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Correctness of the free boundary problem for the microscopic in-situ leaching model

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Abstract. We consider initial boundary value problem for in-situ leaching process of rare metals at the microscopic level. This physical process describes by the Stokes equations for the liquid component coupled with the Lame's equations for the solid skeleton and the diffusion-convection equations for acid concentration. Due to the dissolution of the solid skeleton, the pore space has an unknown (free) boundary. For formulated initial boundary-value problem we prove existence and uniqueness of the classical solution.

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1 Introduction

In the present publication we consider the dissolution of the elastic solid skeleton by acid at the microscopic level. This problem has been considered before only at the macroscopic level by several authors [3, 4, 10]. All of them just postulated the corresponding model without any justification. It is quite understandable, since the main mechanism of the physical process is focused on the unknown (free) boundary between the pore space and the solid skeleton and not spelled out in any way in the proposed macroscopic models. That is why there is a great variety of models, depending on the tastes and preferences of its authors.

Sanchez-Palencia [11] and Burridge and Keller [2] outlined the right way of mathematical modeling as a passage from microstructure to macrostructure. Following this idea we consider at the microscopic level a joint motion of the compressible viscous liquid, described by Stokes equations, and the compressible elastic skeleton, governed by Lame's equations, coupled with diffusion-convection equations for the acid and products of chemical reactions. Fortunately, the dynamics of chemical reaction products are determined after finding the dynamics of the acid

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itself. So, we may restrict ourself only with Lame's, Stokes and diffusion-convection equations for the acid and corresponding boundary and initial conditions.

The problem under consideration is the free boundary problem. In other words, we need to find a solution to the boundary value problem along with the domain, where we are looking for this solution. Note, that free boundary problems are the most difficult part of the theory of partial differential equations [7]. That is why we restrict ourself with one dimensional case.

In Section 2 we state the mathematical problem. In Section 3 we formulate existence (Theorem 1) and uniqueness results. Section 4 is devoted to the proof of Theorem 1.

We use notations of functional spaces and norm there from [5, 6].

2 The problem statement

Let $R(t)$ be unknown function of the variable $t \in (0,T)$, defined the position of the free boundary

$$\Gamma_R = \bigcup_{t=0}^{t=T} \Gamma_R(t) \in Q_T = Q \times (0,T),$$

where $Q = \{x \in \mathbb{R} : 0 < x < L\}$.

Let also $\Omega_{f,R}(t) = \{x : 0 < x < R(t)\}$ be the domain in Q, occupied with the liquid component at the moment t, $\Omega_{s,R}(t) = \{x : R(t) < x < L\}$ be the domain in Q, occupied with the solid component at the moment t, $\Omega_{f,T,R} = \{(x,t) : 0 < t < T,\ 0 < x < R(t)\} = G_{f,R}$ be domain in Q_T, occupied with the liquid component, and $\Omega_{s,T,R} = \{(x,t) : 0 < t < T,\ R(t) < x < L\} = G_{s,R}$ be domain in Q_T, occupied with the solid component, $Q_{f,T} = Q_f \times (0,T)$, $Q_{s,T} = Q_s \times (0,T)$, $Q_f = (0, R_0)$, $Q_s = (R_0, L)$, $0 < R_0 < L$.

In the domain $G_{f,R}$ the liquid satisfies the Stokes equations

$$\mu \frac{\partial^2 v_f}{\partial x^2} - \frac{\partial p_f}{\partial x} = 0, \tag{1}$$

$$\frac{\partial p_f}{\partial t} + \varrho_f\, c_f^2 \frac{\partial v_f}{\partial x} = 0, \tag{2}$$

for the velocity $v_f(x,t)$ and pressure $p_f(x,t)$ of a viscous compressible liquid. The dynamics of the acid concentration $c(x,t)$ in $G_{f,R}$ is described by diffusion-convection equations

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} - v_f \frac{\partial c}{\partial x}. \tag{3}$$

The solid component in $G_{s,R}$ is governed by the Lame's equation

$$(\lambda + c_s^2 \varrho_s) \frac{\partial^2 w_s}{\partial x^2} = 0 \tag{4}$$

for the displacements $w_s(x,t)$.

