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Optimal control of time delay Fredholm integro-differential equations

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Abstract. This paper is devoted to solve a set of non-linear optimal control problems which are touched with time-delay Fredholm integro-differential equations. The serious objective of this work is to contribute an appropriate direct scheme for solving these problems. The technique used in this paper is based upon the Dickson polynomials and collocation points. Getting through the solutions, the states and controls variables can be approximated with Dickson polynomials. Therefore, the optimal control problem with time-delay integro-differential equation transforms into a system of algebraic equations that by solving it, we can obtain the unknown coefficients of the main problem. The residual error estimation of this technique is also investigated. Accuracy amount of the absolute errors have been studied for the performance of this method by solving several non-trivial examples.

Keywords: Optimal control problems, Dickson polynomials, time-delay equation, Fredholm integrao-differential equation, collocation points.
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1 Introduction

Differential equation with time delay is a kind of differential equations that the derivative of the unknown function at present time depending on the values of the function at previous times. This type of equations have been appeared in modeling various problems in electronic, biological, transport systems and control theory. Any system that includes feedback control is almost always accompanied by a time delay because it takes a limited amount of time to sense the information and then react to it. Therefore, time-delay systems are an essential category of systems whose optimal control has been considered by many scientists. Overwhelming research has been done on developing applications of delay problems in engineering fields and physical

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models. In [9], the authors studied two cases in biomedicine with multiple time delays in state and control variables. A hybrid method based on the block-pulse functions and orthonormal Taylor series proposed to solve optimal control of time-delay systems by Dadkhah et al. in [4]. Multiple time delays in theory and applications of optimal control problems are investigated in [10]. Liu et al. in [16], have presented a computational approach to solve time-delay optimal control problems with free terminal time. Haar wavelet method is applied to solve optimal control problems of the time-delayed systems in [19]. You can see more articles in this field in references [5, 17, 23, 24].

More precisely, this topic extended to another type of delay optimal control problems where the time-delay dynamic system is governed by integral equation namely delay integro-optimal control problems (DIOCPs). Although, many computational methods have been proposed to solve optimal control problems with time-delay, but only one paper is devoted to solve DIOCPs [18]. So, our goal is to offer a method that can easily solve these problems based on Dickson polynomials and collocation points. Dickson polynomials are well described by Lidl et al., in [15]. These polynomials are definable over a commutative ring R in which, if $R = C$ be the set of complex numbers, $D_m(t, \alpha)$ is associated with the known Chebyshev polynomials of the first kind $T_m(t)$. Exactly, $D_m(2\cos\theta, 1) = 2T_m(\cos\theta)$ for any real number θ and we have Lucas polynomials when $\alpha = -1$ [3]. Beside, there have been various articles on Dickson polynomials [1, 7, 8, 22, 25]. See some practical articles in this area in [11, 12]. The proposed method allows us to transform the DIOCPs to a system of algebraic equations with matrix form of unknown coefficients for choosing the state and control parameters optimally. The error estimation of this technique is also investigated. The significant merits of this approach are swift calculations, efficiency, ease of implementation and robustness. Indeed, it provides satisfactory results even a small number of the Dickson polynomials is used. Simple operations and ease of implementation are further characteristics of the mentioned polynomials. To attain these aims, the suitable choice of α, the parameter of Dickson polynomials, plays a crucial role to enhance the accuracy of the results evaluated by the current approach.

The overall layout of this manuscript is according to the following pattern. Section 2 explains the basic concepts of Dickson polynomials and their properties. We present a direct approach based on collocation method and Dickson polynomials to solve DIOCPs in Section 3. Also, the function approximation and the operational matrix of Dickson polynomials have been discussed in this section. The error estimation and the convergence analysis of this approach are carried out in Section 4. The numerical results and comparison have brought in Section 5 to substantiate the efficiency of our approach and then, some conclusions are drawn in the last section.

