here by the following expressions

$$\begin{aligned} \Omega_k &= c_{k1}\mathbf{a}\otimes\mathbf{a}+c_{k2}\mathbf{b}\otimes\mathbf{b}+c_{k3}\mathbf{e}\otimes\mathbf{e}, \qquad k=1,2,3, \\ \Omega_4 &= \frac{1}{\sqrt{2}}(\mathbf{e}\otimes\mathbf{a}+\mathbf{a}\otimes\mathbf{e}), \qquad \Omega_5 = \frac{1}{\sqrt{2}}(\mathbf{e}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{e}), \qquad \Omega_6 = \frac{1}{\sqrt{2}}(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a}), \end{aligned} \tag{50}$$

where c_{ij} are dimensionless constants such that $\sum_k c_{ik}c_{kj} = \delta_{ij}$.

The equations of motion of a orthotropic-rhombic beams are very similar to that ones of a monocline beams; they can easily be obtained substituting the expressions of material elastic constants (48)-(50) in equations (28)-(45). The terms coupling extension and torsion warping vanish if the cross section is endowed with two axes of symmetry and the axes of anisotropy are coincident with the inertia principal axes. As a consequence $\mathbf{a}\cdot\mathbf{G}[\mathbf{e}\times\mathbf{J}\mathbf{a}] = \mathbf{b}\cdot\mathbf{G}[\mathbf{e}\times\mathbf{J}\mathbf{b}] = 0$: the torsion does not influence the re-sizing and re-shaping of transversal section; this is the reason why the proposed model is unable to describe the connection between torsion and oval-re-shaping of the section.

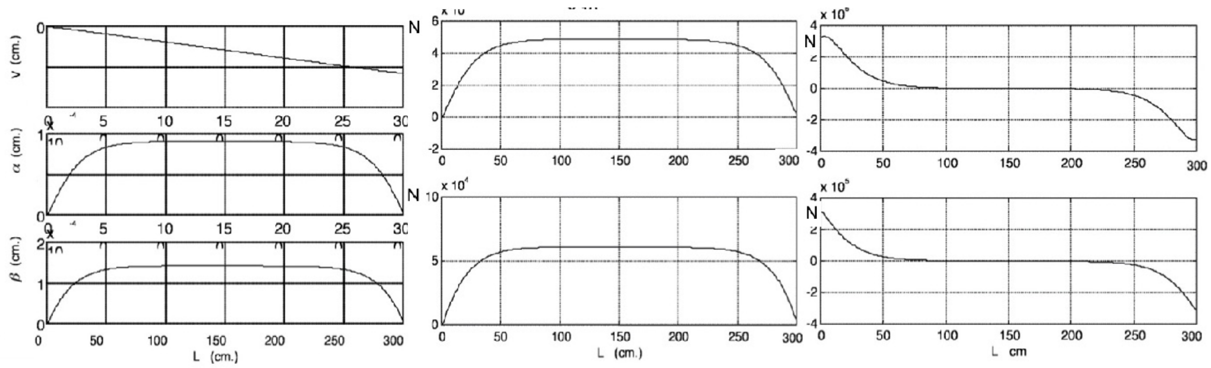

Figure 2: Displacements and internal forces in the first boundary condition.

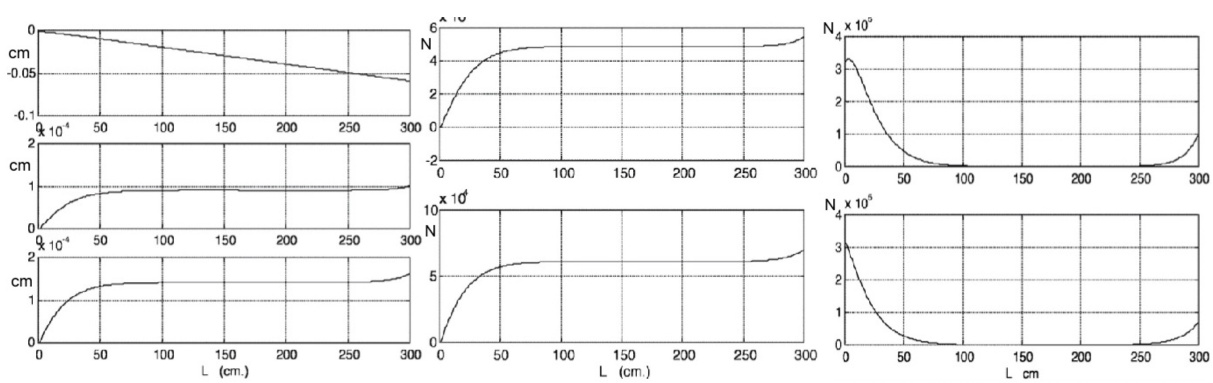

Figure 3: Displacements and internal forces in the second boundary condition.

7 Compression of prismatic beam with rectangular cross section

Some numerical investigations concerning the phenomenon of coupling between the axial displacement of the beam axis and the variation of size of transversal sections are presented here. We treat a material for which the dimensionless constants appearing in (50) are given by $c_{ij} = \frac{\delta_{ij}}{\sqrt{3}}$. We consider a short beam of length $L = 3.00\,m$ with rectangular cross section of dimensions $B = 0.45\,m$ and $H = 0.60\,m$. We also suppose that the axes of anisotropy overlap with the main ones of inertia. The second order inertia tensor has the representation $\mathbf{J} = \frac{1}{12}BH^3\mathbf{a}\otimes\mathbf{a} + \frac{1}{12}B^3H\mathbf{b}\otimes\mathbf{b}$. We reck the beam is in a vertical position and subject to its own weight and to an axial load of compression of $1.5 \cdot 10^5 N$ from the terminal to the initial base. The mechanical characteristics of the constituent material are reported in Table 1 while boundary conditions are collected in Table 2. The beam is constrained in such a way as to prevent rigid motions. The axial motion is hindered in the initial base only, we investigate the behavior of transversal sections in two representative situations for the bases.

