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Mixed fractional differential equation with nonlocal conditions in Banach spaces

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Abstract. This paper is devoted to study the existence of solution for a class of nonlinear differential equations with nonlocal boundary conditions involving the right Caputo and left Riemann–Liouville fractional derivatives. Our approach is based on Darbo's fixed point theorem associated with the Hausdorff measure of noncompactness. The obtained results generalize and extend some of the results found in the literature. Besides, the reported results concerned in the Banach space's sense. In the end, an example illustrates our acquired results.

Keywords: Right Caputo and left Riemann–Liouville fractional derivatives, nonlocal boundary conditions, existence, Banach spaces, Darbo's fixed point theorem, Hausdorff measure of noncompactness.
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1 Introduction

Fractional calculus is an important area of mathematics due to its well-founded theoretical basis, as well as its many applications, see for instance [26,30–34,36]. In consequence, there are many published papers that are devoted to the existence of solutions of nonlinear fractional boundary value problems. For details and examples, see [3,13,21] and their references. On the other hand, the nonlocal boundary value problems have recently received considerable attention as nonlocal condition is more appropriate than the local condition (initial and/or boundary) to describe correctly some physics phenomenons (see for instance [11,12,16,17]). More recently, fractional differential equations involving right Caputo and left Riemann–Liouville fractional derivatives are attracting much attention as an interesting field in fractional differential equations theory and many results are obtained concerning the existence of solutions by the help of different methods to see more applications about the usefulness of this new kind of problems, the reader can be referred to [6–8,15,22–24,27–29,37] and references cited therein. From these points of view, it is imperative to study differential equation with left and right fractional derivatives. Moreover,

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it has been noticed that most of the above-mentioned work on the topic is based on techniques of nonlinear analysis such as Banach fixed point theorem, Schauder's fixed point theorem, and Leray-Schauder nonlinear alternative, etc. But if compactness and Lipschitz condition are not satisfied these results cannot be used. The measure of noncompactness comes handy in such situations. For instance, the celebrated Darbo fixed point theorem and Mönch fixed point theorem are used by several authors with the end goal to establish existence results for nonlinear integral equations (see [1,2,4,5,14,20] and references therein).

In [23] Guezane–Lakoud et al. investigated the existence of solutions for boundary value problems involving both left Riemann–Liouville and right Caputo-type fractional derivatives of the form

$$\begin{cases} {}^{\mathrm{C}}\mathcal{D}^{\alpha}_{1^-}\,{}^{\mathrm{RL}}\mathcal{D}^{\beta}_{0^+}u(t) = f(t,u(t)), \quad t \in J := [0,1], \\ u(0) = u'(0) = u(1) = 0, \end{cases}$$

where ${}^{\mathrm{C}}\mathcal{D}^{\beta}_{1^-}$ and ${}^{\mathrm{RL}}\mathcal{D}^{\alpha}_{0^+}$ denote the right Caputo fractional derivative of order $\alpha \in (0,1)$ and the left Riemann–Liouville fractional derivative of order $\beta \in (1,2)$ respectively, and $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function. By using Krasnoselskii's fixed point theorem, the authors obtained the existence of solutions.

Very recently, in [6], the authors considered the following nonlocal boundary value problem involving both Caputo and Riemann–Liouville fractional derivatives

$$\begin{cases} {}^{\mathrm{C}}\mathcal{D}^{\alpha}_{1^-}\,{}^{\mathrm{RL}}\mathcal{D}^{\beta}_{0^+}u(t) = f(t,u(t)), \quad t \in J := [0,1], \\ u(0) = u'(0) = 0, \qquad u(1) = \delta u(\eta), \quad 0 < \eta < 1, \end{cases}$$

where ${}^{\mathrm{C}}\mathcal{D}^{\beta}_{1^-}$ and ${}^{\mathrm{RL}}\mathcal{D}^{\alpha}_{0^+}$ denote the right Caputo fractional derivative of order $\alpha \in (1,2)$ and the left Riemann–Liouville fractional derivative of order $\beta \in (0,1)$ respectively, and $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. They obtained the existence and uniqueness of solutions by employing some fixed point theorems.

Inspired by the work of the above papers, the present paper aims to establish the existence of solutions for nonlocal boundary value problem involving both Caputo and Riemann–Liouville fractional derivatives in Banach spaces. More precisely, we will consider the following problem

$$\begin{cases} {}^{\mathrm{C}}\mathcal{D}^{\alpha}_{b^-}\,{}^{\mathrm{RL}}\mathcal{D}^{\beta}_{a^+}u(t) = f(t,u(t)), \quad t \in J := [a,b], \\ u(a) = u'(a) = 0, \qquad u(b) = \delta u(\eta), \quad a < \eta < b, \end{cases} \tag{1}$$

where ${}^{\mathrm{C}}\mathcal{D}^{\beta}_{b^-}$ and ${}^{\mathrm{RL}}\mathcal{D}^{\alpha}_{a^+}$ denote the right Caputo fractional derivative of order $\alpha \in (0,1)$ and the left Riemann–Liouville fractional derivative of order $\beta \in (1,2)$ respectively, and $f : [a,b] \times E \longrightarrow E$ is a given function satisfying some assumptions that will be specified later, E is a Banach space with norm $\|\cdot\|$, and δ is a parameter.

