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Theory and application of the power Ailamujia distribution

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Abstract. Statistical modeling is constantly in demand for simple and flexible probability distributions. We are helping to meet this demand by proposing a new candidate extending the standard Ailamujia distribution, called the power Ailamujia distribution. The idea is to extend the adaptability of the Ailamujia distribution through the use of the power transform, introducing a new shape parameter in its definition. In particular, the new parameter is able to produce original non-monotonic shapes for the main functions that are desirable for data fitting purposes. Its interest is also shown through results about stochastic orders, quantile function, moments (raw, incomplete and probability weighted), stress-strength parameter and Tsallis entropy. New classes of distributions based on the power Ailamujia distribution are also presented. Then, we investigate the corresponding statistical model to analyze two kinds of data: complete data and data in presence of censorship. In particular, a goodness-of-fit statistical test allowing the processing of right-censored data is developed. The potential of the new model is demonstrated by its application to four data sets, two being related to the Covid-19 pandemic.

Keywords: Ailamujia distribution, power distribution, moments, stress-strength parameter, entropy, data analysis, Covid-19 pandemic.
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1 Introduction

In order to understand the proposed methodology, a retrospective on the so-called Ailamujia distribution is needed. First, the Ailamujia distribution is an univariate lifetime distribution

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depending on a single parameter, say $\theta > 0$. It is defined by the following cumulative distribution function (cdf):

$$F_o(x, \theta) = 1 - (1 + \theta x)e^{-\theta x}, \quad x \geq 0, \tag{1}$$

and $F_o(x, \theta) = 0$ otherwise. Initially, it considers a parametrization of the form $\theta = 2\alpha$, where $\alpha > 0$, just to have an elegant mathematical mean of the form $1/\alpha$ (without other additional constant). The corresponding probability density function (pdf) is quite simple; it is expressed as

$$f_o(x, \theta) = \theta^2 x e^{-\theta x}, \quad x \geq 0, \tag{2}$$

and $f_o(x, \theta) = 0$ otherwise. The Ailamujia distribution was emphasized by [25]. Technically, as defined by (1), it corresponds to the Erlang distribution with shape parameter 2 and rate parameter θ. With a parametrization of the form $\theta = 1/\gamma$ with $\gamma > 0$, it also appears under the name of length-biased exponential (LBE) distribution in [11]. It has the features to be simple, with pdf presenting right-skewed and bell shapes, and increasing hazard rate function (hrf). Thus, it can be viewed as a suitable alternative to other well established one-parameter lifetime distributions, such as the exponential and Lindley distributions (see [23]). The basics on the Ailamujia distribution can be found in [25], including the estimation of the parameter via the maximum likelihood method.

Over the last years, the Ailamujia distribution has also received a particular treatment. For the statistical developments, we may refer the reader to [28] for the parameter estimation via confidence intervals and statistical tests, Reference [33] showing how the Ailamujia distribution can be applied for maintenance-decision-oriented modeling, Reference [24] for the parameter estimation with Bayesian techniques, and Reference [22] developing the minimax estimation of the parameter under different loss functions.

In order to overcome some limitations of the Ailamujia distribution for modeling purposes, such as the lack of non-monotonic shapes for the hrf, modern extensions of it have been proposed. Among them, let us evoke the weighted version introduced in [17], the size biased version proposed in [30], and the new generalization developed in [18]. We may also refer to [1], [14] and [16] developing extensions of the LBE distribution connected to the Ailamujia distribution. The one in [1] is based on the exponential transmuted transformation, the one in [16] uses the Marshall-Olkin transformation and the one in [14] is based on the odd Weibull transformation of the LBE distribution. All these extensions are applied in concrete scenarios, showing how derivations of the Ailamujia and LBE distributions can be efficient for various data analyses.

In this paper, we propose an extension of the Ailamujia distribution through the use of the power transform. It is based on a quite natural idea: adding a shape parameter $\beta > 0$ to the former Ailamujia distribution in order to increase its overall flexibility. More precisely, we define the power Ailamujia (PA) distribution with parameters β and θ by the distribution of the random variable $Y = X^{1/\beta}$, where X denotes a random variable having the Ailamujia distribution with parameter θ. That is, it constitutes a new two-parameter lifetime distribution specified by the following cdf:

$$F(x, \varpi) = 1 - (1 + \theta x^{\beta})e^{-\theta x^{\beta}}, \quad x \geq 0, \tag{3}$$

and $F(x, \varpi) = 0$ otherwise, where $\varpi = (\beta, \theta)^T$ is the parameter vector. To draw parallels, the PA distribution is to the Ailamujia distribution what the Weibull distribution is to the exponential distribution or what the power Lindley distribution is to the Lindley distribution (see [15]). To make some links with the existing literature, when $\beta = 1$, the PA distribution is reduced to the former Ailamujia distribution. Moreover, the PA distribution is also a special case of the weighted Weibull distribution as introduced by [13], defined with the shape parameter $c = 1$. However, to our knowledge, this special case has not received particular attention. Among the interesting facts of the PA distribution, the following ones are discussed:

(i) The corresponding pdf and hrf are very pliant, presenting monotonic and non-monotonic shapes; the pdf presents decreasing or bell-like shapes, and the hrf has decreasing, concave increasing, convex increasing and upside-down bathtub shapes, contrary to the pdf and hrf of the Ailamujia distribution.

(ii) It enjoys interesting stochastic ordering results.

(iii) The quantile function has a closed-form expression.

(iv) The moments are easily defined via the Euler gamma function and can be used to define various characteristics of importance.

(v) The expression of the stress-strength parameter is tractable, which makes it statistically attractive.

(vi) The Tsallis entropy is quite manageable.

For further probabilistic or statistical aims, some possible classes of distributions derived to the PA distribution are also examined. The inferential complexity is favorable; the PA model is easily implementable and can be used for diverse statistical purposes. We discuss these aspects for the treatment of complete data and data in presence of censorship. The parameters of the model are estimated following the maximum likelihood approach. The behavior of this approach is attested by simulated experiments. In addition, based on the methodology of [5], a goodness-of-fit statistical test allowing the processing of right-censored data is developed. The fits of four practical data sets are performed, including two being related to the Covid-19 pandemic. The obtained results are quite favorable to the PA model.

We organize the paper as follows. Section 2 contains all the essential results on the proposed distribution. Its applicability is discussed in Section 3. Goodness-of-fit tests for complete and right-censored data are developed in Section 4. Finally, the paper concludes in Section 5.

2 Results on the PA distribution

Diverse results on the PA distribution are presented in this section. When possible, the Ailamujia and PA distributions are compared on the considered aspects.

2.1 Analytical study of the pdf

Upon a simple differentiation of $F(x,\varpi)$ in (3), the pdf of the PA distribution with vector of parameters ϖ, or with parameters $\beta > 0$ and $\theta > 0$, is given as

$$f(x,\varpi) = \theta^2 \beta x^{2\beta-1} e^{-\theta x^\beta}, \quad x \geq 0, \tag{4}$$

and $f(x,\varpi) = 0$ otherwise. Diverse analytical results of $f(x,\varpi)$ are now discussed, starting with the asymptotic properties.