At the free boundary $\Gamma_R(t)$ one has the key condition

$$\frac{dR}{dt}(t) = \alpha\, c\big(R(t), t\big), \ \ \alpha = \text{const} > 0, \tag{5}$$

expressing the dynamics of the free boundary, and boundary conditions

$$\mu\,\frac{\partial v_f}{\partial x}\big(R(t),t\big) - p_f\big(R(t),t\big) = (\lambda + c_s^2\varrho_s)\,\frac{\partial w_s}{\partial x}\big(R(t),t\big), \tag{6}$$

$$v_f\big(R(t),t\big) = \frac{dR}{dt}(t), \tag{7}$$

$$D\,\frac{\partial c}{\partial x}\big(R(t),t\big) = -\beta\, c\big(R(t),t\big), \;\; \beta = \text{const} > 0, \tag{8}$$

expressing the laws of conservation of mass and momentum [8,9].

The problem is ended with boundary conditions

$$\mu\,\frac{\partial v_f}{\partial x}(0,t) - p_f(0,t) = -p^0(t), \;\; c(0,t) = c^0 = \text{const} > 0, \;\; w_s(L,t) = 0, \tag{9}$$

at the known boundaries $x = 0$ and $x = L$, and initial conditions

$$c(x,0) = c_0(x), \; x \in Q_f, \; R(0) = R_0, \; 0 < R_0 < L,$$

$$p_f(x,0) = p_f^0(x), \; x \in Q_f, \; p_f\big(x, t_R(x)\big) = p_{f,R}^0(x) = p_f^0\big((t_R(x)\big), \; x \in Q_s. \tag{10}$$

In Eqs. (1)–(10) $t_R(x)$ is the inverse function to the function $x = R(t)$ for $x > R_0$, $\mu = \text{const} > 0$ is a viscosity of the liquid component, $c_f^2 = \text{const} > 0$ is speed of sound in the liquid, $\lambda = \text{const} > 0$ is a Lame's coefficient, $c_s^2 = \text{const} > 0$ is speed of sound in the solid component, $\varrho_f = \text{const} > 0$ is a density of the liquid component at rest, and $\varrho_s = \text{const} > 0$ is a density of the solid component at rest.

Remark 1. *To define p_f in the subdomain $\{(x,t) : R_0 < x < R(t),\, 0 < t < T\}$ of the domain $G_{f,R}$ we have to know "initial" condition $p_{f,R}^0(x)$ at the curve $\{(x,t) : x = R(t),\, 0 < t < T\}$. That is why we introduce the inverse function $t_R(x)$ to the function $x = R(t)$ for $x > R_0$ in Eq. (10).*

We denote the formulated problem without boundary condition (5) as $A(R)$.

The problem $A(R)$ is obviously divided into two successively solved subproblems: the **Dynamic Problem** $A(R)$, consisting of Eqs. (1), (2), (4), and boundary conditions (6)–(7), (9), completed with initial conditions (10), which we solve in the first place, and the **Diffusion-Convection Problem** $A(R)$, consisting of Eq. (3) and boundary conditions (8), (9), completed with initial conditions (10).

3 Main results

Theorem 1. *Let $c_0(x) \geqslant c_* = \text{const} > 0$, $c_0 \in \mathbb{W}_q^{2-\frac{2}{q}}(Q_f)$, $q > 3$, $p_f(x,0) = p_f^0(x)$, $p_f^0(x) \geqslant 0$, $p_f^0 \in \mathbb{W}_q^1(Q)$, $p^0(t) \geqslant 0$, $p^0 \in \mathbb{L}_q(0,T)$, and at $(x,t) = (0,0)$ and $(x,t) = (0,R_0)$ conditions of concordance $c_0(0) = c^0$, $\dfrac{dc_0}{dx}(R_0) = -\beta\, c_0(R_0)$ are satisfied. Then for any $T > 0$ the problem (1)–(10) has a unique weak solution $\{R(t),\, v_f,\, p_f,\, c, w_s\}$, such that $\dfrac{dR}{dt} > 0$, $\dfrac{dR}{dt} \in \mathbb{L}_q(0,T)$,*