2 Dickson polynomials

The Dickson polynomials of degree m, which was firstly derived in [6], for any integer $m \geq 1$ and any element α over finite fields are defined as follows:

$$D_m(t,\alpha) = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m}{m-i} \binom{m-i}{i} (-\alpha)^i t^{(m-2i)}, \qquad -\infty < t < \infty$$

where $\lfloor \frac{m}{2} \rfloor$ is the greatest integer less than or equal to $\frac{m}{2}$. In addition, $D_0(t,\alpha) = 2$, $D_1(t,\alpha) = t$ and for $m > 1$, we have the following recurrence relation [15]:

$$D_m(t,\alpha) = tD_{m-1}(t,\alpha) - \alpha D_{m-2}(t,\alpha), \qquad m \geq 2.$$

Furthermore, $D_m(t,\alpha)$ satisfies the following ordinary differential equations [15]

$$(t^2 - 4\alpha)y'' + ty' - m^2 y = 0, \qquad m = 0, 1, 2, \ldots.$$

3 Method of solution

In this section, we present a direct scheme based on collocation method and Dickson polynomials to solve the following DIOCP:

$$\min \ J(y,u) = \frac{1}{2}\int_0^1 (a\, y^2(t) + b\, u^2(t) + c\, y^2(t-\eta))dt, \tag{1}$$

subject to

$$y'(t) = \sum_{i=1}^{d} \alpha_i(t) y^i(t) + \sum_{j=1}^{e} \beta_j(t) y^j(t-\eta) + \sum_{k=1}^{f} \int_0^1 B_k(t,\tau) y^k(\tau-\eta) d\tau + \gamma(t) u(t), \tag{2}$$

with the initial conditions

$$\begin{aligned} &y(0) = y_0, \\ &y(t) = \phi(t), \qquad -\eta \leq t < 0, \end{aligned} \tag{3}$$

where $y(t), u(t) \in \mathbb{R}$ and a, b, c are nonnegative real numbers. Also, $\alpha_i(t)$, $\beta_i(t)$, $B_k(t,\tau)$ and $\gamma(t)$, $i = 1, 2, \ldots, d, j = 1, 2, \ldots, e$, $k = 1, 2, \ldots, f$, are arbitrary functions and $\phi(t)$ is a known function. Sufficient and necessary conditions for existence solutions of OCP with time delay was studied in [2]. The target of this work is to find the admissible pair $(y(t), u(t))$ that minimizes the cost functional (1), while the dynamic equality constraint (2)-(3) is satisfied. Nonetheless it should be noted that through the difficulty of handling analytically solutions for DIOCPs, finding a numerical method with low computing costs and enough accuracy and performance has become an active research undertaking. For obtain an approximate solution based on the truncated Dickson polynomials, we have:

$$y(t) \simeq y_M(t) = \sum_{p=0}^{M} D_p(t,\alpha) y_p \qquad u(t) \simeq u_M(t) = \sum_{p=0}^{M} D_p(t,\alpha) u_p, \tag{4}$$

where y_p and u_p, $p = 0, 1, 2, \ldots, M$, are the unknown Dickson coefficients. Now, for solving problem (1)-(2), we need to find the approximations presented in (4). For this purpose, we used the following collocation points:

$$t_q = t_0 + (\frac{t_f - t_0}{2M})q, \qquad q = 0, 1, 2, \ldots, 2M. \tag{5}$$

in which $t_0 < t_1 < t_2 < \ldots < t_{2M} = t_f$. To simplify calculations, we rewrite the approximation solution (4) with the following matrix form [11, 12]:

$$\begin{aligned} y(t) &\simeq y_M(t) = D(t,\alpha)Y = S(t)K(\alpha)Y, \\ u(t) &\simeq u_M(t) = D(t,\alpha)U = S(t)K(\alpha)U, \end{aligned} \tag{6}$$

where $Y = [y_0, y_1, \ldots, y_M]^T$ and $U = [u_0, u_1, \ldots, u_M]^T$ are the unknown coefficients, $S(t) = [1, t, t^2, \ldots, t^M]$ and if M is even