Based on (4), when $x \to 0$, we have $f(x,\varpi) \sim \theta^2\beta x^{2\beta-1}$. Thus, for $\beta < 1/2$, we get $\lim_{x\to 0} f(x,\varpi) = +\infty$, for $\beta = 1/2$, we have $\lim_{x\to 0} f(x,\varpi) = \theta^2/2$, and for $\beta > 1/2$, we obtain $\lim_{x\to 0} f(x,\varpi) = 0$. In all cases, we have $\lim_{x\to +\infty} f(x,\varpi) = 0$.

A modes analysis of the PA distribution is performed in the next result.

Proposition 1. *The PA distribution is unimodal and*

- *if $\beta \leq 1/2$, the mode is given as $x_* = 0$,*
- *if $\beta > 1/2$, the mode is defined by*

$$x_{**} = \left(\frac{2\beta - 1}{\theta\beta}\right)^{1/\beta}.$$

Proof. The modes analysis of a distribution consists in studying the existence and definition of the maximum(s) of the corresponding pdf. In the context of the PA distribution, for $x \geq 0$, after developments and simplifications, we get

$$f'(x,\varpi) = -\theta^2\beta x^{2\beta-2} e^{-\theta x^\beta}\left[\beta(\theta x^\beta - 2) + 1\right].$$

Therefore, $f'(x,\varpi) = 0$ implies that $x = x_*$ or $x = x_{**}$, depending on the values of β. Therefore, since $f(x,\varpi) \geq 0$, in view of the limit properties of $f(x,\varpi)$, it is clear that x_* is a maximum for $f(x,\varpi)$ if $\beta \leq 1/2$, and x_{**} is a maximum for $f(x,\varpi)$ if $\beta > 1/2$. This ends the proof of Proposition 1. □

Proposition 1 illustrates the importance of the parameter β in the definition of the modes. If $\beta > 1/2$, this is crucial in understanding the peak observed in the pdf curve.

2.2 Analytical study of the hrf

Based on (3) and (4), the corresponding hrf is given as

$$h(x,\varpi) = \theta^2\beta \frac{x^{2\beta-1}}{1+\theta x^\beta}, \quad x \geq 0, \tag{5}$$

and $h(x,\varpi) = 0$ otherwise.

Some asymptotic results of $h(x,\varpi)$ are now discussed. Based on (5), when $x \to 0$, we have $h(x,\varpi) \sim \theta^2\beta x^{2\beta-1}$. Hence, for $\beta < 1/2$, we obtain $\lim_{x\to 0} h(x,\varpi) = +\infty$, for $\beta = 1/2$, we have

$\lim_{x\to 0} h(x,\varpi) = \theta^2/2$, and for $\beta > 1/2$, we get $\lim_{x\to 0} h(x,\varpi) = 0$. Also, when $x \to +\infty$, we have $h(x,\varpi) \sim \theta\beta x^{\beta-1}$. Hence, for $\beta < 1$, we get $\lim_{x\to+\infty} h(x,\varpi) = 0$, for $\beta = 1$, we obtain $\lim_{x\to+\infty} h(x,\varpi) = \theta$, and for $\beta > 1$, we have $\lim_{x\to+\infty} h(x,\varpi) = +\infty$.

The shapes behavior of $h(x,\varpi)$ is studied in the following proposition.

Proposition 2. *The hrf $h(x,\varpi)$ satisfies the following shapes properties, depending on the values of β:*

- *for $1/2 < \beta < 1$, $h(x,\varpi)$ has a upside-down bathtub shape,*
- *for $\beta \leq 1/2$, $h(x,\varpi)$ is decreasing,*
- *for $\beta \geq 1$, $h(x,\varpi)$ is increasing.*

Proof. The derivative of $h(x,\varpi)$ is the key; for $x \geq 0$, we have

$$h'(x,\varpi) = \theta^2\beta x^{2\beta-2}\frac{2\beta + (\beta-1)\theta x^\beta - 1}{(1+\theta x^\beta)^2}.$$

Therefore, $h'(x,\varpi) = 0$ implies that $x = 0$ or $x = \{(2\beta-1)/[\theta(1-\beta)]\}^{1/\beta}$, depending on the values of β. Now, remark that $x_o = \{(2\beta-1)/[\theta(1-\beta)]\}^{1/\beta}$ is into $[0,+\infty)$ if $1/2 < \beta < 1$. Therefore, since $h(x,\varpi) \geq 0$, in view of the limit properties of $h(x,\varpi)$, the following results hold:

- for $1/2 < \beta < 1$, $h(x,\varpi)$ has a upside-down bathtub shape, attaining its maximum at x_o,
- for $\beta \leq 1/2$, we have $2\beta + (\beta-1)\theta x^\beta - 1 \leq -(\theta/2)x^\beta \leq 0$, implying that $h'(x,\varpi) \leq 0$, and $h(x,\varpi)$ is decreasing with a maximum at $x = 0$,
- for $\beta \geq 1$, we have $2\beta + (\beta-1)\theta x^\beta - 1 \geq 1 > 0$, implying that $h'(x,\varpi) \geq 0$, and $h(x,\varpi)$ is increasing with a minimum at $x = 0$.

This completes the proof of Proposition 2. □

Proposition 2 points out the importance of the parameter β in the versatility of the shapes of the hrf.

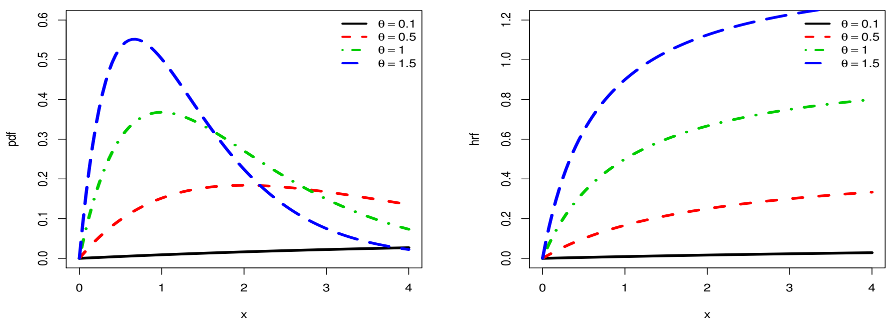
Propositions 1 and 2 give mathematical elements on the flexibility of the PA distribution.

2.3 Graphical comparison

Now, we complete the two above parts by comparing graphically the panel of shapes of the pdfs and hrfs of the standard Ailamujia and PA distributions, respectively. First, Figure 1 presents the possible shapes of the pdf and hrf of the Ailamujia distribution for various values of θ.

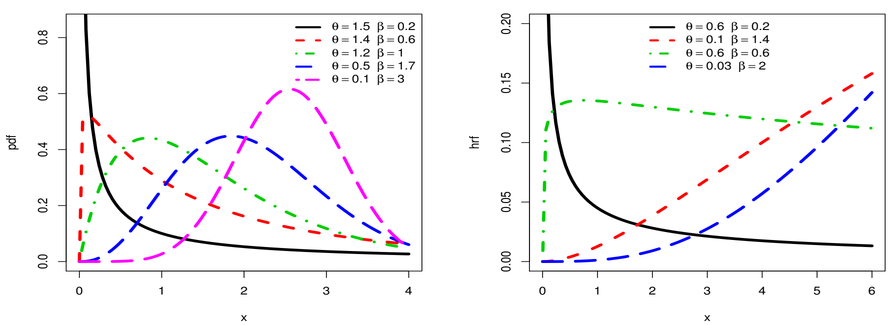
From Figure 1, we see that the pdf of the standard Ailamujia distribution has right-skewed bell shapes, with more or less weight on the tails, and the corresponding hrf is only concave and increasing.