$v_f \in \mathbb{L}_q\big(0,T;\mathbb{W}^1_q(\Omega_f(t))\big)$, $p_f \in \mathbb{L}_q(G_{f,R})$, $c \in \mathbb{W}^{2,1}_q(G_{f,R})$, w_s, $\dfrac{\partial w_s}{\partial x} \in \mathbb{L}_q\big(0,T;\mathbb{L}_\infty(\Omega_s(t)\big) \equiv \mathbb{Y}_s$, *and*

$$\|v_f\|_{q,G_{f,R}} + \|\frac{\partial v_f}{\partial x}\|_{q,G_{f,R}} + \|p_f\|_{q,G_{f,R}} + \|w_s\|_{Y_s} + \|\frac{\partial w_s}{\partial x}\|_{Y_s} \leqslant M\,M_p,$$

$$c_* \leqslant c(x,t) \leqslant M^0_c,\ (x,t)\in G_{f,R},\ \ \|c\|^{(2)}_{q,G_{f,R}} \leqslant M^1_c. \tag{11}$$

where $M_p = \|p^0_f\|_{q,G} + \|p^0\|_{q,(0,T)}$, $M^0_c = \max\{c^0, |c_0|^{(0)}_{G_{f,R}}\}$, $M^1_c = \|c_0\|^{(2-\frac{2}{q})}_{q,Q_f}$, *and the constant* M *depend only* L *and* T *and finite for finite* L *and* T*.*

4 Proof of Theorem 1

First of all we rewrite the problem in the new variables

$$t=t,\quad x=\frac{R(t)}{R_0}\,y,\ \ \text{for}\,(x,t)\in G_{f,R},$$

$$t=t,\ \ x=\frac{L-R(t)}{L-R_0}\,y + L\,\frac{R(t)-R_0}{L-R_0},\ \ \text{for}\,(x,t)\in G_{s,R},$$

transforming domains $G_{f,R}$ and $G_{s,R}$ onto domains $Q_{f,T}$ and $Q_{s,T}$ correspondingly.

One has for new unknown functions $\bar v_f(y,t)=v_f(x,t)$, $\bar p_f(y,t)=p_f(x,t)$, $\bar c(y,t)=c(x,t)$, $\bar w_s(y,t)=w_s(x,t)$,

$$\frac{\partial}{\partial y}\left(\mu\,\frac{R_0}{R}\frac{\partial \bar v_f}{\partial y} - \bar p_f\right) = 0,\ \ (y,t)\in Q_{f,T},$$

$$\mu\,\frac{R_0}{R}\frac{\partial \bar v_f}{\partial y}(0,t) - \bar p_f(0,t) = -\bar p^0_f(0,t),\ \ \bar v_f(R_0,t) = \frac{dR}{dt}(t),$$

$$\mu\,\frac{R_0}{R}\frac{\partial \bar v_f}{\partial y}(R_0,t) - \bar p_f(R_0,t) = (\lambda + c^2_s\varrho_s)\,\frac{\partial \bar w_s}{\partial y}(R_0,t),$$

$$(\lambda + c^2_s\varrho_s)\,\frac{\partial^2 \bar w_s}{\partial y^2} = 0,\ \ (y,t)\in Q_{s,T},\ \ \bar w_s(L,t)=0, \tag{12}$$

$$\frac{\partial \bar c}{\partial t} = D\,\frac{R_0^2}{R^2}\,\frac{\partial^2 \bar c}{\partial y^2} + \left(y\,\frac{dR}{dt} + \bar v_f\right)\frac{R_0}{R}\,\frac{\partial \bar c}{\partial y},\ \ (y,t)\in Q_{f,T},$$

$$\bar c(0,t) = c^0,\ \ D\,\frac{R_0}{R}\frac{\partial \bar c}{\partial y}(R_0,t) = -\beta\,\bar c(R_0,t),\ \ t\in(0,T),$$

$$\bar c(y,0) = c_0(y),\ \ y\in Q_f. \tag{13}$$

Let $M^1_r = \alpha\,M^0_c$ and

$$\mathfrak{M}(T) = \{r,\ \frac{dr}{dt}\in\mathbb{L}_q(0,T):\ r(0)=R_0,\ \frac{dr}{dt}(t)\ \geqslant\ c_*,\ \|\frac{dr}{dt}\|_{q,(0,T)} \leqslant\ M^1_r\}.$$

For any $r\in\mathfrak{M}(T)$ we consider a problem $A(r)$, consisting of Eqs. (1)–(4) in the known domains $G_{f,r}$ and $G_{s,r}$, boundary conditions (6)–(9) at known boundaries $S^0_T = \{(x,t):\ x=0,\ 0<t<T\}$, $S^L_T = \{(x,t):\ x=L,\ 0<t<T\}$, and $\Gamma(r)$, and initial conditions (10).