To the best of our knowledge, this is the first paper dealing with a nonlocal three-point boundary conditions involving right Caputo and left Riemann–Liouville differential equations of fractional order in Banach spaces E. As in Banach space (in general in any infinite-dimensional linear space) a closed and bounded set is not necessarily compact set, mere continuity of the function f doesn't guarantee the existence of a solution of differential equations. The arguments are based on Darbo's fixed point theorem combined with the technique of measures of noncompactness to establish the existence of a solution for BVP (1).

The structure of this paper is as follows. The next section provides the definitions and preliminary results that we will need to prove our main results. Then, we present the existence results in Section 3. In Section 4, we give an example to illustrate the obtained results. The last section concludes this paper.

2 Preliminaries

We start this section by introducing some necessary definitions and basic results required for further developments.

Let $C(J,E)$ be the Banach space of all continuous functions u from J into E with the supremum (uniform) norm

$$\|u\|_\infty = \sup\{\|u(t)\|, t \in J\}.$$

By $L^1(J)$ we denote the space of Bochner-integrable functions $u : J \to E$, with the norm

$$\|u\|_1 = \int_0^T \|u(t)\|\mathrm{dt}.$$

Next, we define the Hausdorff measure of noncompactness and give some of its important properties.

Definition 1. *[10] Let E be a Banach space and $\mathcal{B}$ a bounded subsets of E. Then the Hausdorff measure of non-compactness of $\mathcal{B}$ is defined by*

$$\chi(\mathcal{B}) = \inf\big\{\varepsilon > 0 : \mathcal{B} \textit{ has a finite cover by closed balls of radius } \varepsilon\big\}.$$

To discuss the problem in this paper, we need the following lemmas.

Lemma 1. *Let $\mathcal{A}, \mathcal{B} \subset E$ be bounded. Then the Hausdorff measure of non-compactness has the following properties. For more details and the proof of these properties see [10].*

1. $\mathcal{A} \subset \mathcal{B} \Longrightarrow \chi(\mathcal{A}) \leq \chi(\mathcal{B})$;
2. $\chi(\mathcal{A}) = 0 \Longleftrightarrow \mathcal{A}$ *is relatively compact;*
3. $\chi(\mathcal{A} \cup \mathcal{B}) = \max\{\chi(\mathcal{A}), \chi(\mathcal{B})\}$;
4. $\chi(\mathcal{A}) = \chi(\overline{\mathcal{A}}) = \chi(conv(\mathcal{A}))$, *where* $\overline{\mathcal{A}}$ *and* $conv\mathcal{A}$ *represent the closure and the convex hull of* $\mathcal{A}$ *, respectively;*
5. $\chi(\mathcal{A} + \mathcal{B}) \leq \chi(\mathcal{A}) + \chi(\mathcal{B})$, *where* $\mathcal{A} + \mathcal{B} = \{x + y : x \in \mathcal{A}, y \in \mathcal{B}\}$;
6. $\chi(\lambda\mathcal{A}) \leq |\lambda|\chi(\mathcal{A})$, *for any* $\lambda \in \mathbb{R}$;
7. *If the map* $Q : D(Q) \subseteq E \to Z$ *is Lipschitz continuous with constant* k, *then* $\chi_Z(QB) \leq k\chi(\mathcal{B})$ *for any bounded subset* $\mathcal{B} \subseteq D(Q)$, *where* Z *is a Banach space. and* $\chi_{Z(.)}$ *is the Hausdorff measure of noncompactness associated with* Z.

Lemma 2. *[9] (Generalized Cantor's intersection) If $\{\mathcal{W}_n\}_{n=1}^\infty$ is a decreasing sequence of bounded and closed nonempty subsets of E and $\lim\limits_{n\to\infty} \chi(\mathcal{W}_n) = 0$, then $\bigcap_{n=1}^\infty \mathcal{W}_n$ is nonempty and compact in E*

Lemma 3. *[10] If $\mathcal{W} \subseteq C(J,E)$ is bounded and equicontinuous, then $\chi(\mathcal{W}(t))$ is continuous on J, and $\chi(\mathcal{W}) = \sup_{t\in J} \chi(\mathcal{W}(t))$.*

We call $\mathcal{B} \subset L^1(J,E)$ uniformly integrable if there exists $\eta \in L^1(J,\mathbb{R}^+)$ such that $\|u(s)\| \leq \eta(s)$, for all $u \in \mathcal{B}$ and a.e. $s \in J$.