Figure 2 presents diverse shapes of the pdf and hrf of the PA distribution for various values of θ and β, illustrating some special cases of Propositions 1 and 2.



(a) (b)

Figure 1: Plots of the (a) pdf and (b) hrf of the standard Ailamujia distribution.



(a) (b)

Figure 2: Plots of the (a) pdf and (b) hrf of the PA distribution.

In Figure 2, we see that the pdf of the PA distribution has decreasing or bell-like shapes, with various features on the skewness and kurtosis. Also, the corresponding hrf has decreasing, concave increasing, convex increasing and upside-down bathtub shapes.

Based on these figures, it is clear that the functions of the PA distribution are more pliant to those of the former Ailamujia distribution, mainly thanks to the presence of the shape parameter β.

2.4 Stochastic ordering results

The PA and Weibull distributions are involved in a simple stochastic order dominance, as stated in the result below.

Proposition 3. *The PA distribution first-order stochastically dominates the Weibull distribu-*

tion, that is, for any $x \in \mathbb{R}$, we have

$$F(x,\varpi) \le G(x,\varpi),$$

where $F(x,\varpi)$ is given in (3) and $G(x,\varpi)$ is the cdf of the Weibull distribution with parameters β and θ, i.e., $G(x,\varpi) = 1 - e^{-\theta x^{\beta}}$ with $x \ge 0$, and 0 otherwise.

Proof. From (3), the inequality holds for $x < 0$ since $F(x,\varpi) = G(x,\varpi) = 0$. For $x \ge 0$, since $1 + \theta x^{\beta} \ge 1$, we immediately get

$$F(x,\varpi) = 1 - (1+\theta x^{\beta})e^{-\theta x^{\beta}} \le 1 - e^{-\theta x^{\beta}} = G(x,\varpi).$$

The desired first-order stochastic property is established. □

Proposition 3 shows that the PA and Weibull distributions do not intersect in the stochastic sense; the PA model can be applied when the Weibull model falls from some statistical purposes.

Another stochastic ordering aspect of the PA distribution is described in the next result. It shows that the PA distribution enjoys a simple likelihood ratio-order. See [8] for the essentials on this probabilistic concept.

Proposition 4. *Let Y_1 and Y_2 be two random variables following the PA distribution with parameters $\beta > 0$, and $\theta_1 > 0$ and $\theta_2 > 0$, respectively. Then, if $\theta_1 \ge \theta_2$, Y_2 is greater to Y_1 according to the likelihood ratio-ordering, i.e., $Y_1 \le_{lr} Y_2$. Mathematically, this is equivalent to say that the function $\hbar(x,\varpi_1,\varpi_2) = f(x,\varpi_1)/f(x,\varpi_2)$, where $\varpi_1 = (\beta,\theta_1)^T$ and $\varpi_2 = (\beta,\theta_2)^T$, is decreasing with respect to x.*

Proof. The proof is mainly based on the expression in (4) and some simplification. Indeed, for $x \ge 0$, we have

$$\hbar(x,\varpi_1,\varpi_2) = \left(\frac{\theta_1}{\theta_2}\right)^2 e^{-(\theta_1-\theta_2)x^{\beta}},$$

which is clearly decreasing if $\theta_1 \ge \theta_2$ by composition. The desired result is proved. □

The likelihood ratio-order described in Proposition 4 has numerous consequences, implying inequalities on diverse probabilistic functions and moments. The essentials are available in [8].

2.5 Quantile function

The quantile function of the PA distribution is derived to the inverse function of $F(x,\varpi)$. It is specified in the next theorem.

Theorem 1. *The quantile function of the PA distribution is expressed as*

$$Q(y,\varpi) = \left(-\frac{1}{\theta}\left[W(e^{-1}(y-1))+1\right]\right)^{1/\beta}, \quad y \in (0,1),$$

where $W(x)$ denotes the well-known Lambert function.

Proof. By its definition, $Q(y, \varpi)$ satisfies $F[Q(y, \varpi), \varpi] = y$ with $y \in (0, 1)$. From this equality, by using (3) and the property of the Lambert function, i.e., $W(x)e^{W(x)} = x$, the following chain of equivalences holds:

$$
\begin{aligned}
1 - (1 + \theta Q(y, \varpi)^{\beta})e^{-\theta Q(y,\varpi)^{\beta}} = y &\Leftrightarrow (1 + \theta Q(y, \varpi)^{\beta})e^{-\theta Q(y,\varpi)^{\beta}} = 1 - y \\
&\Leftrightarrow -(1 + \theta Q(y, \varpi)^{\beta})e^{-(1+\theta Q(y,\varpi)^{\beta})} = e^{-1}(y - 1) \\
&\Leftrightarrow -(1 + \theta Q(y, \varpi)^{\beta}) = W(e^{-1}(y - 1)) \\
&\Leftrightarrow \theta Q(y, \varpi)^{\beta} = -W(e^{-1}(y - 1)) - 1 \\
&\Leftrightarrow Q(y, \varpi) = \left(-\frac{1}{\theta}\left[W(e^{-1}(y - 1)) + 1\right]\right)^{1/\beta}.
\end{aligned}
$$

The desired result is obtained. □

The closed-form expression of the quantile function in Theorem 1 is an advantage of the PA distribution. From it, we can obtain the main quartiles and define some measures of skewness and kurtosis. It is also fundamental for obtaining simulated values from the PA distribution in a simple way.

2.6 Moments

Let Y be a random variable following the PA distribution with a vector of parameters ϖ, that is, with cdf and pdf specified by (3) and (4), respectively. The s^{th} moment of Y is presented below.

Proposition 5. *Let s be an integer. Then, the s^{th} moment of Y is given as*

$$m_s = E(Y^s) = \theta^{-s/\beta}\Gamma\left(\frac{s}{\beta} + 2\right), \tag{6}$$

where $\Gamma(x)$ denotes the standard Euler gamma function.

Proof. The transfer formula gives

$$m_s = \int_0^{+\infty} x^s f(x, \varpi)dx = \theta^2\beta \int_0^{+\infty} x^{s+2\beta-1}e^{-\theta x^{\beta}}dx.$$

One can remark that there is no problem of convergence for this integral. By making the change of variable $y = \theta x^{\beta}$, the desired result follows:

$$m_s = \theta^{-s/\beta}\int_0^{+\infty} x^{s/\beta+1}e^{-y}dy = \theta^{-s/\beta}\Gamma\left(\frac{s}{\beta} + 2\right).$$

The proof of Proposition 5 ends. □

From Proposition 5, the mean of Y is given as

$$\mu = m_1 = \theta^{-1/\beta}\Gamma\left(\frac{1}{\beta} + 2\right)$$

and the variance of Y is obtained as

$$\sigma^2 = m_2 - m_1^2 = \theta^{-2/\beta}\left[\Gamma\left(\frac{2}{\beta}+2\right) - \Gamma\left(\frac{1}{\beta}+2\right)^2\right].$$

As a complementary indicator, the coefficient of variation of Y is defined by

$$CV = \frac{\sigma}{\mu} = \sqrt{\frac{\Gamma\left(\frac{2}{\beta}+2\right)}{\Gamma\left(\frac{1}{\beta}+2\right)^2} - 1}.$$

We can notice that CV only depends on β; θ has no influence on it. The s^{th}non-central moment of Y is defined by $m_s^* = E[(Y-\mu)^s]$. That is, by applying the binomial formula, we can express it as

$$m_s^* = \theta^{-s/\beta} s! \sum_{k=0}^{s} \frac{1}{k!(s-k)!}(-1)^{s-k}\Gamma\left(\frac{1}{\beta}+2\right)^{s-k}\Gamma\left(\frac{k}{\beta}+2\right).$$

From the first four non-central moments of Y, we can define the general coefficient of Y as $C_s = m_s^*/\sigma^s$. The so-called coefficients of skewness and kurtosis correspond to C_3 and C_4, respectively.