Lemma 1. *Under conditions of Theorem 1 for any $r \in \mathfrak{M}$ the Dynamic Problem $A(r)$ has a unique solution.*

Proof. It is easy to see, that

$$\mu\,\frac{\partial v_f}{\partial x}(x,t) - p_f(x,t) = (\lambda + c_s^2 \varrho_s)\,\frac{\partial w_s}{\partial x}(x,t) = -p^0(t);$$

$$p_f(x,t) = p_f^0(x)\,e^{-\delta\,t} + \delta\int_0^t e^{-\delta(t-\tau)}p^0(\tau)\,d\tau,\ \delta = \frac{\varrho_f\,c_f^2}{\mu};$$

$$v_f(x,t) = \frac{dr}{dt}(t) + \frac{1}{\mu}\left(p_f(x,t) - p^0(t)\right),\ w_s(x,t) = -\frac{p^0(t)}{(\lambda + \varrho_s\,c_s^2)}(x-L)), \tag{14}$$

and functions v_f, p_f and w_s satisfy estimates (11) with the function $r(t)$ instead of the function $R(t)$. □

Lemma 2. *Under conditions of Theorem 1 for any $r \in \mathfrak{M}$ the Diffusion-Convection Problem $A(r)$ has a a unique solution $c \in \mathbb{W}_q^{2,1}(G_{f,r})$, where $c(x,t)$ satisfies estimates (11) with corresponding change of the domain $G_{f,R}$ to the domain $G_{f,r}$.*

Proof. To prove theorem we consider approximate problems $A(r_n)$ in the domain $Q_{f,T}$ for the function $\bar{c}_n$:

$$\frac{\partial \bar{c}_n}{\partial t} = D\,\frac{r_0^2}{(r_n)^2}\frac{\partial^2 \bar{c}_n}{\partial y^2} + \left(\frac{y}{r_n}\,\frac{dr_n}{dt} + \bar{v}_{f,n}\,\frac{R_0}{r_n}\right)\,\frac{\partial \bar{c}_n}{\partial y},\ (y,t) \in Q_{f,T},$$

$$\bar{c}_n(0,t) = c^0,\ \ D\,\frac{R_0}{R}\frac{\partial \bar{c}_n}{\partial y}(R_0,t) = -\beta\,\bar{c}_n(R_0,t),\ t \in (0,T),$$

$$\bar{c}_n(y,0) = c_{0,n}(y),\ y \in Q_f. \tag{15}$$

Here $\bar{v}_{f,n}$ is the solution to the Dynamic problem $A(r_n)$,

$$r_n \in \mathbb{H}^{1+\frac{\gamma}{2}}[0,T],\ \frac{dr_n}{dt}(t) \geqslant 0, \text{ for } 0<t<T,\ \bar{v}_{f,n} \in \mathbb{H}^{\gamma,\frac{\gamma}{2}}(\overline{Q}_{f,T}),\ c_{0,n} \in \mathbb{H}^{2+\gamma}(\overline{Q}_f),$$

$$\|r_n - r\|^{(1)}_{q,(0,T)} + \|\bar{v}_{f,n} - \bar{v}_f\|_{q,Q_{f,T}} + \|c_{0,n} - c_0\|^{(2-\frac{2}{q})}_{q,Q_f} \to 0 \text{ as } n \to \infty.$$

Theorems 5.3 and 5.4 from §5, Chapter IV in [5] state that the first boundary value problem (5.2), (5.3) and the problem with directional derivative (5.2), (5.4) have unique solution in the spaces $\mathbb{H}^{k+\gamma,\frac{k+\gamma}{2}}$, $k \in \mathbb{N}$ – integer, $k \geqslant 0$. But there is a remark that if the boundary of the domain under consideration consists of several disjoint components, then all the results are also true for problems with different types of boundary conditions on different components.

That is exactly our case, and we may state, that the problem (15) has a unique solution $\bar{c}^n \in \mathbb{H}^{2+\gamma,\frac{2+\gamma}{2}}(\bar{Q}_{f,T})$.