Lemma 4. *[25] If $\{u_n\}_{n=1}^{\infty} \subset L^1(J,E)$ is uniformly integrable, then $\chi(\{u_n\}_{n=1}^{\infty})$ is measurable, and*

$$\chi\left(\left\{\int_a^t u_n(s)\mathrm{ds}\right\}_{n=1}^{\infty}\right) \leq 2\int_a^t \chi(\{u_n(s)\}_{n=1}^{\infty})\mathrm{ds}.$$

Lemma 5. *[18] If $\mathcal{W}$ is bounded, then for each ε, there is a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{W}$, such that*

$$\chi(\mathcal{W}) \leq 2\chi(\{u_n\}_{n=1}^{\infty}) + \varepsilon.$$

Definition 2. *[38] A function $f : [a,b]\times E \longrightarrow E$ is said to satisfy the Carathéodory conditions, if the following hold*

- *$f(t,u)$ is measurable with respect to t for $u \in E$,*
- *$f(t,u)$ is continuous with respect to $u \in E$ for $t \in J$.*

Definition 3. *[10] The mapping $\mathcal{N} : \Omega \subset E \longrightarrow E$ is said to be a χ-contraction, if there exists a positive constant $k < 1$ such that*

$$\chi(\mathcal{N}(\mathcal{W})) \leq k\chi(\mathcal{W}),$$

for every bounded subset $\mathcal{W}$ of Ω.

A useful fixed point result for our goals is the following, proved in [10,19].

Theorem 1. (*Darbo and Sadovskii*) *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $\mathcal{N} : \Omega \to \Omega$ be a continuous operator. If $\mathcal{N}$ is a χ-contraction. Then $\mathcal{N}$ has at least one fixed point.*

Let us recall some preliminary concepts of fractional calculus related to our work

Definition 4. *[30] The fractional left and right Riemann–Liouville integrals of order α are defined as*

$$\begin{aligned}
{}^{\mathrm{RL}}\mathcal{I}_{a^+}^{\alpha} u(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)\,\mathrm{ds},\\
{}^{\mathrm{RL}}\mathcal{I}_{b^-}^{\alpha} u(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}u(s)\,\mathrm{ds},
\end{aligned}$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t}t^{\alpha-1}\mathrm{dt}, \quad \alpha > 0.$$

Definition 5. *The left Riemann-Liouville fractional derivative and the right Caputo fractional derivative of order $\alpha > 0$, of a function $u \in AC^n([a,b])$ are, respectively,*

$$
\begin{aligned}
{}^{\mathrm{RL}}\mathcal{D}^{\alpha}_{a^+}u(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left({}^{\mathrm{RL}}\mathcal{I}^{n-\alpha}_{a^+}u\right)(t),\\
{}^{\mathrm{C}}\mathcal{D}^{\alpha}_{b^-}u(t) &= (-1)^n\, {}^{\mathrm{RL}}\mathcal{I}^{n-\alpha}_{b^-}u^{(n)}(t),
\end{aligned}
$$

where $u^{(n)}(t) = \frac{\mathrm{d}^n u(t)}{\mathrm{d}t^n}$.

Lemma 6. *[30] If $u \in AC^n([a,b])$, then*

$$
{}^{\mathrm{RL}}\mathcal{I}^{\alpha}_{a^+}\,{}^{\mathrm{RL}}\mathcal{D}^{\alpha}_{a^+}u(t) = u(t) - \sum_{j=0}^{n-1}\frac{u^{(j)}(a)}{j!}(t-a)^j,
$$

and

$$
{}^{\mathrm{RL}}\mathcal{I}^{\alpha}_{b^-}\,{}^{\mathrm{C}}\mathcal{D}^{\alpha}_{b^-}u(t) = u(t) - \sum_{j=0}^{n-1}\frac{(-1)^j u^{(j)}(b)}{j!}(b-t)^j.
$$

Remark 1. *Note that for an abstract function $u : J \longrightarrow E$, the integrals which appear in the previous definitions are taken in Bochner's sense (see, for instance, [35]).*

3 Main results

Let us recall the definition and lemma of a solution for problem (1). First of all, we define what we mean by a solution for the boundary value problem (1).

Definition 6. *A function $u \in C(J,E)$ is said to be a solution of Eq. (1) if u satisfies the equation ${}^{\mathrm{C}}\mathcal{D}^{\alpha}_{b^-}\,{}^{\mathrm{RL}}\mathcal{D}^{\beta}_{a^+}u(t) = f(t,u(t))$, a.e. on J, and the condition $u(0) = u'(0) = 0$, $u(b) = \delta u(\eta)$.*

For the existence of solutions for the problem (1) we need the following lemma.