Also, the incomplete moments of Y can be determined in a similar way to the moments; by following the proofs of Proposition 5, we establish that the s^{th} incomplete moments of Y at $t \geq 0$ is given by

$$m_s(t) = E(Y^s I(Y \leq t)) = \theta^{-s/\beta}\gamma\left(\frac{s}{\beta}+2, \theta t^\beta\right),$$

where $\gamma(a, x)$ denotes the lower incomplete gamma function. Basically, the moments of Y follow by applying $t \to +\infty$. Also, from this formula, one can define various probabilistic measures, such as the mean residual life, Bonferroni, Zenga and Lorenz curves, and so on. See, for instance, [10].

We complete this part by exhibiting the probability weighted moments of Y, which reveal to be quite manageable. First, for any integers s and u, let us define the $(s,u)^{th}$ probability weighted moment of Y as $m_{s,u} = E(Y^s(1 - F(Y, \varpi))^u)$. Owing to the binomial formula and Proposition 5 with $s + k\beta$ instead of s (the formula remaining true for any real number, not only integer), we get

$$m_{s,u} = \theta^{-s/\beta}\frac{1}{(u+1)^{s/\beta+2}}\sum_{k=0}^{u}\binom{u}{k}\frac{1}{(u+1)^k}\Gamma\left(\frac{s}{\beta}+k+2\right).$$

The moments of Y are deduced by taking $u = 0$. More generally, this result is of interest for further moments analysis of the order statistics, among others. See, for instance, [4].

2.7 Stress-strength parameter

The stress-strength parameter measures the probability that a random strength exceeds the random stress in a certain system. Thus, we can model the strength and stress of the system by two independent random variables, say Y_1 and Y_2, and the stress-strength parameter is defined by $R = P(Y_1 \geq Y_2)$. We may refer to [31] for further detail. Here, we express it in a special case of the PA distribution.

Theorem 2. *Assume that Y_1 and Y_2 follow the PA distribution with parameters $\beta > 0$, and $\theta_1 > 0$ and $\theta_2 > 0$, respectively. Then, we have*

$$R = 1 - \frac{\theta_1^2(\theta_1 + 3\theta_2)}{(\theta_1 + \theta_2)^3}.$$

Proof. From the independence of Y_1 and Y_2, (3) and (4), by denoting $\varpi_1 = (\beta, \theta_1)^T$ and $\varpi_2 = (\beta, \theta_2)^T$, we can write

$$\begin{aligned} R = P(Y_1 \geq Y_2) &= \int_0^{+\infty} F(x, \varpi_2) f(x, \varpi_1) dx \\ &= \int_0^{+\infty} \left[1 - (1 + \theta_2 x^\beta) e^{-\theta_2 x^\beta}\right] \theta_1^2 \beta x^{2\beta - 1} e^{-\theta_1 x^\beta} dx \\ &= 1 - \theta_1^2 \beta \int_0^{+\infty} x^{2\beta-1} e^{-(\theta_1+\theta_2)x^\beta} dx - \theta_2 \theta_1^2 \beta \int_0^{+\infty} x^{3\beta-1} e^{-(\theta_1+\theta_2)x^\beta} dx. \end{aligned}$$

By applying the change of variable $y = (\theta_1 + \theta_2)x^\beta$, and using $\Gamma(2) = 1$ and $\Gamma(3) = 2$, we get

$$R = 1 - \frac{\theta_1^2}{(\theta_1+\theta_2)^2}\Gamma(2) - \frac{\theta_2\theta_1^2}{(\theta_1+\theta_2)^3}\Gamma(3) = 1 - \frac{\theta_1^2}{(\theta_1+\theta_2)^2} - \frac{2\theta_2\theta_1^2}{(\theta_1+\theta_2)^3} = 1 - \frac{\theta_1^2(\theta_1+3\theta_2)}{(\theta_1+\theta_2)^3}.$$

The stated formula is obtained. □

This expression is quite simple, and can be the object of a future statistical study (proceeding as in [21], for instance). As a last remark, when $\theta_1 = \theta_2$, we obtain the expected $R = 1/2$.

2.8 Tsallis entropy

Let Y be a random variable following the PA distribution with vector of parameters ϖ, that is with cdf and pdf specified by (3) and (4), respectively. The randomness of Y can be evaluated through the use of various entropy measures (see [3]). Here, we focus our attention on the Tsallis entropy of Y defined by

$$T_\gamma = \frac{1}{\gamma - 1}\left(1 - \int_0^{+\infty} f(x, \varpi)^\gamma dx\right),$$

where $\gamma \neq 1$ and $\gamma > 0$. A closed-form expression of T_γ is now investigated.

Theorem 3. *The Tsallis entropy of the PA distribution exists for $\gamma(2\beta - 1) > -1$, and it is given by*

$$T_\gamma = \frac{1}{\gamma - 1}\left[1 - \theta^{(\gamma-1)/\beta}\beta^{\gamma-1}\gamma^{-2\gamma+(\gamma-1)/\beta}\Gamma\left(\frac{\gamma}{\beta}(2\beta - 1) + \frac{1}{\beta}\right)\right].$$

Proof. First, let us study the convergence of T_γ. When $x \to 0$, we have $f(x,\varpi)^\gamma \sim \theta^{2\gamma}\beta^\gamma x^{(2\beta-1)\gamma}$ and, for all the possible values of the parameters, we have $\lim_{x\to+\infty} x^2 f(x,\varpi)^\gamma = 0$. The Riemann integral criteria implies that T_γ converges if and only if $(2\beta-1)\gamma > -1$. In this case, we have

$$\int_0^{+\infty} f(x,\varpi)^\gamma dx = \theta^{2\gamma}\beta^\gamma \int_0^{+\infty} x^{\gamma(2\beta-1)} e^{-\gamma\theta x^\beta} dx.$$

By making the change of variable $y = \gamma\theta x^\beta$, after some developments, we get

$$\int_0^{+\infty} f(x,\varpi)^\gamma dx = \theta^{(\gamma-1)/\beta}\beta^{\gamma-1}\gamma^{-2\gamma+(\gamma-1)/\beta}\Gamma\left(\frac{\gamma}{\beta}(2\beta-1)+\frac{1}{\beta}\right).$$

We finally establish that, for $\gamma \neq 1$, $\gamma > 0$ and $\gamma(2\beta-1) > -1$,

$$T_\gamma = \frac{1}{\gamma-1}\left[1-\theta^{(\gamma-1)/\beta}\beta^{\gamma-1}\gamma^{-2\gamma+(\gamma-1)/\beta}\Gamma\left(\frac{\gamma}{\beta}(2\beta-1)+\frac{1}{\beta}\right)\right].$$

The desired result is get. □

The effects of all the parameters impact on this entropy measure; β has a crucial role in this regard, mainly for the gamma term.