At the same time, this solution is a solution from the space $\mathbb{W}_q^{2,1}(Q_{f,T})$ and the following estimates

$$c_* \leqslant \bar{c}_n(y,t) \leqslant M_c^0,\ (y,t) \in Q_{f,T},\ \|\bar{c}_n\|^{(2)}_{q,Q_{f,T}}\| \leqslant M\,\|\bar{c}_{0,n}\|^{(2-\frac{2}{q})}_{q,Q_f}, \tag{16}$$

are true.

Now we consider the boundary value problem for the difference $\widetilde{c} = \bar{c}_n - \bar{c}_m$:

$$\frac{\partial \widetilde{c}}{\partial t} = D\left(\frac{r_0}{r_n}\right)^2 \frac{\partial^2 \widetilde{c}}{\partial y^2} + \left(\frac{y}{r^n}\frac{dr^n}{dt} + \bar{v}_f^n \frac{R_0}{r^n}\right)\frac{\partial \widetilde{c}}{\partial y} + a_1 \frac{d\widetilde{r}}{dt} + a_2\,\widetilde{r},\ \ (y,t) \in Q_{f,T},$$

$$\widetilde{c}(0,t) = 0,\ \ D\,\frac{R_0}{r_n}\frac{\partial \widetilde{c}}{\partial y}(R_0,t) + \beta\,\widetilde{c}(R_0,t) = b\,\widetilde{r},\ \ t \in (0,T),$$

$$\widetilde{c}(y,0) = c_{0,n}(y) - c_{0,n}(y),\ \ y \in Q_f;$$

$$a_1 = \left(\frac{y}{r_n}\frac{dr_n}{dt} + \bar{v}_{f,n}\frac{R_0}{r_n} - \frac{y}{r_m}\frac{dr_m}{dt} - \bar{v}_{f,m}\frac{R_0}{r_m}\right)\frac{\partial \bar{c}_m}{\partial y},$$

$$a_2 = D\left(\left(\frac{r_0}{r_n}\right)^2 - \left(\frac{r_0}{r_m}\right)^2\right)\frac{\partial^2 \bar{c}_m}{\partial y^2},\ \ b = D\left(\frac{R_0}{r_n} - \frac{R_0}{r_m}\right)\frac{\partial \bar{c}_m}{\partial y}.$$

Applying estimates (16) we get

$$\|\bar{c}_n - \bar{c}_m\|^{(2)}_{q,Q_{f,T}} \leqslant M\left(\|\bar{c}_{0,n} - \bar{c}_{0,n}\|^{(2-\frac{2}{q})}_{q,Q_f} + \|\bar{v}_{f,n} - \bar{v}_{f,m}\|_{q,Q_{f,T}} + \|r_n - r_m\|^{(1)}_{q,(0,T)}\right). \tag{17}$$

In turn, taken into account representation (14) we eventually obtain

$$\|\bar{c}_n - \bar{c}_m\|^{(2)}_{q,Q_{f,T}} \leqslant M\left(\|\bar{c}_{0,n} - \bar{c}_{0,n}\|^{(2-\frac{2}{q})}_{q,Q_f} + \|r_n - r_m\|^{(1)}_{q,(0,T)}\right). \tag{18}$$

Thus, the sequence $\{\bar{c}_n\}$ is a fundamental one in the space $\mathbb{W}^{2,1}_q(Q_{f,T})$ and, due to completeness in this space, converges to some function $\bar{c}$ from $\mathbb{W}^{2,1}_q(Q_{f,T})$, that satisfies the problem (13) and estimates

$$c_* \leqslant \bar{c}(y,t) \leqslant M^0_c,\ \ (y,t) \in Q_{f,T},\ \ \|\bar{c}\|^{(2)}_{q,Q_{f,T}} \leqslant M\,\|\bar{c}_0\|^{(2-\frac{2}{q})}_{q,Q_f},$$

are valid. □

Lemmas 1 and 2 result in Lemma 3.

