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Distribution of eigenvalues for sub-skewtriagonal Hankel matrices

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Abstract. We investigate the eigenvalue distribution of banded Hankel matrices with non-zero skew diagonals. This work uses push-forward of an arcsine density, block structures and generating functions. Our analysis is done by a combination of Chebyshev polynomials, Laplacian determinant expansion and mathematical induction.

Keywords: Hankel, eigenvalue, distribution, generating function.
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1 Introduction

The computation of eigenvalues of large matrices is a matter of major importance in many scientific and engineering applications such as quantum chemistry, structural dynamics, chemical reactions, electrical networks, Markov chain techniques, control theory, magneto hydrodynamics, combustion processes and cell biology [2, 7, 8]. A number of matrices which appear in most applications have special structures than others, such as Toeplitz matrices, circulant matrices and Hankel matrices. These are typically dense matrices, but their entries depend on fewer limitations than size of matrices.

In [2], Angerer and Zamora analyzed the spectral of a statistical mechanics matrix which appears in the context of a special toy model of contractile structures from cell biology. We try to generalize this problem for special group of Hankel matrices. As in [7], we first introduce the notation

$$f = f(\omega) = \omega^n \sum_{m=1}^{k} f_m \omega^{m-1},$$

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where $f \in L^\infty$, f_m is the mth Fourier coefficients of

$$F(z) = \sum_{m=1}^{\infty} f_m z^m, \tag{1}$$

and the series in the right-hand side converges absolutely to $F(z)$ on the open unit disk $|z| < 1$. The purpose of this paper is to analyze an $n \times n$ banded Hankel matrices with non-zero skew diagonals ($n \geq k$) of the form

$$H_n(f) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & f_1 \\ 0 & 0 & 0 & \cdots & 0 & f_1 & f_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & f_1 & \cdots & f_{k-2} & f_{k-1} \\ 0 & 0 & f_1 & f_2 & \cdots & f_{k-1} & f_k \\ 0 & f_1 & f_2 & \cdots & f_{k-1} & f_k & 0 \\ f_1 & f_2 & \cdots & f_{k-1} & f_k & 0 & 0 \end{pmatrix}. \tag{2}$$

Our aim is to study the eigenvalues of this matrix in the special case $n \to \infty$. The main reason to do this work is that the square of this matrix is almost Toeplitz. Eigenvalues for banded Toeplitz matrices are given in [3, 5, 6, 9, 10]. Angerer [1] derives a weak limit law for eigenvalues of Hankel matrices with three non-zero skew diagonals:

$$H_n(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & f_1 \\ 0 & 0 & \cdots & 0 & f_1 & f_2 \\ 0 & 0 & \cdots & f_1 & f_2 & f_3 \\ 0 & \vdots & f_1 & f_2 & f_3 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & f_1 & f_2 & \cdots & 0 & 0 \\ f_1 & f_2 & f_3 & 0 & \cdots & 0 \end{pmatrix}.$$

Let

$$\rho_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k}, \tag{3}$$

be the empirical distribution of its eigenvalues, and

$$\frac{d\rho(x)}{dx} = \frac{1}{2\pi\sqrt{1-x^2}}, \tag{4}$$

be one-half of the arcsine distribution on $(-1, 1)$. Then he proved that, in the sense of weak convergence of measures, $\lim_{n\to\infty} \rho_n =: \rho^\sharp$ exists, and decomposes into the push-forward of the density (4) under the mapping

$$\varphi = \sqrt{(f_1 - f_3)^2 + f_2^2 + 2f_2(f_1 + f_3)x + 4f_1 f_3 x^2}.$$

In fact, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n} w(\lambda_i)=\frac{1}{2\pi}\int_{-1}^{1}\frac{w(\varphi(x))}{\sqrt{1-x^2}}dx+\frac{1}{2\pi}\int_{-1}^{1}\frac{w(-\varphi(x))}{\sqrt{1-x^2}}dx, \tag{5}$$

for every continuous function $w:\mathbb{C}\to\mathbb{C}$ with the compact support. He also presented a conjecture without proof for the eigenvalue distribution of Hankel matrices with more than three non-zero skew diagonals, matrix $H_n(f)$ in (2). This motivated us to present a proof for this conjecture.