2.9 On some new classes of distributions

We now present some general classes of distributions based on common approaches and the PA distribution. First, consider a generic continuous distribution having a cdf denoting by $G(x)$, possibly of several parameters that we do not formalize. Then, the following classes of distributions are natural to consider for further statistical purposes.

- The odd-power Ailamujia (OPA) class of distributions defined by the following cdf:

$$F_{OPA}(x,\varpi) = 1-\left[1+\theta\left(\frac{G(x)}{1-G(x)}\right)^\beta\right]e^{-\theta[G(x)/(1-G(x))]^\beta}, \quad x \in \mathbb{R},$$

  where ϖ denotes a vector of parameters containing β, θ and the eventual parameter(s) of $G(x)$. The expression of $F_{OPA}(x,\varpi)$ comes from the T-X transformation by [2], the PA distribution and the use of the odd function.

- The log-power Ailamujia (LPA) class of distributions specified by the following cdf:

$$F_{OPA}(x,\varpi) = 1-\left\{1+\theta\left[-\log(1-G(x))\right]^\beta\right\}e^{-\theta[-\log(1-G(x))]^\beta}, \quad x \in \mathbb{R}.$$

  This definition is derived to the T-X transformation by [2], the PA distribution and the use of the logarithmic function.

- The truncated-power Ailamujia (TPA) class of distributions defined by the following cdf:

$$F_{TPA}(x,\varpi)=\frac{1}{1-(1+\theta)e^{-\theta}}\left[1-(1+\theta G(x)^{\beta})e^{-\theta G(x)^{\beta}}\right],\quad x\in\mathbb{R}.$$

  This expression comes from the truncation transformation over $(0,1)$ as described in [26] and the PA distribution.

These three classes of distributions can be generalized by replacing $G(x)$ by its exponentiated version, i.e., $G(x)^{\alpha}$ with $\alpha>0$. More details on the techniques allowing to construct useful classes of distributions can be found in [10].

3 Maximum likelihood method

The PA model is characterized by the cdf and pdf specified by (3) and (4), respectively. The parameters β and θ are supposed to be unknown. As the first step in any data analysis through the PA model, we need to estimate these parameters via the data. These data are supposed to be observations of random variable Y following the PA distribution. Here, the observations can be of two complementary natures: complete or right-censored. Among the available methods in the literature, we focus on the more popular one: the maximum likelihood method. The complete theory of this method with various applications can be found in [9].

3.1 Complete observations case

The basics of the maximum likelihood method in the context of the PA model for the complete observations case are described below. By denoting $x_1,\ldots,x_n$ the observations of Y, the maximum likelihood estimates (MLEs) of β and θ, denoted by $\widehat{\beta}$ and $\widehat{\theta}$, respectively, are obtained from the following maximization definition:

$$\widehat{\varpi}=\operatorname{argmax}_{\varpi\in[0,+\infty)^2}\ell(\varpi),$$

where $\widehat{\varpi}=(\widehat{\beta},\widehat{\theta})^T$, and $\ell(\varpi)$ denotes the log-likelihood function defined by

$$\ell(\varpi)=\sum_{i=1}^{n}\log[f(x_i,\varpi)]=2n\log\theta+n\log\beta+(2\beta-1)\sum_{i=1}^{n}\log x_i-\theta\sum_{i=1}^{n}x_i^{\beta}.$$

The MLEs are thus the solutions of the following score equations:

$$\frac{\partial\ell}{\partial\beta}(\varpi)=\frac{n}{\beta}+2\sum_{i=1}^{n}\log x_i-\theta\sum_{i=1}^{n}x_i^{\beta}\log x_i=0,\quad \frac{\partial\ell}{\partial\theta}(\varpi)=\frac{2n}{\theta}-\sum_{i=1}^{n}x_i^{\beta}=0.$$

That is, $\widehat{\beta}$ and $\widehat{\theta}$ satisfied

$$\frac{n}{\widehat{\beta}}+2\sum_{i=1}^{n}\log x_i-\left[\frac{1}{2n}\sum_{i=1}^{n}x_i^{\widehat{\beta}}\right]^{-1}\sum_{i=1}^{n}x_i^{\widehat{\beta}}\log x_i=0,\quad \widehat{\theta}=\left[\frac{1}{2n}\sum_{i=1}^{n}x_i^{\widehat{\beta}}\right]^{-1},$$

but the exact formulas for $\widehat{\beta}$ and $\widehat{\theta}$ involving $x_1, \ldots, x_n$ are not available. In practice, this is however not an obstacle: we can easily determine the numerical values of $\widehat{\beta}$ and $\widehat{\theta}$ through the use of a current statistical software. Here, the R software is used[1].

In order to check the efficiency of the obtained MLEs, a short simulation work is now proposed. We generated $N = 10000$ complete samples of observations from the PA model with parameters $\beta = 1.5$ and $\theta = 0.4$ with a fixed size, repeating this procedure for different sizes. Then, we calculate the MLEs of the unknown parameters and their mean squared errors (MSEs). These numerical quantities are given in Table 1.

Table 1: Mean simulated values of the MLEs and MSEs for the completed case.

$N = 10000$	$n = 15$	$n = 25$	$n = 50$	$n = 130$	$n = 350$	$n = 500$
$\widehat{\beta}$	1.5212	1.5192	1.5169	1.5103	1.5074	1.5012
MSE	0.0076	0.0067	0.0048	0.0032	0.0019	0.0010
$\widehat{\theta}$	0.3788	0.3807	0.3843	0.3890	0.3901	0.3994
MSE	0.0089	0.0074	0.0052	0.0038	0.0028	0.0015

In Table 1, we see that the MLEs agree closely with the true parameter values. This confirms the consistency of the obtained MLEs.

We now consider several criteria allowing us to compare numerically the fits of several statistical models. Among these criteria, there are the Anderson-Darling (AD), Cramér-von Mises (CVM), Kolmogorov-Smirnov (KS) and its p-value, as well as Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC). The lower the values of AD, CVM, KS, AIC, CAIC, BIC and HQIC, and the greater values of KS p-value, the better the fit.

We now apply our methodology to the vinyl chloride data set given below: {5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2}. All the details on this data set are available in [7], [6] and [20]. Also, as competitors of the PA model, we consider the classic Weibull (W) model, power Lindley (PL) model, classic gamma (Ga) model and Ailamujia (A) model.

Table 2 presents the values of CVM, AD, KS, p-value and MLEs for the mentioned models.

Since it has the lowest CVM, AD, KS and the greatest p-value, the PA model is the more adequate to fit the data. We complete this analysis with Table 3, displaying the values of $-\widehat{\ell}$, AIC, CAIC, BIC and HQIC.

From Table 3, we see that the PA model is the best, having the lowest AIC, CAIC, BIC and HQIC. Finally, Figure 3 illustrates the obtained fits of the considered models, showing how good the fits of the PA model is.