Lemma 3. *Under conditions of Theorem 1 for any $r \in \mathfrak{M}$ the Problem $A(r)$ has a a unique solution $\{\,v_f,\, p_f,\, c, w_s\}$, $v_f \in \mathbb{L}_q\big(0,T;\mathbb{W}^1_q(\Omega_f(t)\big)$, $p_f \in \mathbb{L}_q(G_{f,r})$, $c \in \mathbb{W}^{2,1}_q(G_{f,r})$, w_s, $\dfrac{\partial w_s}{\partial x} \in \mathbb{Y}_s$, satisfies estimates (11) with the function $r(t)$ instead of the function $R(t)$.*

Let $F_v(r) \overset{def}{=} \bar{v}_f(y,t)$, $F_p(r) \overset{def}{=} \bar{p}_f(y,t)$, and $F_c(r) \overset{def}{=} \bar{c}(y,t)$, where $\{\bar{v}_f,\, \bar{p}_f,\, \bar{c}\}$ be the solution to the problem $A(r)$.

Lemma 4. *Operator $F_c : \mathbb{W}^1_q(0,T) \to \mathbb{W}^{2,1}_q(Q_{f,T})$ is continuous.*

Proof. Applying formulas (17) and (14), estimate (18), and embedding $\mathbb{W}^{2,1}_q(Q_{f,T}) \to \mathbb{C}(\overline{Q}_{f,T})$ (Lemma 3,3, §3, chapter II, [5]) we get

$$F_p(r) = p_f(x,t) = p^0_f(x)\, e^{-\delta\, t} + \delta \int_0^t e^{-\delta(t-\tau)} p^0(\tau)\, d\tau,$$

$$F_v(r) = v_f(x,t) = \frac{dr}{dt}(t) + \frac{1}{\mu}\big(p_f(x,t) - p^0(t)\big);$$

$$F_v(r_1) - F_v(r_2) = \frac{dr_1}{dt}(t) - \frac{dr_2}{dt}(t),$$

$$\|F_v(r_1) - F_v(r_2)\|_{q,Q_{f,T}} = \|r_1 - r_2\|^{(1)}_{q,(0,T)},$$

$$F_c(r_1) - F_c(r_2) = R_1(t) - R_2(t) = \int_0^t \big(\bar{c}_1(R_0,\tau) - \bar{c}_2(R_0,\tau)\big) d\tau,$$

$$|R_1(t) - R_2(t)| \leqslant \alpha\, T\, |\bar{c}_1 - \bar{c}_2|^{(0)}_{Q_{f,T}},$$

$$|\frac{dR_1}{dt}(t) - \frac{dR_2}{dt}(t)| = \alpha\, |\bar{c}_1(R_0,t) - \bar{c}_2(R_0,t)|,$$

$$\|R_1 - R_2\|^{(1)}_{q,(0,T)} \leqslant \alpha\, T^{\frac{1}{q}}\, |\bar{c}_1 - \bar{c}_2|^{(0)}_{Q_{f,T}} \leqslant$$

$$\alpha\, T^{\frac{1}{q}}\, M\, \|\bar{c}_1 - \bar{c}_2\|^{(2)}_{q,Q_{f,T}} \leqslant \alpha\, T^{\frac{1}{q}}\, M\, \|r_1 - r_2\|^{(1)}_{q,(0,T)}, \tag{19}$$

which completes the proof of the lemma. □

Now we define the operator, acting by formula

$$F(r) = R_0 + \alpha \int_0^t \bar{c}(R_0,\tau) d\tau \equiv R(t), \tag{20}$$

where $\bar{c}(y,t)$ is a solution to the problem $A(r)$. By construction $F(r)$ transforms the set $\mathfrak{M}$ into itself:

$$R(0) = R_0,\ \ \frac{dR}{dt}(t) = \alpha\, \bar{c}(R_0,t) \,\geqslant\, 0,\ \ |\frac{dR}{dt}(t)| \,\leqslant\, \alpha\, M^0_c = M^1_r,\ \ \|R\|^{(1)}_{q,(0,T)} \,\leqslant\, M^1_r.$$

Lemma 5. *The problem (1)–(10) has a unique weak solution* $\{R(t),\, v_f,\, p_f,\, c, w_s\}$, $\dfrac{dR}{dt} > 0$, $R \in \mathbb{W}^1_q(0,T_1)$, $v_f \in \mathbb{L}_q\big(0,T_1;\mathbb{W}^1_q(\Omega_f(t))\big)$, $p_f \in \mathbb{L}_q(\Omega_{f,T_1,R})$, $c \in \mathbb{W}^{2,1}_q(\Omega_{f,T_1,R})$, w_s, $\dfrac{\partial w_s}{\partial x} \in \mathbb{L}_q\big(0,T_1;\mathbb{L}_\infty(\Omega_s(t)\big) \equiv \mathbb{Y}_s$, *where* $T_1 < (\alpha\, M)^{-q}$.