$$K^T(\alpha) = \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0}(-\alpha)^0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1}(-\alpha)^1 & 0 & \frac{2}{2}\binom{2}{0}(-\alpha)^0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1}(-\alpha)^1 & 0 & \frac{3}{3}\binom{3}{0}(-\alpha)^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{M}{M/2}\binom{M/2}{M/2}(-\alpha)^{M/2} & 0 & \frac{M}{(M/2)+1}\binom{(M/2)+1}{(M/2)-1}(-\alpha)^{(M/2)-1} & 0 & \cdots & \frac{M}{M}\binom{M}{0}(-\alpha)^0 \end{bmatrix},$$

and if M is odd

$$K^T(\alpha) = \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0}(-\alpha)^0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1}(-\alpha)^1 & 0 & \frac{2}{2}\binom{2}{0}(-\alpha)^0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1}(-\alpha)^1 & 0 & \frac{3}{3}\binom{3}{0}(-\alpha)^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{M}{\lceil M/2\rceil}\binom{\lceil M/2\rceil}{\lfloor M/2\rfloor}(-\alpha)^{\lfloor M/2\rfloor} & 0 & \frac{M}{\lceil M/2\rceil+1}\binom{\lceil M/2\rceil+1}{\lfloor M/2\rfloor-1}(-\alpha)^{\lfloor M/2\rfloor-1} & \cdots & \frac{M}{M}\binom{M}{0}(-\alpha)^0 \end{bmatrix}.$$

Now, for the matrix form of derivative we have [11, 12]:

$$\begin{aligned} y'(t) &\simeq y'_M(t) = D'(t,\alpha)Y = S(t)BK(\alpha)Y, \\ u'(t) &\simeq u'_M(t) = D'(t,\alpha)U = S(t)BK(\alpha)U, \end{aligned} \tag{7}$$

in which

$$\begin{aligned} D(t,\alpha) &= [D_0(t,\alpha), D_1(t,\alpha), \ldots, D_M(t,\alpha)], \\ D'(t,\alpha) &= [D'_0(t,\alpha), D'_1(t,\alpha), \ldots, D'_M(t,\alpha)], \end{aligned}$$

and

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & M \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, by putting $t \to t - \eta$ in Eq. (6), we will have:

$$y(t-\eta) \simeq y_M(t-\eta) = S(t-\eta)K(\alpha)Y = S(t)C(-\eta)K(\alpha)Y, \tag{8}$$

in which

$$C(\eta)=\begin{bmatrix} \binom{0}{0}(-\eta)^0 & \binom{1}{0}(-\eta)^1 & \binom{2}{0}(-\eta)^2 & \ldots & \binom{M}{0}(-\eta)^M \\ 0 & \binom{1}{1}(-\eta)^0 & \binom{2}{0}(-\eta)^1 & \ldots & \binom{M}{1}(-\eta)^{M-1} \\ 0 & 0 & \binom{2}{2}(-\eta)^0 & \ldots & \binom{M}{2}(-\eta)^{M-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{M}{M}(-\eta)^0 \end{bmatrix}.$$

Also, we obtain the fundamental matrix relations for nonlinear parts of problem (1)-(2). Therefore, for the nonlinear part $y^2(t)$ we have [12,13]:

$$y^2(t)\simeq D(t,\alpha)\bar{D}(t,\alpha)\bar{Y}=S(t)K(\alpha)\bar{S}(t)\bar{K}(\alpha)\bar{Y}, \tag{9}$$

where

$$\bar{S}(t)=\operatorname{diag}[S(t)]_{(M+1)\times(M+1)^2}=\begin{bmatrix} S(t_0) & 0 & \ldots & 0 \\ 0 & S(t_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & S(t_M) \end{bmatrix}_{(M+1)\times(M+1)^2},$$

and

$$\bar{K}(\alpha)=\operatorname{diag}[K(\alpha)]_{(M+1)^2\times(M+1)^2}, \qquad \bar{Y}=\begin{bmatrix} y_0Y \\ y_1Y \\ \vdots \\ y_MY \end{bmatrix}_{(M+1)^2\times 1}.$$