The paper is organized as follows. The conjecture is presented in Section 2. In Section 3, the proof of theorem is started by some of the elementary steps such as a block structure for sub-skewtriagonal Hankel matrices. In Section 4, the determinant is expanded by Laplace expansion and generating functions. Finally in Section 5 the theorem is proved by mathematical induction.

2 Main theorem

In this section, we introduce the next theorem which has a main role in this paper.

Theorem 1. *Let $H_n(f)$, $n\ge k$ be an $n\times n$ (complex) Hankel matrix by symbol $f=f(\omega)=\omega^n\sum_{m=1}^{k} f_m\omega^{m-1}$ for some $k\ge 1$. Let*

$$\rho_n:=\frac{1}{n}\sum_{k=1}^{n}\delta_{\lambda_k}, \tag{6}$$

be the empirical distribution of its eigenvalues. We choose for ρ a measure so that its density is introduced by

$$\frac{d\rho(x)}{dx}=\frac{1}{2\pi\sqrt{1-x^2}}, \tag{7}$$

which is one-half of the arcsine distribution on $(-1,1)$. Then $\lim_{n\to\infty}\rho_n=:\rho^\sharp$ exists in the sense of weak convergence of measures and decomposes into the push-forward of the density (7) under each of the following mappings

$$\varphi^+,\varphi^-:[-1,1]\to\mathbb{C}, \tag{8}$$

$$\varphi_k(x):=\varphi^+(x)=\sqrt{T\times J_k\times H_k(f)\times F^t},\quad \varphi^-:=-\varphi^+,$$

$$F=(f_1,f_2,\ldots,f_k),\quad T=(1,2T_1(x),2T_2(x),\ldots,2T_{k-1}(x)),$$

where $T_k(x)$'s are Chebyshev polynomials of the first kind, J_k is the appropriate exchange matrix and $H_k(f)$ is a $k\times k$ Hankel matrix,

$$H_k(f)=\begin{pmatrix} 0 & 0 & \cdots & 0 & f_1\\ 0 & 0 & \cdots & f_1 & f_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & f_1 & \cdots & f_{k-2} & f_{k-1}\\ f_1 & f_2 & \cdots & f_{k-1} & f_k \end{pmatrix}. \tag{9}$$

The empirical distribution (6) of its eigenvalues, when $n \to \infty$, will tend towards the push-forward of the density (7) by the two mappings (8). Therefore for every continuous function $w : \mathbb{C} \to \mathbb{C}$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} w(\lambda_i) = \frac{1}{2\pi}\int_{-1}^{1} \frac{w(\varphi^+(x))}{\sqrt{1-x^2}}dx + \frac{1}{2\pi}\int_{-1}^{1} \frac{w(\varphi^-(x))}{\sqrt{1-x^2}}dx. \tag{10}$$

In other words, the majority of eigenvalues of a large matrix $H_n(f)$ tends to cluster around the set $\varphi^+([-1,1]) \cup \varphi^-([-1,1])$ with the compact support.

We will give the proof in the Sections 3, 4 and 5.

3 A block structure for sub-skewtriagonal Hankel matrices

Let $k = n$ and (n,n)-entry of the matrix $H_n(f)$ in (9) be zero, namely

$$G_n(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & f_1 \\ 0 & 0 & \cdots & f_1 & f_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & f_1 & \cdots & f_{n-2} & f_{n-1} \\ f_1 & f_2 & \cdots & f_{n-1} & 0 \end{pmatrix}.$$