Lemma 7. *Let $h \in C(J,E)$ and $(b^\beta - \delta\eta^\beta) \neq 0$. Then the solution of the linear fractional differential equation supplemented with nonlocal boundary conditions*

$$
\begin{cases}
{}^{\mathrm{C}}\mathcal{D}^{\alpha}_{b^-}\,{}^{\mathrm{RL}}\mathcal{D}^{\beta}_{a^+}u(t) = f(t,u(t)), \quad t \in J,\\
u(a) = u'(a) = 0, \qquad u(b) = \delta u(\eta).
\end{cases}
\tag{2}
$$

is equivalent to the fractional integral equation given by

$$
\begin{aligned}
u(t) &= {}^{\mathrm{RL}}\mathcal{I}^{\beta}_{a^+}\,{}^{\mathrm{RL}}\mathcal{I}^{\alpha}_{b^-}h(t) + \frac{t^\beta}{b^\beta - \delta\eta^\beta}\left[\delta\,{}^{\mathrm{RL}}\mathcal{I}^{\beta}_{a^+}\,{}^{\mathrm{RL}}\mathcal{I}^{\alpha}_{b^-}h(\eta) - {}^{\mathrm{RL}}\mathcal{I}^{\beta}_{a^+}\,{}^{\mathrm{RL}}\mathcal{I}^{\alpha}_{b^-}h(b)\right]\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_s^b (t-s)^{\beta-1}(\tau-s)^{\alpha-1}h(\tau)\mathrm{d}\tau\mathrm{d}s\\
&\quad + \frac{\delta t^\beta}{(b^\beta - \delta\eta^\beta)\Gamma(\alpha)\Gamma(\beta)}\int_a^\eta\int_s^b (\eta-s)^{\beta-1}(\tau-s)^{\alpha-1}h(\tau)\mathrm{d}\tau\mathrm{d}s\\
&\quad - \frac{t^\beta}{(b^\beta - \delta\eta^\beta)\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_s^b (b-s)^{\beta-1}(\tau-s)^{\alpha-1}h(\tau)\mathrm{d}\tau\mathrm{d}s.
\end{aligned}
\tag{3}
$$

Proof. We first apply the right fractional integrals ${}^{\mathrm{RL}}\mathcal{I}_{b^-}^{\alpha}$ to the first equality of (2) and then the left fractional integrals ${}^{\mathrm{RL}}\mathcal{I}_{a^+}^{\alpha}$ to the resulting equations, and using the properties of Caputo and Riemann–Liouville fractional derivatives, we get

$$u(t) = {}^{\mathrm{RL}}\mathcal{I}_{a^+}^{\beta}\, {}^{\mathrm{RL}}\mathcal{I}_{b^-}^{\alpha} h(t) + \frac{c_0 t^{\beta}}{\Gamma(\beta+1)} + c_1(t-a)^{\beta-1} + c_2(t-a)^{\beta-2}. \tag{4}$$

Using the conditions $u(a) = 0, u'(a) = 0$, in (4) yields $c_1 = 0, c_2 = 0$. In consequence, the equation (4) reduce to the form:

$$u(t) = {}^{\mathrm{RL}}\mathcal{I}_{a^+}^{\beta}\, {}^{\mathrm{RL}}\mathcal{I}_{b^-}^{\alpha} h(t) + \frac{c_1 t^{\beta}}{\Gamma(\beta+1)}. \tag{5}$$

By the boundary condition $u(b) = \delta u(\eta)$, we find that

$$c_1 = \frac{\Gamma(\beta+1)}{b^{\beta} - \delta\eta^{\beta}} \left[\delta {}^{\mathrm{RL}}\mathcal{I}_{a^+}^{\beta}\, {}^{\mathrm{RL}}\mathcal{I}_{b^-}^{\alpha} h(\eta) - {}^{\mathrm{RL}}\mathcal{I}_{a^+}^{\beta}\, {}^{\mathrm{RL}}\mathcal{I}_{b^-}^{\alpha} h(b)\right],$$

which, on substituting in (5), leads to the solution (3). The converse follows by direct computation. The proof is completed. □

In the following, for computational convenience we put

$$\mathcal{M}_{\psi} = \frac{\|\psi\|\,(b-a)^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta+1)} \left[(b-a)^{\beta} + \frac{|\delta| b^{\beta}\,(\eta-a)^{\beta}}{|b^{\beta} - \delta\eta^{\beta}|} + \frac{b^{\beta}\,(b-a)^{\beta}}{|b^{\beta} - \delta\eta^{\beta}|}\right], \tag{6}$$

and

$$\mathcal{L}_{\psi} = \frac{\|\psi\|\,(b-a)^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta)} \left[(b-a)^{\beta-1} + \frac{|\delta| b^{\beta-1}\,(\eta-a)^{\beta}}{|b^{\beta} - \delta\eta^{\beta}|} + \frac{b^{\beta-1}\,(b-a)^{\beta}}{|b^{\beta} - \delta\eta^{\beta}|}\right]. \tag{7}$$

Now, we shall present our main result concerning the existence of solutions of problem (1). Let us introduce the following hypotheses

1. (H1) the function $f : [a, b] \times E \longrightarrow E$ satisfy Carathéodory conditions;
2. (H2) There exist function $\psi \in L^{\infty}(J, \mathbb{R}_+)$ such that

$$\|f(t, u(t))\| \le \psi(t)(1 + \|u\|), \quad \text{for all } u \in C(J, E);$$

3. (H3) For each bounded set $\mathcal{W} \subset E$, and each $t \in J$, the following inequality holds

$$\chi(g(t, \mathcal{W})) \le \psi(t)\chi(\mathcal{W}).$$

Now, we shall prove the following theorem concerning the existence of solutions of problem (1)

Theorem 2. *Assume that the hypotheses (H1)–(H3) are satisfied. If*

$$\mathcal{M}_{\psi} < 1, \tag{8}$$

then the problem (1) has at least one solution defined on J.