Similarly, for nonlinear part $y^3(t)$ we have [11,12]:

$$y^3(t)\simeq D(t,\alpha)\bar{D}(t,\alpha)\bar{\bar{D}}(t,\alpha)\bar{\bar{Y}}=S(t)K(\alpha)\bar{S}(t)\bar{K}(\alpha)\bar{\bar{S}}(t)\bar{\bar{K}}(\alpha)\bar{\bar{Y}}, \tag{10}$$

where

$$\begin{aligned} \bar{\bar{S}}(t) &= \operatorname{diag}[\bar{S}(t_i)]_{(M+1)^2\times(M+1)^3}, \\ \bar{\bar{K}}(\alpha) &= \operatorname{diag}[\bar{K}(\alpha)]_{(M+1)^3\times(M+1)^3}, \\ \bar{\bar{Y}} &= [y_0\bar{Y}\ y_1\bar{Y}\ \cdots\ y_M\bar{Y}]^T_{(M+1)^3\times 1}. \end{aligned}$$

Based on the above approximations, we have:

$$\int_0^1 y^2(t)dt\simeq\int_0^1 S(t)K(\alpha)\bar{S}(t)\bar{K}(\alpha)\bar{Y}dt=K(\alpha)\bar{K}(\alpha)\mathbb{S}\bar{Y}, \tag{11}$$

in which

$$\mathbb{S}=\int_0^1 S(t)\bar{S}(t)dt=[s_{mn}]; \qquad s_{mn}=\frac{1}{m+n+1},\ m,n=0,1,\ldots,M.$$

Similarly,

$$\int_0^1 u^2(t)dt \simeq \int_0^1 S(t)K(\alpha)\bar{S}(t)\bar{K}(\alpha)\bar{U}dt = K(\alpha)\bar{K}(\alpha)\mathbb{S}\bar{U}. \tag{12}$$

$$\begin{aligned}\int_0^1 y^2(t-\eta)dt &\simeq = \int_0^1 S(t)C(-\eta)K(\alpha)\bar{S}(t)\bar{C}(-\eta)\bar{K}(\alpha)\bar{Y}dt \\ &= C(-\eta)K(\alpha)\bar{C}(-\eta)\bar{K}(\alpha)\mathbb{S}\bar{Y}.\end{aligned} \tag{13}$$

With the use of approximations (11)-(13), the performance index (1) is transformed to the following form:

$$\begin{aligned}\min \quad J(Y,U) &= \frac{1}{2}(aK(\alpha)\bar{K}(\alpha)\mathbb{S}\bar{Y} + bK(\alpha)\bar{K}(\alpha)\mathbb{S}\bar{U} + cC(-\eta)K(\alpha)\bar{C}(-\eta)\bar{K}(\alpha)\mathbb{S}\bar{Y}) \\ &\cong G(\bar{Y},\bar{U}).\end{aligned} \tag{14}$$

Now, by assuming $d, e, f = 3$ and substituting the approximations (4)-(13) in equation (2), will have:

$$\Lambda(t,Y,U) = S(t)BK(\alpha)Y - V_1(t) - V_2(t) - V_3(t) - \gamma(t)S(t)K(\alpha)U, \tag{15}$$

where

$$\begin{aligned}V_1(t) &= \alpha_1(t)S(t)K(\alpha)Y + \alpha_2(t)S(t)K(\alpha)\bar{S}(t)\bar{K}(\alpha)\bar{Y} + \alpha_3(t)S(t)K(\alpha)\bar{S}(t)\bar{K}(\alpha)\bar{\bar{S}}(t)\bar{\bar{K}}(\alpha)\bar{\bar{Y}}, \\ V_2(t) &= \beta_1(t)S(t)C(-\eta)K(\alpha)Y + \beta_2(t)S(t)C(-\eta)K(\alpha)\bar{S}(t)\bar{C}(-\eta)\bar{K}(\alpha)\bar{Y} \\ &\quad + \beta_3(t)S(t)C(-\eta)K(\alpha)\bar{S}(t)\bar{C}(-\eta)\bar{K}(\alpha)\bar{\bar{S}}(t)\bar{\bar{C}}(-\eta)\bar{\bar{K}}(\alpha)\bar{\bar{Y}},\end{aligned}$$