We set

$$J_n \times G_n(f) = \left(\begin{array}{cccc|c} f_1 & f_2 & \cdots & f_{n-1} & 0 \\ 0 & f_1 & \cdots & f_{n-2} & f_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f_1 & f_2 \\ \hline 0 & 0 & \cdots & 0 & f_1 \end{array}\right) = \left(\begin{array}{c|c} A & C \\ \hline \mathbf{0} & f_1 \end{array}\right),$$

and

$$J_n \times H_n(f) = \left(\begin{array}{cccc|c} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ 0 & f_1 & \cdots & f_{n-2} & f_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f_1 & f_2 \\ \hline 0 & 0 & \cdots & 0 & f_1 \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline \mathbf{0} & f_1 \end{array}\right).$$

These blocks help us to find two mappings in Theorem 1 with the compact forms

$$\varphi_n(x) = \sqrt{(T^\star, 2T_{n-1}(x)) \times \left(\begin{array}{c|c} A & B \\ \hline \mathbf{0} & f_1 \end{array}\right) \times \begin{pmatrix} F^\star \\ f_n \end{pmatrix}}, \tag{11}$$

where

$$T = (\underbrace{1, 2T_1(x), 2T_2(x), \ldots, 2T_{n-2}}_{T^\star}, 2T_{n-1}(x)),$$

$$F^t = (\underbrace{f_1, f_2, \ldots, f_{n-1}}_{F^{\star t}}, f_n)^t.$$

After some computations similar to (11) for the matrix $G_n(f)$, we get

$$\phi_n(x) = \sqrt{(T^\star, 0) \times \left(\begin{array}{c|c} A & C \\ \hline \mathbf{0} & f_1 \end{array}\right) \times \begin{pmatrix} F^\star \\ 0 \end{pmatrix}}. \tag{12}$$

We compute the difference between $\varphi_n(x)^2$ and $\phi_n(x)^2$

$$\begin{aligned} \varphi_n(x)^2 - \phi_n(x)^2 &= (T^\star B + 2T_{n-1}(x) f_1) f_n \\ &= [f_n + 2T_1(x) f_{n-1} + \ldots + 2T_{n-1}(x) f_1] f_n. \end{aligned} \tag{13}$$

Then

$$|\varphi_n(x)^2 - \phi_n(x)^2| = |f_n + 2T_1(x) f_{n-1} + 2T_2(x) f_{n-2} + \ldots + 2T_{n-1}(x) f_1||f_n|,$$

and we have

$$|\varphi_n(x)^2 - \phi_n(x)^2| \leq 2 \sum_{i=1}^{n} |f_i||f_n|.$$

Since the series in Eq. (1) is absolutely convergent, we deduce

$$\lim_{n\to\infty} |\varphi_n(x)^2 - \phi_n(x)^2| = 0. \tag{14}$$

From Eqs. (13), (14) and for $n \geq k$ in the Hardy-Hilbert space [7], we write

$$\lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} \varphi_n(x). \tag{15}$$

4 Determinant expansion

We find the determinants of the matrices $H_n(f)$ and $G_n(f)$ which are denoted by

$$D_n(\lambda) = \begin{vmatrix} -\lambda & 0 & \cdots & 0 & f_1 \\ 0 & -\lambda & \cdots & f_1 & f_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & f_1 & \cdots & f_{n-2} - \lambda & f_{n-1} \\ f_1 & f_2 & \cdots & f_{n-1} & f_n - \lambda \end{vmatrix},$$

and

$$S_n(\lambda) = \begin{vmatrix} -\lambda & 0 & \cdots & 0 & f_1 \\ 0 & -\lambda & \cdots & f_1 & f_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & f_1 & \cdots & f_{n-2}-\lambda & f_{n-1} \\ f_1 & f_2 & \cdots & f_{n-1} & -\lambda \end{vmatrix},$$

respectively. Our aim is to find a relationship between determinants as the coefficients of rational generating functions. This helps us to recover a linear system of recursion equations [11]. These generating functions are

$$h(z) = \sum_{n=0}^{\infty} D_n(\lambda) z^n, \quad g(z) = \sum_{n=0}^{\infty} S_n(\lambda) z^n. \tag{16}$$