Proof. Consider the operator $\mathcal{N}: C(J,E) \longrightarrow C(J,E)$ defined by:

$$\begin{aligned}\mathcal{N}u(t) = &\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_s^b (t-s)^{\beta-1}(\tau-s)^{\alpha-1} f(\tau,u(\tau))\mathrm{d}\tau\mathrm{ds} \\ &+ \frac{\delta t^\beta}{(b^\beta - \delta\eta^\beta)\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \int_s^b (\eta-s)^{\beta-1}(\tau-s)^{\alpha-1} f(\tau,u(\tau))\mathrm{d}\tau\mathrm{ds} \\ &- \frac{t^\beta}{(b^\beta - \delta\eta^\beta)\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_s^b (b-s)^{\beta-1}(\tau-s)^{\alpha-1} f(\tau,u(\tau))\mathrm{d}\tau\mathrm{ds}. \end{aligned} \tag{9}$$

It is obvious that $\mathcal{N}$ is well defined due to (H1) and (H2). Then, fractional integral equation (3) can be written as the following operator equation

$$u = \mathcal{N}u. \tag{10}$$

Thus, the existence of a solution for Eq. (1) is equivalent to the existence of a fixed point for operator $\mathcal{N}$ which satisfies operator equation (10). Define a bounded closed convex set

$$\mathcal{B}_\mathcal{R} = \left\{ w \in C(J,E) : \|w\|_\infty \le \mathcal{R} \right\},$$

with $\mathcal{R} > 0$, such that

$$\mathcal{R} \ge \frac{\mathcal{M}_\psi}{1 - \mathcal{M}_\psi}.$$

In order to satisfy the hypotheses of the Darbo fixed point theorem, we split the proof into four steps.

Step 1: The operator $\mathcal{N}$ maps the set $\mathcal{B}_\mathcal{R}$ into itself. By the assumption (H2), we have

$$\begin{aligned} \|\mathcal{N}u(t)\| \le\ & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_s^b (t-s)^{\beta-1}(\tau-s)^{\alpha-1} \|f(\tau,u(\tau))\|\mathrm{d}\tau\mathrm{ds} \\ &+ \frac{|\delta| t^\beta}{|b^\beta - \delta\eta^\beta|\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \int_s^b (\eta-s)^{\beta-1}(\tau-s)^{\alpha-1} \|f(\tau,u(\tau))\|\mathrm{d}\tau\mathrm{ds} \\ &+ \frac{t^\beta}{|b^\beta - \delta\eta^\beta|\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_s^b (b-s)^{\beta-1}(\tau-s)^{\alpha-1} \|f(\tau,u(\tau))\|\mathrm{d}\tau\mathrm{ds} \\ \le\ & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_s^b (t-s)^{\beta-1}(\tau-s)^{\alpha-1} \psi(\tau)(1+\|u(\tau)\|)\mathrm{d}\tau\mathrm{ds} \\ &+ \frac{|\delta| t^\beta}{|b^\beta - \delta\eta^\beta|\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \int_s^b (\eta-s)^{\beta-1}(\tau-s)^{\alpha-1} \psi(\tau)(1+\|u(\tau)\|)\mathrm{d}\tau\mathrm{ds} \\ &+ \frac{t^\beta}{|b^\beta - \delta\eta^\beta|\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_s^b (b-s)^{\beta-1}(\tau-s)^{\alpha-1} \psi(\tau)(1+\|u(\tau)\|)\mathrm{d}\tau\mathrm{ds} \end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{\psi\|(1+\|u\|)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_s^b (t-s)^{\beta-1}(\tau-s)^{\alpha-1}\mathrm{d}\tau\mathrm{ds} \\
&+\frac{|\delta|t^{\beta}\psi\|(1+\|u\|)}{|b^{\beta}-\delta\eta^{\beta}|\Gamma(\alpha)\Gamma(\beta)}\int_a^{\eta}\int_s^b (\eta-s)^{\beta-1}(\tau-s)^{\alpha-1}\mathrm{d}\tau\mathrm{ds} \\
&+\frac{t^{\beta}\psi\|(1+\|u\|)}{|b^{\beta}-\delta\eta^{\beta}|\Gamma(\alpha)\Gamma(\beta)}\int_a^{b}\int_s^b (b-s)^{\beta-1}(\tau-s)^{\alpha-1}\mathrm{d}\tau\mathrm{ds}.
\end{aligned}$$