Geometric characteristics		
B=0.45 m	H=0.60 m	L=3.00 m
Mechanical characteristics		
$\varrho = 2000\,kg/m^3$	$E = 4.00 \times 10^{10} Pa$	
$C_1 = 2.00 \times 10^{10} Pa$	$C_2 = 1.60 \times 10^{10} Pa$	
$D_{\mathbf{a}} = 1.00 \times 10^{10} Pa$	$D_{\mathbf{b}} = 9.15 \times 10^{10} Pa$	
$G_{\mathbf{a}} = 1.33 \times 10^{10} Pa$	$D_{\mathbf{b}} = 1.07 \times 10^{10} Pa$	
Normal force p=150 kN		

Table 1: Geometric and mechanical characteristics of the beam

First boundary conditions	
$\alpha(0) = 0$	$\alpha(L) = 0$
$\beta(0) = 0$	$\beta(L) = 0$
$v(L) = 0$	$r(L) = p$
Second boundary conditions	
$\alpha(0) = 0$	$\mathtt{A}(L) = 0$
$\beta(0) = 0$	$\mathtt{B}(L) = 0$
$v(L) = 0$	$r(L) = p$

Table 2: Boundary conditions for the proposed problems

First condition of compression: both bases are clamped to impede the motion in the plane of the section. It is possible to see that all the shaft of the beam is subject to a sectional deformation (Figure 2, left side); this deformation increases in the first third of the length, is sensibly constant in the second third and decreases in the last third. The same behavior appears for both measures of the section deformation, their difference is due to material anisotropy. The state of the zero-forces (Figure 2, central side) is analogous. The diagram of re-sizing-forces is reported in the same figure, (Figure 2, right side); it is important to note that the last third has an inversion of sign of both re-sizing forces.

Second condition of compression: the motion in the plane of the cross section is hindered in the initial base as in the former case but it is allowed in the terminal base. Scanning the parameters of extension of transversal section we note that a further extension of the sections is present in a neighborhood of terminal base (Figure 3, left side); a similar behavior is exhibited by the zero-forces (Figure 3, central side); conversely the re-sizing-forces decay to zero in the same neighborhood of the terminal base (Figure 3, right side).

Traction A simple case of traction is scanned reporting only the results for the axial displacement and for the parameters of extension of transversal sections which is subject to a contraction in this case. As it is possible to see in Figure 4, left side, this event is symmetrical to the first one. We note, beside, the analogy with the shape of a thin plate subject to an axial extension and a subsequent sectional contraction (Figure 4, right side[1]).

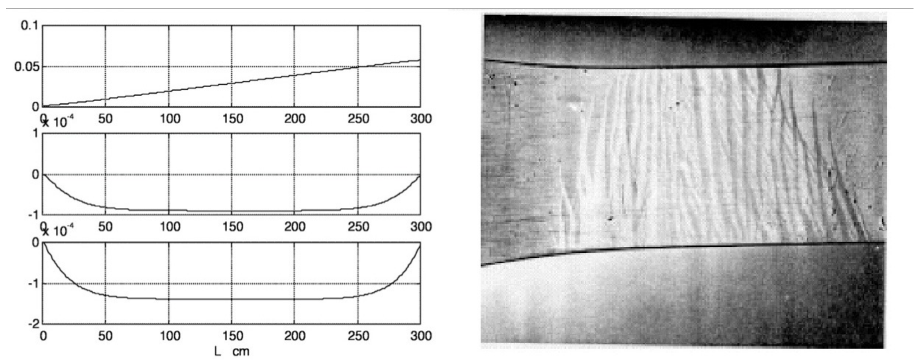

Figure 4: The problem of traction for the same beam, in both bases, the sectional motion is hindered.

8 Conclusions

The proposed model can easily represent the behavior of a beam subject to a wide range of deformations. The introduction of deformability of the section in its own plane via the two functions α and β allows the description of the coupling between the extension of the axis and the re-sizing of the section, consistent with the experimental study. The use of the functions γ and δ, associated with a quadratic form in the coordinates of the plane section, allows to depict the re-shaping of the section, consequent to the bending of the axis. The coefficients of the motion equations are the geometric and inertial characteristics of the section, referred to a frame with origin in the section's center of mass and main axes of inertia coincident with that ones of anisotropy of the material.

9 Some suggestions for further investigations

A first improvement of this model can be performed considering beams with cross sections without two axes of symmetry, the center of rotation (*twist* or *shear center*) does not coincide

[1] http://www.ems.psu.edu/?ryba/310/proposal%20example.html

with the center of mass in this case, moreover several properties of the warping shape function vanish in a frame with an origin other than the center of mass. The structure of equations will be much more complex because various kinds of inertial tensors are present on them. It is also necessary to extend the theory to beams with curvilinear axis, widening in this manner the theory to ring-rods and other curvilinear mono-dimensional structures. The linear theory is only feasible in the ambit of small motions and deformations and does not represent the widest cases of large deformations which occur in very soft materials. It is necessary to remember that the beam theory is now used not only in the conventional ambits of civil and mechanical engineering but also in other frames where the assumptions of linearity may be too strong, it is sufficient to remind the soft pipes in plastic materials and the studies in bio-mechanical engineering. A more refined kinematic analysis will also allow the description of the oval-re-shaping of the transversal section for beams with two axes of symmetry and subject to a torque only.

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