and

$$\begin{aligned}V_3(t) &= \int_0^1 B_1(t,\tau)S(\tau)C(-\eta)K(\alpha)Yd\tau + \int_0^1 B_2(t,\tau)S(\tau)C(-\eta)K(\alpha)\bar{S}(\tau)\bar{C}(-\eta)\bar{K}(\alpha)\bar{Y}d\tau \\ &\quad + \int_0^1 B_3(t,\tau)S(\tau)C(-\eta)K(\alpha)\bar{S}(\tau)\bar{C}(-\eta)\bar{K}(\alpha)\bar{\bar{S}}(\tau)\bar{\bar{C}}(-\eta)\bar{\bar{K}}(\alpha)\bar{\bar{Y}}d\tau.\end{aligned}$$

By substituting the collocation point (5) into (15), we obtain:

$$\Lambda_i \cong \Lambda(t_i,Y,U) \cong 0, \qquad i = 1,\ldots,2M, \tag{16}$$

with the initial condition

$$\Lambda_0 \cong S(0)K(\alpha)Y - y_0 = 0. \tag{17}$$

To get the approximate solutions of optimization problem (1)-(2), we can adopt the Lagrange multipliers method for minimizing (14) subject to the conditions given in (16)-(17) as

$$J^*(Y,U,\lambda) = G(\bar{Y},\bar{U}) + \Lambda\lambda, \tag{18}$$

where $\lambda = [\lambda_0, \lambda_1, \ldots, \lambda_{2M}]$ is the vector of unknown lagrange multipliers and $\Lambda = [\Lambda_0, \Lambda_1, \ldots, \Lambda_{2M}]$. For the optimality of functional (18) we obtain the following necessary conditions:

$$\frac{\partial J^*}{\partial Y} = 0, \qquad \frac{\partial J^*}{\partial U} = 0, \qquad \frac{\partial J^*}{\partial \lambda} = 0. \tag{19}$$

The equations (19) can be solved with any software. In this paper we used *Mathematica* package for it.

4 Residual error estimation

Let P_N be the set of all Dickson polynomials of degree at most N. If $f(t)$ be a function in $L^2[0,1]$, since P_N is a finite space, $f(t)$ has a best approximation out of P_N like as $f_0(t)$ such that:

$$\forall g \in P_N : \qquad ||f - f_0||_2 \leq ||f - g||_2.$$

Suppose that $f_n \in P_N$, then there exist coefficients $c_k, k = 0, 1, \ldots, n$, so that

$$f_n(t) \approx \sum_{k=0}^{n} c_k D_k(t, \alpha) = C_n^T D_n,$$

where $c_k, k = 0, 1, \ldots, n$, are real valued unknown coefficients and D_n is the vector of Dickson polynomials defined as:

$$C_n^T = [c_0, c_1, \ldots, c_n], \qquad D_n^T = [D_0(t, \alpha), D_1(t, \alpha), \ldots, D_n(t, \alpha)].$$

Theorem 1. *Let $f \in L^2[0,1]$ be approximated by f_n in terms of the Dickson polynomials $\{D_k(t, \alpha)\}_{k=0}^n$ that is, $f_n(t) = \sum_{k=0}^n c_k D_k(t, \alpha)$. If $e_n(t) = ||f(t) - f_n(t)||$ then $\lim_{n\to\infty} e_n(t) = 0$.*

Proof. Using the Taylor expansion, we define the following approximation of f out of P_N as follows:

$$\tilde{f}(x) = \sum_{k=0}^{n} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0^+).$$

Then we have:

$$|f(x) - \tilde{f}(t)| \leq \frac{t^{n+1}}{\Gamma(n+2)} \sup_{0 \leq t \leq 1} |f^{(n+1)}(t)|.$$

Let $L = \sup_{0 \leq t \leq 1} |f^{(n+1)}(t)|$. Because $\tilde{f}(t)$ is the best approximation of f, so

$$\|f - f_n\|_2 \leq \|f - \tilde{f}\|_2 = \int_0^1 |f(t) - \tilde{f}(t)| dx \leq \frac{L}{\Gamma(n+2)} \int_0^1 t^{n+1} dx = \frac{L}{(n+2)!}.$$