Also note that

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_s^b (t-s)^{\beta-1}(\tau-s)^{\alpha-1}\mathrm{d}\tau\mathrm{ds}\leq \frac{(b-a)^{\alpha}\,(b-a)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)},$$

where we have used the fact that $(b-s)^{\alpha}\leq (b-a)^{\alpha}$ for $0<\alpha\leq 1$. Using the above arguments, we have

$$\begin{aligned}
\|\mathcal{N}u(t)\| &\leq \frac{\|\psi\|(1+\|u\|)\,(b-a)^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left[(b-a)^{\beta}+\frac{|\delta|b^{\beta}\,(\eta-a)^{\beta}}{|b^{\beta}-\delta\eta^{\beta}|}+\frac{b^{\beta}\,(b-a)^{\beta}}{|b^{\beta}-\delta\eta^{\beta}|}\right] \\
&\leq \frac{\|\psi\|(1+\mathcal{R})\,(b-a)^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\left[(b-a)^{\beta}+\frac{|\delta|b^{\beta}\,(\eta-a)^{\beta}}{|b^{\beta}-\delta\eta^{\beta}|}+\frac{b^{\beta}\,(b-a)^{\beta}}{|b^{\beta}-\delta\eta^{\beta}|}\right] \\
&= \mathcal{M}_{\psi}(1+\mathcal{R})\leq \mathcal{R}.
\end{aligned}$$

Thus $\|\mathcal{N}u\|\leq\mathcal{R}$. This proves that $\mathcal{N}$ transforms the ball $\mathcal{B}_{\mathcal{R}}$ into itself. Furthermore for any $u\in\mathcal{B}_{\mathcal{R}}$ and $t\in J$, we have

$$\begin{aligned}
\|(\mathcal{N}u)'(t)\| \leq\ & \frac{1}{\Gamma(\alpha)\Gamma(\beta-1)}\int_a^t\int_s^b (t-s)^{\beta-2}(\tau-s)^{\alpha-1}\psi(\tau)(1+\|u(\tau)\|)\mathrm{d}\tau\mathrm{ds} \\
&+\frac{|\delta|\beta t^{\beta-1}}{|b^{\beta}-\delta\eta^{\beta}|\Gamma(\alpha)\Gamma(\beta)}\int_a^{\eta}\int_s^b (\eta-s)^{\beta-1}(\tau-s)^{\alpha-1}\psi(\tau)(1+\|u(\tau)\|)\mathrm{d}\tau\mathrm{ds} \\
&+\frac{\beta t^{\beta-1}}{|b^{\beta}-\delta\eta^{\beta}|\Gamma(\alpha)\Gamma(\beta)}\int_a^{b}\int_s^b (b-s)^{\beta-1}(\tau-s)^{\alpha-1}\psi(\tau)(1+\|u(\tau)\|)\mathrm{d}\tau\mathrm{ds}.
\end{aligned}$$

Some computations give

$$\begin{aligned}
\|(\mathcal{N}u)'(t)\| &\leq \frac{\|\psi\|(1+\mathcal{R})\,(b-a)^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta)}\left[(b-a)^{\beta-1}+\frac{|\delta|b^{\beta-1}\,(\eta-a)^{\beta}}{|b^{\beta}-\delta\eta^{\beta}|}+\frac{b^{\beta-1}\,(b-a)^{\beta}}{|b^{\beta}-\delta\eta^{\beta}|}\right] \\
&:= \mathcal{L}_{\psi}(1+\mathcal{R}).
\end{aligned}$$

Step 2: The operator $\mathcal{N}$ is continuous. Suppose that $\{u_n\}$ is a sequence such that $u_n\to u$ in

$\mathcal{B}_{\mathcal{R}}$ as $n \to \infty$. Since f satisfies (H1), for each $t \in J$, we get

$$\begin{aligned}
&\|\mathcal{N}u_n(t) - \mathcal{N}u(t)\| \\
&\quad \le \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_s^b (t-s)^{\beta-1}(\tau-s)^{\alpha-1}\|f(\tau,u_n(\tau)) - f(\tau,u(\tau))\|\mathrm{d}\tau\mathrm{ds} \\
&\qquad + \frac{|\delta|t^\beta}{|b^\beta - \delta\eta^\beta|\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \int_s^b (\eta-s)^{\beta-1}(\tau-s)^{\alpha-1}\|f(\tau,u_n(\tau)) - f(\tau,u(\tau))\|\mathrm{d}\tau\mathrm{ds} \\
&\qquad + \frac{t^\beta}{|b^\beta - \delta\eta^\beta|\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_s^b (b-s)^{\beta-1}(\tau-s)^{\alpha-1}\|f(\tau,u_n(\tau)) - f(\tau,u(\tau))\|\mathrm{d}\tau\mathrm{ds} \\
&\quad \le \frac{(b-a)^\alpha}{\Gamma(\alpha+1)\Gamma(\beta+1)} \left[(b-a)^\beta + \frac{|\delta|b^\beta\,(\eta-a)^\beta}{|b^\beta - \delta\eta^\beta|} + \frac{b^\beta\,(b-a)^\beta}{|b^\beta - \delta\eta^\beta|}\right] \|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\|.
\end{aligned}$$