Now, via Laplace expansion with respect to the their last rows, the following results are deduced:

$$D_n(\lambda) = \begin{vmatrix} -\lambda & 0 & \cdots & 0 & f_1 \\ 0 & -\lambda & \cdots & f_1 & f_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & f_1 & \cdots & f_{n-2}-\lambda & f_{n-1} \\ f_1 & f_2 & \cdots & f_{n-1} & f_n-\lambda \end{vmatrix} = (-1)^{n+1} f_1 \begin{vmatrix} 0 & \cdots & 0 & 0 & f_1 \\ -\lambda & \cdots & 0 & f_1 & f_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & f_1 & \cdots & f_{n-3} & f_{n-2} \\ f_1 & \cdots & f_{n-3} & f_{n-2}-\lambda & f_{n-1} \end{vmatrix}$$

$$+ \cdots + (-1)^{2n-1} f_{n-1} \begin{vmatrix} -\lambda & 0 & \cdots & 0 & f_1 \\ 0 & -\lambda & \cdots & f_1 & f_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f_{n-4}-\lambda & f_{n-2} \\ 0 & f_1 & \cdots & f_{n-3} & f_{n-1} \end{vmatrix}$$

$$+ (f_n-\lambda) \begin{vmatrix} -\lambda & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \cdots & 0 & f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f_{n-4}-\lambda & f_{n-3} \\ 0 & f_1 & \cdots & f_{n-3} & f_{n-2}-\lambda \end{vmatrix}$$

and

$$S_n(\lambda) = \begin{vmatrix} -\lambda & 0 & \cdots & 0 & f_1 \\ 0 & -\lambda & \cdots & f_1 & f_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & f_1 & \cdots & f_{n-2}-\lambda & f_{n-1} \\ f_1 & f_2 & \cdots & f_{n-1} & -\lambda \end{vmatrix} = (-1)^{n+1} f_1 \begin{vmatrix} 0 & \cdots & 0 & 0 & f_1 \\ -\lambda & \cdots & 0 & f_1 & f_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & f_1 & \cdots & f_{n-3} & f_{n-2} \\ f_1 & \cdots & f_{n-3} & f_{n-2}-\lambda & f_{n-1} \end{vmatrix}$$

$$+\ldots+(-1)^{2n-1}f_{n-1}\begin{vmatrix} -\lambda & 0 & \cdots & 0 & f_1 \\ 0 & -\lambda & \cdots & f_1 & f_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f_{n-4}-\lambda & f_{n-2} \\ 0 & f_1 & \cdots & f_{n-3} & f_{n-1} \end{vmatrix}$$

$$+(-\lambda)\begin{vmatrix} -\lambda & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \cdots & 0 & f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f_{n-4}-\lambda & f_{n-3} \\ 0 & f_1 & \cdots & f_{n-3} & f_{n-2}-\lambda \end{vmatrix}.$$

The difference between these determinants are given by

$$D_n(\lambda)=S_n(\lambda)+f_n\underbrace{\begin{vmatrix} -\lambda & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & \cdots & 0 & f_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f_{n-4}-\lambda & f_{n-3} \\ 0 & f_1 & \cdots & f_{n-3} & f_{n-2}-\lambda \end{vmatrix}}_{E}. \tag{17}$$

After expanding E with respect to the first row, we get

$$D_n(\lambda)=S_n(\lambda)-\lambda f_nD_{n-2}(\lambda). \tag{18}$$

Multiplying the Eq. (18) by z^n and z^k and summing up from $n,k=2$ to infinity, gives

$$\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}D_nz^nz^k=\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}(S_n-\lambda f_nD_{n-2})z^nz^k,$$

which results in

$$\sum_{n=2}^{\infty}D_nz^n\sum_{k=2}^{\infty}z^k=\sum_{n=2}^{\infty}S_nz^n\sum_{k=2}^{\infty}z^k-\lambda\sum_{n=2}^{\infty}f_nz^n\sum_{k=2}^{\infty}D_{k-2}z^k.$$