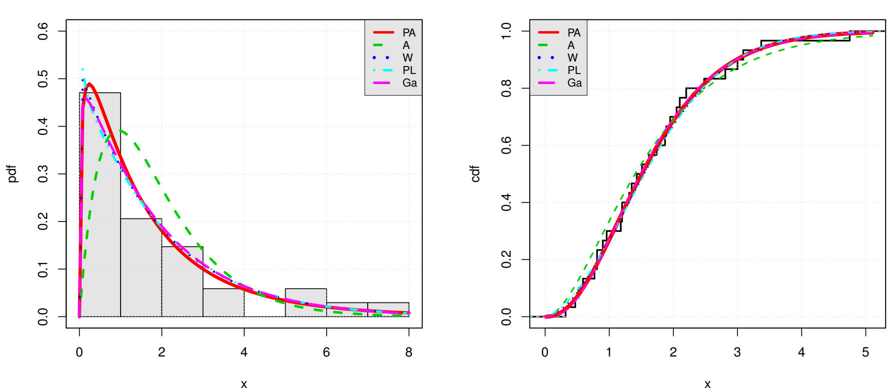
[1] R Development Core Team, R: *A language and environment for statistical computing*, R Foundation for Statistical Computing, Vienna, Austria (2005), ISBN 3-900051-07-0, URL: http://www.R-project.org.

Table 2: Goodness-of-fit measures of the models for the vinyl chloride data.

Model	CVM	AD	KS	p-value	MLEs	
PA	0.0334	0.2191	0.0822	0.9756	1.4416	0.6819
(θ, β)					(0.2030)	(0.0868)
A	0.0465	0.3017	0.2182	0.0785	1.0641	-
(θ)					(0.1290)	-
W	0.0462	0.3000	0.0918	0.9366	0.5262	1.0102
(α, β)					(0.1176)	(0.1326)
PL	0.0531	0.3431	0.0943	0.9226	0.8832	0.9139
(λ, η)					(0.1094)	(0.1452)
Ga	0.0459	0.2975	0.0973	0.9041	1.0626	0.5654
(a, b)					(0.2281)	(0.1535)

Table 3: The values of $-\widehat{\ell}$, AIC, CAIC, BIC and HQIC for the vinyl chloride data.

Model	$-\widehat{\ell}$	AIC	CAIC	BIC	HQIC
PA	55.0126	114.0253	114.4124	117.0781	115.0664
A	60.6572	123.3146	123.4396	124.8409	123.8351
W	55.4496	114.8992	115.2863	117.9520	115.9403
PL	55.7599	115.5198	115.9069	118.5726	116.5609
Ga	55.4131	114.8263	115.2134	117.8790	115.8674



(a) (b)

Figure 3: Plots of the (a) estimated pdfs and (b) estimated cdfs of the considered models.

3.2 Right censored observations case

Let $c_1, \ldots, c_n$ be n right censored observations with fixed censoring time τ. Hence, each c_i can be written as $c_i = (x_i, \delta_i)$, where $\delta_i = 0$ if x_i is a censoring time and $\delta_i = 1$ if x_i is a failure time, where $x_1, \ldots, x_n$ denotes observations from a random variable following the PA distribution. The censorship is assumed to be non informative. Then, the MLEs of β and θ, denoted by $\widehat{\beta}$ and $\widehat{\theta}$, respectively, are obtained such that

$$\widehat{\varpi} = \operatorname{argmax}_{\varpi \in [0,+\infty)^2} \ell(\varpi),$$

where $\widehat{\varpi} = (\widehat{\beta}, \widehat{\theta})^T$, and $\ell(\varpi)$ denotes the log-likelihood function defined by

$$\ell(\varpi) = \sum_{i=1}^{n} \delta_i \log[h(c_i, \varpi)] + \sum_{i=1}^{n} \log[S(c_i, \varpi)],$$

where $S(x, \varpi) = 1 - F(x, \varpi)$, that is

$$\ell(\varpi) = \sum_{i=1}^{n} \delta_i \left[\log\beta + 2\log\theta + (2\beta - 1)\log c_i - \log(1 + \theta c_i^{\beta})\right] + \sum_{i=1}^{n} \log(1 + \theta c_i^{\beta}) - \theta \sum_{i=1}^{n} c_i^{\beta}.$$

The MLEs are thus the solutions of the following score equations:

$$\frac{\partial \ell}{\partial \beta}(\varpi) = \sum_{i=1}^{n} \delta_i \left[\frac{1}{\beta} + 2\log c_i - \frac{\theta c_i^{\beta} \log c_i}{1 + \theta c_i^{\beta}}\right] + \sum_{i=1}^{n} \frac{\theta c_i^{\beta} \log c_i}{1 + \theta c_i^{\beta}} - \theta \sum_{i=1}^{n} c_i^{\beta} \log c_i = 0,$$

and

$$\frac{\partial \ell}{\partial \theta}(\varpi) = \sum_{i=1}^{n} \delta_i \left[\frac{2}{\theta} - \frac{c_i^{\beta}}{1 + \theta c_i^{\beta}}\right] + \sum_{i=1}^{n} \frac{c_i^{\beta}}{1 + \theta c_i^{\beta}} - \sum_{i=1}^{n} c_i^{\beta} = 0.$$

The exact formulas for $\widehat{\beta}$ and $\widehat{\theta}$ involving $x_1, \ldots, x_n$ are not available, but numerical methods can be used for evaluation, such as the Newton-Raphson or Monte Carlo methods. For these methods, we refer the reader to the book of [12].

We complete this part by a short simulation work. As for the complete case, we generate $N = 10000$ samples of right-censored observations with fixed size from the PA model with parameters $\beta = 1.5$ and $\theta = 0.4$, repeating the operation for different sample sizes. Then, we calculate the MLEs of the unknown parameters and their mean squared errors (MSEs). The obtained numerical quantities are given in Table 4.

In Table 4, we see that the MLEs agree closely with the true parameter values. This confirms the consistency of the obtained MLEs. Application on real data will be provided later, with the use of an adequate statistical test.

4 Goodness-of-fit tests

To check the validity of any statistical model, different criteria may be used. When data are complete, the most common tests are those based on empirical functions such as Kolmogorov-Smirnov, Anderson-Darling and others statistics, or the likelihood ratio test, AIC, BIC or chi-square test. When the parameters are unknown, they are replaced by their estimates.

Table 4: Mean simulated values of the MLEs and MSEs for the censored data case.

$N = 10000$	$n = 15$	$n = 25$	$n = 50$	$n = 130$	$n = 350$	$n = 500$
$\widehat{\beta}$	1.4824	1.4863	1.4884	1.4912	1.4936	1.4983
MSE	0.0088	0.0072	0.0057	0.0048	0.0032	0.0019
$\widehat{\theta}$	0.4298	0.4274	0.4203	0.4134	0.4089	0.4022
MSE	0.0067	0.0056	0.0042	0.0037	0.0024	0.0012

Among these goodness-of-fit tests, we are interesting in the well known Nikulin Rao Robson (NRR) statistic Υ^2 which is based on maximum likelihood estimates on initial non grouped data. This statistic proposed by [27] and [29] recovers information lost while grouping data and follows a chi-square distribution. However, the presence of censorship makes all classical goodness-of-fit tests inapplicable. Therefore, researchers gave different modifications of existing statistics. Ten years ago, Reference [5] developed a modified NRR statistic for continuous distributions with unknown parameters and right censoring. As this version of the NRR statistic recovers also all information lost while regrouping data, it can be used to fit data from survival analysis, reliability and other fields where data are generally censored.

In this section, we construct modified chi-square goodness-of-fit test statistics for fitting complete and right censored data to the proposed model.