By using the Lebesgue dominated convergence theorem, we know that

$$\|\mathcal{N}u_n(t) - \mathcal{N}u(t)\| \to 0, \quad \text{as} \quad n \to +\infty,$$

for any $t \in J$. Therefore, we get that

$$\|\mathcal{N}u_n - \mathcal{N}u\| \to 0, \quad \text{as} \quad n \to +\infty,$$

which implies the continuity of the operator $\mathcal{N}$.
Step 3: The operator $\mathcal{N}$ is equicontinuous. For any $a < t_1 < t_2 < b$ and $u \in \mathcal{B}_{\mathcal{R}}$, we get

$$\|\mathcal{N}(u)(t_2) - \mathcal{N}(u)(t_1)\| \le \int_{t_1}^{t_2} \|(\mathcal{N}u)'(s)\|\mathrm{ds} \le (1+\mathcal{R})\mathcal{L}_\psi|t_2 - t_1|.$$

where $\mathcal{L}_\psi$ is given by (7). As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero independently of $u \in \mathcal{B}_{\mathcal{R}}$. Hence, we conclude that $\mathcal{N}(\mathcal{B}_{\mathcal{R}}) \subseteq C(J,E)$ is bounded and equicontinuous.
Step 4: Our aim in this step is to show that $\mathcal{N}$ is χ-contraction on $\mathcal{B}_{\mathcal{R}}$. For every bounded subset $\mathcal{W} \subset \mathcal{B}_{\mathcal{R}}$ and $\varepsilon > 0$ using Lemma 5 and the properties of χ, there exist sequences $\{u_k\}_{k=1}^\infty \subset \mathcal{W}$ such that

$$\begin{aligned}
\chi\left(\mathcal{N}\mathcal{W}(t)\right) \le 2\chi \Bigg\{ &\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_s^b (t-s)^{\beta-1}(\tau-s)^{\alpha-1} f\left(\tau, \{u_k(\tau)\}_{k=1}^\infty\right) \mathrm{d}\tau\mathrm{ds} \\
&+ \frac{\delta t^\beta}{(b^\beta - \delta\eta^\beta)\Gamma(\alpha)\Gamma(\beta)} \int_a^\eta \int_s^b (\eta-s)^{\beta-1}(\tau-s)^{\alpha-1} f\left(\tau, \{u_k(\tau)\}_{k=1}^\infty\right) \mathrm{d}\tau\mathrm{ds} \\
&- \frac{t^\beta}{(b^\beta - \delta\eta^\beta)\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_s^b (b-s)^{\beta-1}(\tau-s)^{\alpha-1} f\left(\tau, \{u_k(\tau)\}_{k=1}^\infty\right) \mathrm{d}\tau\mathrm{ds} \Bigg\} + \varepsilon.
\end{aligned}$$

Next, by Lemma 4 and the properties of χ and (H3) we have

$$\begin{aligned}(\mathcal{N}\mathcal{W}(t)) &\leq 4\left\{\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_s^b (t-s)^{\beta-1}(\tau-s)^{\alpha-1}\chi\left(f\left(\tau,\{u_k(\tau)\}_{k=1}^{\infty}\right)\right)\mathrm{d}\tau\mathrm{ds}\right.\\ &\quad+\frac{|\delta|t^{\beta}}{|b^{\beta}-\delta\eta^{\beta}|\Gamma(\alpha)\Gamma(\beta)}\int_a^{\eta}\int_s^b (\eta-s)^{\beta-1}(\tau-s)^{\alpha-1}\chi\left(f\left(\tau,\{u_k(\tau)\}_{k=1}^{\infty}\right)\right)\mathrm{d}\tau\mathrm{ds}\\ &\quad\left.+\frac{t^{\beta}}{|b^{\beta}-\delta\eta^{\beta}|\Gamma(\alpha)\Gamma(\beta)}\int_a^{b}\int_s^b (b-s)^{\beta-1}(\tau-s)^{\alpha-1}\chi\left(f\left(\tau,\{u_k(\tau)\}_{k=1}^{\infty}\right)\right)\mathrm{d}\tau\mathrm{ds}\right\}+\varepsilon\\ &\leq 4\left\{\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_s^b (t-s)^{\beta-1}(\tau-s)^{\alpha-1}\psi(\tau)\chi\left(\{u_k(\tau)\}_{k=1}^{\infty}\right)\mathrm{d}\tau\mathrm{ds}\right.\\ &\quad+\frac{|\delta|t^{\beta}}{|b^{\beta}-\delta\eta^{\beta}|\Gamma(\alpha)\Gamma(\beta)}\int_a^{\eta}\int_s^b (\eta-s)^{\beta-1}(\tau-s)^{\alpha-1}\psi(\tau)\chi\left(\{u_k(\tau)\}_{k=1}^{\infty}\right)\mathrm{d}\tau\mathrm{ds}\\ &\quad\left.+\frac{t^{\beta}}{|b^{\beta}-\delta\eta^{\beta}|\Gamma(\alpha)\Gamma(\beta)}\int_a^{b}\int_s^b (b-s)^{\beta-1}(\tau-s)^{\alpha-1}\psi(\tau)\chi\left(\{u_k(\tau)\}_{k=1}^{\infty}\right)\mathrm{d}\tau\mathrm{ds}\right\}+\varepsilon\\ &\leq 4\mathcal{M}_{\psi}\chi(\mathcal{B})+\varepsilon.\end{aligned}$$