Then

$$\sum_{n=0}^{\infty}[D_nz^n-D_0-D_1z]\sum_{k=0}^{\infty}[z^k-1-z]=\sum_{n=0}^{\infty}[S_nz^n-S_0-S_1z]\sum_{k=0}^{\infty}[z^k-1-z]$$

$$-\lambda\sum_{n=0}^{\infty}[f_nz^n-f_0-f_1z]z^2\sum_{k=2}^{\infty}[D_{k-2}z^{k-2}].$$

Without loss of generality, we set $D_0(\lambda)=S_0(\lambda)=1$, $D_1(\lambda)=f_1-\lambda$, $S_1(\lambda)=-\lambda$ for $n\geq 2$. Then

$$[h(z)-D_0-D_1z][\frac{1}{1-z}-1-z]=[g(z)-S_0-S_1z][\frac{1}{1-z}-1-z]-\lambda z^2[F(z)-f_0-f_1z]h(z),$$

and

$$h(z) = \frac{g(z) + f_1 z}{1 + \lambda(1-z)[F(z) - f_0 - f_1 z]}. \tag{19}$$

We know that $\frac{F(z)-f_0-f_1 z}{z^2}$ is ordinary power series generating function (*ops*) for the sequence $\{a_{n+2}\}_0^\infty$ and it converges on $|z| < 1$ for Eq. (1),

$$\frac{F(z) - f_0 - f_1 z}{z^2} \overset{ops}{\leftrightarrow} \{a_{n+2}\}_0^\infty.$$

For more details on the ordinary power series generating function see [11]. This ordinary power series will be used in next section.

5 An inductive proof of the main theorem

We can prove the theorem in Section 2 by mathematical induction. In the first step of induction, we see that Theorem 1 is true for $k = 3$, since the matrix takes the form:

$$H_n(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & f_1 \\ 0 & 0 & \cdots & 0 & f_1 & f_2 \\ 0 & 0 & \cdots & f_1 & f_2 & f_3 \\ 0 & \vdots & f_1 & f_2 & f_3 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & f_1 & f_2 & \cdots & 0 & 0 \\ f_1 & f_2 & f_3 & 0 & \cdots & 0 \end{pmatrix},$$

see [1]. Namely, if $\rho_n := \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k}$ is the empirical distribution of its eigenvalues [1, 2], then $\lim_{n\to\infty} \rho_n =: \rho^\sharp$ exists in the sense of weak convergence of measures and decomposes into the push-forward of the density (7) under each of mappings

$$\varphi_3(x) = \sqrt{(f_1 - f_3)^2 + f_2^2 + 2f_2(f_1 + f_3)x + 4f_1 f_3 x^2},$$

and

$$-\varphi_3(x) = -\sqrt{(f_1 - f_3)^2 + f_2^2 + 2f_2(f_1 + f_3)x + 4f_1 f_3 x^2}.$$

In other words, we have Eq. (5) for every continuous function $w : \mathbb{C} \to \mathbb{C}$ with the compact support. It shows that the function $w(X)$ generally has a different distribution from X.

Now, we assume that the theorem is true for $k = n - 1$ in (2) or matrix $G_n(f)$ in Section 3. Then we will show that it is also true for $k = n$. By our hypothesis in the last section, there exists a function $\varphi_{n-1} = \phi_n$ that Eq. (10) is true for it. It is easy to see that $S_n(\lambda)$ is non-zero for sufficiently large values of λ. We show that $(-\lambda)^n S_n(\lambda^{-1})$ is a polynomial of degree

n in λ. It is nonzero for sufficiently small value of λ. Also, $\log((-\lambda)^n S_n(\lambda^{-1}))$ is analytic and single-valued in a small neighbourhood of 0. We have

$$-\frac{\partial \log((-\lambda)^n S_n(\lambda^{-1}))}{\partial \lambda} = \sum_{k=1}^{n} \frac{\lambda_k}{1-\lambda_k \lambda} = \sum_{r=0}^{\infty} \left(\sum_{k=1}^{n} \lambda_k^{r+1} \right) \lambda^r, \tag{20}$$

see [1,2]. $\sum_{k=1}^{n} \lambda_k^{r+1}$ is n times the $(r+1)$-th empirical moment of the measure ρ_n. Its moment generating function is shown by

$$m_n(\lambda) := \sum_{r=0}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \lambda_k^r \right) \lambda^r = -\lambda \frac{\partial \log \sqrt[n]{(-1)^n S_n(\lambda^{-1})}}{\partial \lambda}.$$