When n increases, the error quickly tends to zero [20]. □

According to this theorem, the approximations of $f(t)$ with Dickson polynomials are converging. Now, an error analysis dependent on residual function is implemented to improve the Dickson polynomials solutions. By using the proposed method we can obtain the residual function on $t \in [0,1]$ as

$$R_N(t) = G(y_i, u_i) + \lambda \Lambda(y_i, u_i), \tag{20}$$

such that

$$y(0) = y_0, \qquad \sum_{i=0}^{m} y_i(0) - y_0 = 0. \tag{21}$$

Let us now construct the residual error analysis for the Dickson polynomials. The error function $e_N(t)$ is obtained by

$$e_{N1}(t) = y(t) - y_N(t),$$
$$e_{N2}(t) = u(t) - u_N(t).$$

So the maximum absolute error can be evaluated as

$$e_N(t) = \max_{0\leq t\leq 1} |e_{N1}(t) + e_{N2}(t)|. \quad (22)$$

Accordingly, by equation (20) and (22) the error equation is of the form

$$L(e_N(t)) = \max |L(y(t)) - L(y_N(t)) + L(u(t)) - L(u_N(t))| = -R_N(t), \quad (23)$$

subject to the initial conditions (21). Thus, we constitute the error problem by equations (21) and (23) and obtain the estimated error function $e_{N,M}(t)$ as follows:

$$e_{N,M}(t) = \sum_{n=0}^{M} c_n^* D_n(t,\alpha), \qquad (M > N). \quad (24)$$

The $e_{N,M}(t)$ is the Dickson polynomials solution of the error problem (23) with condition (21). Therefore, by using (24), the solution based on Dickson polynomials will be obtained as follows:

$$y_{N,M} = y_N(t) + e_{N,M}(t), u_{N,M} = u_N(t) + e_{N,M}(t),$$

and the corrected error functions are obtained as

$$e_{1N,M}(t) = y(t) - y_{N,M}(t),$$
$$e_{2N,M}(t) = u(t) - u_{N,M}(t).$$

The corrected errors are obtained after using the residual error analysis. Authors in [14] have been shown that by increasing the number of M, these residual error functions tend to zero wich indicate a characteristic behavior of the residual function. According to the above discussion, the approximations of a function with Dickson polynomials are converging. It is also easy to conclude that by increasing the number of Dickson polynomials, the error of the derivative defined by operational matrix in (7), tends to zero.

Theorem 2. *Suppose $y_M(t)$ and $u_M(t)$ are approximations of functions $y(t)$ and $u(t)$, respectively, using Dickson polynomials of degree M, defined in Eq. 4. Then (y_M, u_M) converge to the exact solutions of problem (1)-(3) as M, the degree of the Dickson polynomials, tends to infinity.*

Proof. This theorem is easily proved by the given discussion in [21]. □

5 Application

We would test introducing method by several examples. We show the efficiency of this method by solving four non-trivial examples. Since the exact solution is often unknown, we need to estimate the error function and investigate the reliability of the method numerically. So, the accuracy of the approximate solutions is studied by substituting the solutions into Eq. (2) as follows:

$$\begin{aligned} E_M(t) = \ & |y'_M(t) - \sum_{i=1}^{d} \alpha_i(t) y_M^i(t) - \sum_{j=1}^{e} \beta_j(t) y_M^j(t-\eta) \\ & - \sum_{k=1}^{f} \int_0^1 B_k(t,\tau) y_M^k(\tau-\eta) d\tau - \gamma(t) u_M(t)|. \end{aligned} \quad (25)$$

Table 1: Numerical results of J_M^* for Example 1.