4.1 NRR statistic: generality

Here, a general setting is adopted. Let $x_1, \ldots, x_n$ be observations from a random variable X. We consider the null hypothesis H_0 according to which a the distribution of a random variable X has a certain cdf denoted by $F(x, \varpi)$, where $\varpi = (\varpi_1, \ldots, \varpi_s)^T$ is the parameter vector and s is the number of the model parameters. That is $H_0 : P(X \leq x) = F(x, \varpi)$. Consider r equiprobable grouping intervals $I_1, \ldots, I_r$, where $I_j =]a_{j-1}, a_j]$, $I_i \cap I_j = \emptyset, i \neq j$, and $\bigcup_{j=1}^{r} I_j = \mathbb{R}$, such as, for $j = 1, \ldots, r$,

$$p_j(\varpi) = F(a_j, \varpi) - F(a_{j-1}, \varpi) = \frac{1}{r}.$$

Now, let $\upsilon = (\upsilon_1, \ldots, \upsilon_r)^T$ be the number of observed x_i grouping into these intervals I_j and

$$R(\varpi) = \left(\frac{\upsilon_1 - np_1(\varpi)}{\sqrt{np_1(\varpi)}}, \frac{\upsilon_2 - np_2(\varpi)}{\sqrt{np_2(\varpi)}}, \ldots, \frac{\upsilon_r - np_r(\varpi)}{\sqrt{np_r(\varpi)}} \right)^T.$$

The NRR statistic Υ^2 proposed by [27] and [29] is defined by

$$\Upsilon^2 = R^2(\widehat{\varpi}) + \frac{1}{n} l(\widehat{\varpi})^T \left(I(\widehat{\varpi}) - J(\widehat{\varpi})\right)^{-1} l(\widehat{\varpi}),$$

where $I(\widehat{\varpi})$ and $J(\widehat{\varpi})$ are the estimated information matrices on non-grouped and grouped data, respectively, and $\widehat{\varpi}$ is the vector of the MLEs. The elements of the vector $l(\widehat{\varpi}) = (l_s(\widehat{\varpi}))^T_{1\times s}$

are specified by

$$l_s(\widehat{\varpi}) = \sum_{j=1}^{r} \frac{\nu_j}{p_j} \frac{\partial p_j(\widehat{\varpi})}{\partial \widehat{\varpi}_s}.$$

Then, the distribution of the random version of Υ^2 is a chi-square distribution with $r-1$ degrees of freedom.

To construct the test statistic Υ^2 corresponding to the PA model with a parameter vector $\varpi = (\beta, \theta)^T$, we calculate the MLEs, say $\widehat{\varpi} = (\widehat{\beta}, \widehat{\theta})^T$, and the limit intervals a_j, the derivatives $\partial p_j(\varpi)/\partial \varpi_s$ and the estimated information matrices $I(\widehat{\varpi})$ and $J(\widehat{\varpi})$. They are not given in this paper but we make them available to users. Finally, we obtain the statistic Υ^2 which allows to verify if data belong to the PA distribution.

4.2 NRR statistic with right censorship

Now, suppose that we deal with right censored observations $c_1, \ldots, c_n$ with fixed censoring time τ, recalling that each c_i can be written as $c_i = (x_i, \delta_i)$, where $\delta_i = 0$ if x_i is a censoring time and $\delta_i = 1$ if x_i is a failure time, and that the censorship is assumed to be non informative. Then, [5] introduced a modification of the NRR statistic as described above. First, it is based on the vector

$$Z_j = \frac{1}{\sqrt{n}}(U_j - e_j), \quad j = 1, \ldots, r,$$

with $r \succ s$, where U_j and e_j are the observed and expected numbers of failures to fall into the grouping intervals I_j, the statistic Υ'^2 is defined by $\Upsilon'^2 = Z^T \widehat{\Sigma}^- Z$, where $Z = (Z_1, \ldots, Z_r)^T$, Σ is the covariance matrix of Z and $\widehat{\Sigma}^-$ is a generalized inverse of Σ. For calculation purposes, the authors write this statistic as follow

$$\Upsilon'^2 = \sum_{j=1}^{r} \frac{(U_j - e_j)^2}{U_j} + Q$$

with the quadratic form Q obtained as

$$Q = \widehat{W}^T \widehat{G}^- \widehat{W}, \qquad \widehat{A}_j = \frac{1}{n} U_j, \qquad U_j = \sum_{i: c_i \in I_j} \delta_i,$$

$$\widehat{G} = [\widehat{g}_{ll'}]_{s \times s}, \quad \widehat{g}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^{r} \widehat{C}_{lj} \widehat{C}_{l'j} \widehat{A}_j^{-1},$$

$$\widehat{C}_{lj} = \frac{1}{n} \sum_{i: c_i \in I_j} \delta_i \frac{\partial \log[h(c_i, \widehat{\varpi})]}{\partial \varpi},$$

$$\widehat{i}_{ll'} = \frac{1}{n} \sum_{i=1}^{n} \delta_i \frac{\partial \log[h(c_i, \widehat{\varpi})]}{\partial \varpi_l} \frac{\partial \log[h(c_i, \widehat{\varpi})]}{\partial \varpi_{l'}},$$

$$\widehat{W}_l = \sum_{j=1}^{r} \widehat{C}_{lj} \widehat{A}_j^{-1} Z_j, \qquad \widehat{W} = (\widehat{W}_1, \ldots, \widehat{W}_s)^T,$$

where $l, l' = 1, \ldots, s$, and $\widehat{\varpi}$ represents the MLE of ϖ on initial data. Under the null hypothesis H_0, under a non-degenerated, the limit distribution behind the statistic Υ'^2 is a chi-square with $k = \text{rank}(\Sigma)$ degrees of freedom. For more details on modified chi-square tests, one may refer to the book of [32].

For testing the null hypothesis that a right censored sample is described by the PA model, we develop Υ'^2 corresponding to this distribution. At that end, we have to compute the maximum likelihood estimators $\widehat{\varpi} = (\widehat{\beta}, \widehat{\theta})^T$ on initial data (see Section 3), the estimated information matrix $\widehat{i}_{ll'}$ which can be deduced from the score functions and the estimated limit intervals $\widehat{a}_j$.

To apply this test statistic, the expected failure times e_j to fall into the grouping intervals I_j must be the same for any j, so the estimated interval limits $\widehat{a}_j$ are equal to

$$\widehat{a}_j = H^{-1}\left(\frac{E_j - \sum_{l=1}^{m-1} H\left(c_l, \widehat{\varpi}\right)}{n-m+1}, \widehat{\varpi}\right), \qquad \widehat{a}_r = \max\left(c_{(n),}\tau\right),$$

where $E_r = \sum_{i=1}^{n} H\left(c_i, \widehat{\varpi}\right)$, $E_j = (j/r)E_r$ and m the minimal integer such that $E_j \in [b_{m-1}, b_m]$ with

$$b_m = (n-m)H\left(c_{(m)}, \hat{\varpi}\right) + \sum_{l=1}^{m} H(c_{(l)}, \hat{\varpi}),$$

$c_{(1)}, \ldots, c_{(n)}$ denoting the ordered values of $c_1, \ldots, c_n$.