Since everything is analytic here, then

$$m^\sharp(\lambda) := \lim_{n\to\infty} m_n(\lambda) = -\lambda \frac{\partial \log \left(\lim_{n\to\infty} \sqrt[n]{(-1)^n S_n(\lambda^{-1})} \right)}{\partial \lambda} = -\lambda \frac{\partial \log \sigma(\lambda)}{\partial \lambda} |_{\lambda=\lambda^{-1}}, \tag{21}$$

that $\lim_{n\to\infty} \sqrt[n]{(-1)^n S_n(\lambda^{-1})}$ is the reciprocal radius of convergence of the generating function $g(z)$ in Section 4. Also, $\lim_{n\to\infty} \sqrt[n]{(-1)^n S_n(\lambda^{-1})}$ is a zero of denominator $g(z)$ in λ^{-1} and

$$\lim_{n\to\infty} \sqrt[n]{S_n(\lambda)} = \frac{1}{\sigma(\lambda)} \to \lim_{n\to\infty} \sqrt[n]{S_n(\lambda^{-1})} = \frac{1}{\sigma(\lambda^{-1})} = \sigma^{-1}(\lambda^{-1}).$$

$m^\sharp(\lambda)$ is the moment generating function of the measure $\rho^\sharp$. Set $w(z) = z^r$ in the Theorem 1,

$$m^\sharp(\lambda) = \lim_{n\to\infty} \sum_{r=0}^{\infty} \frac{1}{\pi} \int_{-1}^{1} \frac{\lambda^{2r} \phi_n^{2r}(x)}{\sqrt{1-x^2}} dx = \lim_{n\to\infty} \frac{1}{\pi} \int_{-1}^{1} \frac{1}{1-\lambda^2 \phi_n^2(x)} \frac{1}{\sqrt{1-x^2}} dx. \tag{22}$$

On the other hand, by Eq. (21), we deduce

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{-1}^{1} \frac{1}{1-\lambda^2 \phi_n^2(x)} \frac{1}{\sqrt{1-x^2}} dx = -\lambda \frac{\partial \log \sigma(\lambda)}{\partial \lambda} |_{\lambda=\lambda^{-1}}. \tag{23}$$

The proof will be completed, if we show that Eq. (23), for the function φ_n, is true. Then, we need the zero $\sigma_D(\lambda)$. Now, from the Section 4, we obtain

$$h(z) = \frac{g(z) + f_1 z}{1 + \lambda(1-z)[F(z) - f_0 - f_1 z]},$$

namely the denominator of both generating functions are approximately similar except for the zero $1 + \lambda(1-z)[F(z) - f_0 - f_1 z]$,

$$1 + \lambda(1-z)[F(z) - f_0 - f_1 z] = 0 \to \frac{1}{\lambda} = (1-z)[F(z) - f_0 - f_1 z],$$

and $\sigma_D(\lambda) = \sigma_S(\lambda) \cap \sigma_F(\lambda)$. Therefore, Eqs. (1) and (19) show that $F(z)$ is convergent on the open unit disk $|z| < 1$. Then, it is a zero of the denominator $g(z)$ which is closest to the origin.

Eq. (15) shows that $\lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} \varphi_n(x)$, therefore

$$\lim_{n\to\infty} \frac{1}{\pi} \int_{-1}^{1} \frac{1}{1-\lambda^2 \varphi_n^2(x)} \frac{1}{\sqrt{1-x^2}} dx = -\lambda \frac{\partial \log \sigma(\lambda)}{\partial \lambda} |_{\lambda=\lambda^{-1}}.$$

Thus, the inductive step is proved. □

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