Proof. Estimate (19) shows that operator F is a contraction operator in the space $\mathbb{W}^{(1)}_q(0,T_1)$ for $T_1 < (\alpha\, M)^{-q}$. Applying Banach fixed point theorem [1] we find a unique fixed point $R \in \mathbb{W}^1_q(0,T_1)$, of the operator F: $R = F(R)$, which define a unique solution to the problem (1)–(10) for $0 < t < T_1$. □

Lemma 6. *The problem (1)–(10) has a unique weak solution* $\{R(t),\, v_f,\, p_f,\, c, w_s\}$, $\dfrac{dR}{dt} > 0$, $R \in \mathbb{W}^1_q(0,T)$, $v_f \in \mathbb{L}_q\big(0,T;\mathbb{W}^1_q(\Omega_f(t))\big)$, $p_f \in \mathbb{L}_q(\Omega_{f,T,R})$, $c \in \mathbb{W}^{2,1}_q(\Omega_{f,T,R})$, w_s, $\dfrac{\partial w_s}{\partial x} \in \mathbb{L}_q\big(0,T;\mathbb{L}_\infty(\Omega_s(t)\big) \equiv \mathbb{Y}_s$ *for any* $T > 0$.

Proof. We may start from the moment $t = T_1$ and solve the problem (1)–(10) for $t > T_1$ with "initial" data $c(x,T_1) \in \mathbb{W}_q^{2-\frac{2}{q}}\big(\Omega_f(T_1)\big)$ and $p_f^{T_1}(x)$, where $p_f^{T_1}(x) = p_f(x,T_1)$ for $0 < x < R(T_1)$ and $p_f^{T_1}(x) = p_f^0(x)$ for $R(T_1) < x < L$. Step by step we find moments $0 < T_1 < \cdots < T_n < \cdots$ with $0 < R(T_1) < \cdots < R(T_n) < \cdots < L$.

We solve the problem, if for some n, $R(T_n) = L$, or $T_n = T$. Otherwise there exists a strictly monotone sequence $\{T_n\}$, such that $\lim\limits_{n\to\infty} T_n = T_* < T$ and $\lim\limits_{n\to\infty} R(T_n) = R(T_*) = R_* < L$, and some subsequence $\{\bar{c}_n\}$ (we keep for simplicity the same indices) weakly convergent in $\mathbb{W}_q^{2-\frac{2}{q}}(Q_f)$ to some function $\bar{c}_* \in \mathbb{W}_q^{2-\frac{2}{q}}(Q_f)$.

The last fact means that we may solve the problem (1)–(10) for $t > T_*$, that contradicts to our supposition and prove the theorem. □

5 Conclusion

In this paper we have proved the existence and uniqueness of a weak solution on an arbitrary time interval $(0,T)$ for a microscopic mathematical model of in-situ leaching of a rare metal. Our proof was based on the Banach fixed point theorem. Let $x = R(t)$ be a free boundary, separating domains $\Omega_{f,R}(t)$ and $\Omega_{f,R}(t)$, occupied by the liquid and solid components correspondingly.

We fix a set $\mathfrak{M}$ of functions $r(t)$ in the functional space $\mathfrak{W}^1_q(0,T)$ and for any $r \in \mathfrak{M}$, solve the problem $A(r)$, where $A(r)$ is the original problem in the known domains $\Omega_{f,r}(t)$ and $\Omega_{f,r}(t)$, occupied by liquid and solid components, respectively, but without an additional boundary condition.

The missing boundary condition forms an operator F, whose fixed points define the solution to the original problem.

Using differential properties of corresponding solutions, embedding theorems and we prove the F is a contruction operator on some small interval. Therefore, due to Banach's fixed point theorem there exists a unique fixed point, that determines the solution to the original problem.

Step by step we prove that our solution exists on an arbitrary time interval.

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