As the last inequality is true, for every $\varepsilon>0$, we infer

$$\chi(\mathcal{N}\mathcal{W})=\sup_{t\in J}\chi(\mathcal{N}\mathcal{W}(t))\leq 4\mathcal{M}_{\psi}\chi(\mathcal{B}).$$

Using the condition (8), we claim that $\mathcal{N}$ is a χ-contraction on $\mathcal{B}_{\mathcal{R}}$. By Theorem 1, there is a fixed point u of $\mathcal{N}$ on $\mathcal{B}_{\mathcal{R}}$, which is a solution of (1). This completes the proof. □

4 An example

In this section we give an example to illustrate the usefulness of our main result. Let

$$E=c_0=\{u=(u_1,u_2,\dots,u_n,\dots):u_n\to 0\ (n\to\infty)\},$$

be the Banach space of real sequences converging to zero, endowed its usual norm $\|u\|_{\infty}=\sup_{n\geq 1}|u_n|$.

Example 1. Consider the following boundary value problem of a fractional differential posed in c_0:

$$\begin{cases}{}^{\mathrm{C}}\mathcal{D}^{\frac{1}{2}}_{\frac{1}{2}^-}{}^{\mathrm{RL}}\mathcal{D}^{\frac{3}{2}}_{0^+}u(t)=f(t,u(t)), & t\in J, J:=\left[0,\frac{1}{2}\right],\\ u(0)=u'(0)=0, \quad u\left(\frac{1}{2}\right)=\frac{1}{5}u\left(\frac{1}{4}\right).\end{cases}\tag{11}$$

Note that, this problem is a particular case of BVP (1), where

$$\alpha=\frac{1}{2},\quad \beta=\frac{3}{2},\quad a=0,\quad b=\frac{1}{2},\quad \eta=\frac{1}{2},\quad \delta=\frac{1}{5},$$

and $f:J\times c_0\longrightarrow c_0$ given by

$$f(t,u)=\left\{\frac{1}{(t^2+2)^2}\left(\frac{1}{n^2}+\ln(1+|u_n|)\right)\right\}_{n\geq 1},\quad \text{for } t\in J, u=\{u_n\}_{n\geq 1}\in c_0.$$

It is clear that condition (H1) hold, and as

$$\|f(t,u)\| = \left\| \frac{1}{(t^2+2)^2} \left(\frac{1}{n^2} + \ln(1+|u_n|) \right) \right\| \leq \frac{1}{(t^2+2)^2} (1+\|u\|) = \psi(t)(1+\|u\|).$$

Therefore, assumption (H2) of the Theorem 2 is satisfied with $\psi(t) = \frac{1}{(t^2+2)^2}, t \in J$. On the other hand, for any bounded set $\mathcal{W} \subset c_0$, we have

$$\chi(f(t,\mathcal{W})) \leq \frac{1}{(t^2+2)^2}\chi(\mathcal{W}), \text{ for each } t \in J.$$

Hence (H3) is satisfied. We shall check that condition (8) is satisfied. Indeed $4\mathcal{M}_\psi = 0.7314 < 1$, and $(1+\mathcal{R})\mathcal{M}_\psi \leq \mathcal{R}$, thus

$$\mathcal{R} \geq \frac{\mathcal{M}_\psi}{1-\mathcal{M}_\psi} = 2.7229,$$

Then $\mathcal{R}$ can be chosen as $\mathcal{R} = 3 \geq 2.7229$. Consequently, Theorem 2 implies that problem (11) has at least one solution $u \in C(J, c_0)$.

5 Conclusions

We have proved the existence of solutions for certain classes of nonlinear differential equations involving the right Caputo and left Riemann–Liouville fractional derivatives with nonlocal conditions in Banach spaces. The problem is issued by applying Darbo's fixed point theorem combined with the technique of Hausdorff measure of noncompactness. We also provide an example to make our results clear.

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