We set $e_j = E_r/r$. After that, we calculate the components of the estimated matrix $\widehat{C}$ which are obtained as follows

$$\widehat{C}_{1j} = \frac{1}{n}\sum_{i:c_i\in I_j}^{n} \delta_i \left[\frac{1}{\widehat{\beta}} + 2\log c_i - \frac{\widehat{\theta}c_i^{\widehat{\beta}}\log c_i}{1+\widehat{\theta}c_i^{\widehat{\beta}}}\right],$$

$$\widehat{C}_{2j} = \frac{1}{n}\sum_{i:c_i\in I_j}^{n} \delta_i \left[\frac{2}{\widehat{\theta}} - \frac{c_i^{\widehat{\beta}}}{1+\widehat{\theta}c_i^{\widehat{\beta}}}\right]$$

and the estimated matrix $\widehat{W}$ is derived from the matrix $\widehat{C}$. Therefore, the test statistic can be obtained easily as

$$\Upsilon'^2 = \sum_{j=1}^{r}\frac{(U_j - e_j)^2}{U_j} + \widehat{W}^T\left[\widehat{\imath}_{ll'} - \sum_{j=1}^{r}\widehat{C}_{lj}\widehat{C}_{l'j}\widehat{A}_j^{-1}\right]_{l\times l'}^{-1}\widehat{W}.$$

To confirm the theory and the practicability of the proposed tests for fitting data sets to the PA distribution, we use numerical simulations. We generate $N = 10000$ samples of complete observations from the PA model with parameters $\beta = 1.5$ and $\theta = 0.4$, considering different sample sizes. In this framework, we calculate the test statistic Υ^2. The obtained values with the theoretical levels of significance ($\epsilon = 1\%$, 5% and 10%) are summarized in Table 5.

Now, we reconduct the above simulation study but with right-censored observations. Thus, we determine Υ'^2. The obtained results are presented in Table 6.

Table 5: Simulated levels of significance for Υ^2 against their theoretical values.

$N = 10000$	$n = 25$	$n = 50$	$n = 130$	$n = 350$	$n = 500$
$\varepsilon = 1\%$	0.0063	0.0071	0.0082	0.0094	0.0103
$\varepsilon = 5\%$	0.0438	0.0453	0.0462	0.0478	0.0490
$\varepsilon = 10\%$	0.0959	0.0968	0.0979	0.0983	0.1013

Table 6: Simulated levels of significance for Υ'^2 against their theoretical values 30% of censorship.

$N = 10000$	$n = 25$	$n = 50$	$n = 130$	$n = 350$	$n = 500$
$\varepsilon = 1\%$	0.0055	0.0062	0.0076	0.0085	0.0094
$\varepsilon = 5\%$	0.0462	0.0474	0.0482	0.0493	0.0509
$\varepsilon = 10\%$	0.0942	0.0961	0.0973	0.0986	0.0997

As we can see in Tables 5 and 6, the empirical proportions of rejection of the null hypothesis H_0 are very close to the corresponding theoretical ones. Therefore, the test statistics Υ^2 and Υ'^2 proposed in this work can be used to check the adequacy of data to the PA model in a satisfactory manner.

4.3 Applications

The usefulness of the PA model through the previous methodology is illustrated by three examples from different areas. The first one concerns censored data from survival analysis, so we use Υ'^2 to fit these data to hypothesized distributions. For the complete data case, Υ^2 is constructed for testing if the two other examples are modeled by the proposed model.

Example 1. We consider sample data from 50 patients with acute myeloid leukemia, reported to the International Register of Bone Marrow Transplants. These patients had an allogeneic bone marrow transplant where the HLA (Histocompatibility Leukocyte Antigen) homolog marrow was used to rebuild their immune systems. The ordered data of this study are contained in the following set: {0.030, 0.493, 0.855, 1.184, 1.283, 1.480, 1.776, 2.138, 2.500, 2.763, 2.993, 3.224, 3.421, 4.178, 4.441*, 5.691, 5.855*, 6.941*, 6.941, 7.993*, 8.882, 8.882, 9.145*, 11.480, 11.513, 12.105*, 12.796, 12.993*, 13.849*, 16.612*, 17.138*, 20.066, 20.329*, 22.368*, 26.776*, 28.717*, 28.717*, 32.928, 33.783*, 34.211, 34.770*, 39.539, 41.118*, 45.033, 46.053*, 46.941, 48.289*, 57.401*, 58.322, 60.625*}, where the symbol * represents censorship.

We use the statistic test provided above to verify if the distribution behind these data can be modeled by the PA distribution. First, the MLEs values are given as

$$\widehat{\varpi} = (\widehat{\beta}, \widehat{\theta})^T = (1.5236, 0.8462)^T .$$

If we consider $r = 5$ intervals I_j for grouping the data, preliminary calculations of the statistic Υ'^2 are given in Table 7.

Table 7: Numerical values of $\widehat{a}_j$, U_j, $\widehat{C}_{1j}$, $\widehat{C}_{2j}$ and e_j.

$\widehat{a}_j$	2.645	7.523	16.789	30.256	60.625
U_j	9	10	11	8	13
$\widehat{C}_{1j}$	-1.3463	0.9463	-3.4676	-1.6744	0.7647
$\widehat{C}_{2j}$	0.9346	1.2634	0.8867	1.0346	1.1346
e_j	4.3986	4.3986	4.3986	4.3986	4.3986

Then, we deduce the value of the statistic Υ'^2, which is equal to $\Upsilon'^2 = 7.0353$. Since $\Upsilon'^2 = 7.0353 \leq \chi^2_5 = 11.0705$ (for significance level $\varepsilon = 5\%$) the null hypothesis H_0 is not rejected. This remains true even for the other significance levels, thus which allows us to say that these data are adequately fitted the proposed PA model.

Example 2. The empirical importance of the proposed model is proved by two applications to COVID-19 data sets, containing complete observations.

- **Algeria**

  In the first application, we take into account the daily new infected cases in Algeria. The data is available at

  https://www.worldometer.info/coronavirus/country/algeria/

  and contains daily new cases between May 24 and June 7, 2020. The considered data set is: {104, 115, 104, 98, 107, 113, 119, 127, 133, 137, 140, 160, 194, 197, 193}.

- **Iran**

  In the second application, we consider the new daily deaths in Iran. Because Iran is one of the most affected countries due to the COVID-19 epidemic. The data is available at

  https://www.worldometer.info/coronavirus/country/iran/

  and contains the daily new cases between February 21 and March 10, 2020. The considered data set is: {2, 2, 2, 4, 4, 3, 7, 8, 9, 11, 12, 11, 15, 16, 16, 21, 49, 43, 54}.

  The MLEs of the parameters of the PA model are obtained for these two datasets as follows

$$\widehat{\varpi}_1 = (\widehat{\beta}, \widehat{\theta})^T = (2.1963, 0.8234)^T, \quad \widehat{\varpi}_2 = (\widehat{\beta}, \widehat{\theta})^T = (1.8236, 0.7451)^T,$$

  respectively.

  We also calculate the test statistics Υ^2 to fit these data sets to the competing models. The results are the following:

$$\Upsilon^2_1 = 8.6238, \quad \Upsilon^2_2 = 7.1286,$$

  respectively. As, in each case, $\Upsilon^2 \prec \chi^2_5 = 11.0705$ (for the significance level $\varepsilon = 5\%$) the null hypothesis H_0 is not rejected. The PA model is thus adequate to fit these data.

5 Concluding notes

We develop a new two-parameter distribution, extending the so-called Ailamujia distribution by the use of the power transform. We demonstrate its significant gain in terms of functional pliancy, discuss some measures of importance and show that it has enough qualities to be considered as an efficient statistical model for data analysis. In particular, the complete and right-censored data cases are considered, with the development of a new statistical test in this regard. The results on four practical data sets are quite favorable to the PA model, motivating its use for various purposes, including those beyond the scope of this paper.

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