Itr	$\alpha=-1$	$\alpha=0$	$\alpha=0.1$	$\alpha=1$	Marzban et al. [18]
$M=4$	2.1843	2.18504	2.18469	2.1843	-
$M=6$	1.45606	1.45609	1.45609	1.68073	1.264683014665444244
$M=8$	9.89804	0.935525	0.938045	2.33095	1.26468301466544423585106710

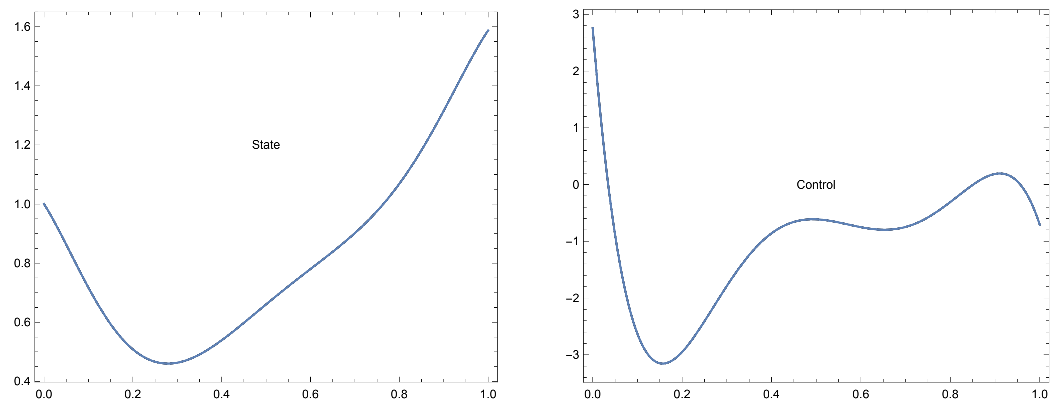

Figure 1: The black dash-line and blue curves denote the approximate solutions with $M=8$ and $M=10$, respectively, for Example 1.

We expect that $E_M(t)=0$ on the collocation points. Indeed, the more accurate of the proposed method will be obtained for the approximation solutions when $E_M(t)$ much be close to zero. To evaluate the benefits and validity of this method, consider the following examples. Also, we refer the interested readers to [14] to view the determination of the optimal value of α.

Example 1. For the first example, we consider

$$\min J(y,u)=\frac{1}{2}\int_0^1 (y^2(t)+u(t)^2)dt,$$

subject to

$$y'(t)=y(t-\frac{1}{3})+\int_0^1 y(\tau-\frac{1}{3})d\tau+\int_0^1 (t+\tau)y^2(\tau-\frac{1}{3})d\tau+u(t),$$

with the boundary conditions $y(t)=1$, for $-\frac{1}{3}\leq t\leq 0$.

Numerical results by applying the offered method for J_m^* have been reported in Table 1 and compared with the results of [18]. From the perspective of cost values, our suggested approach is somewhat more effective by increasing the number of M (especially for $\alpha=0$ and $\alpha=0.1$). The graphs of $y(t)$ and $u(t)$ for $\alpha=0.1$ and $M=8,10$ has been shown in Figure 1. The accuracy of the approximate solutions can be easily investigated by increasing the number of M and considering $\alpha=0$ in Table 2.

Table 2: Accuracy errors for various iteration with $\alpha = 0$ for Example 1.

t	0	0.2	0.4	0.6	0.8	1
E_4	3.53569×10^{-4}	4.02488×10^{-5}	6.11943×10^{-7}	3.37581×10^{-5}	3.66674×10^{-5}	2.77752×10^{-5}
E_6	5.31621×10^{-8}	1.93004×10^{-7}	3.32832×10^{-7}	4.72663×10^{-7}	6.12491×10^{-7}	7.52321×10^{-7}
E_8	1.06273×10^{-7}	2.059994×10^{-7}	3.06483×10^{-7}	4.07034×10^{-7}	5.07531×10^{-7}	6.08054×10^{-7}

Table 3: Numerical results of J_M^* for Example 2.

Itr	$\alpha = -1$	$\alpha = 0.2$	$\alpha = 1$
$M = 4$	3.5502	3.4954	3.49469
$M = 8$	2.6645	2.72597	2.92514

Example 2. As a second example let us consider the following non-linear DIOCP:

$$\min J(y, u) = \frac{1}{2} \int_0^1 (y^2(t) + y^2(t - \frac{1}{3}) + u^2(t))dt,$$

subject to

$$y'(t) = y^2(t - \frac{1}{3}) + \int_0^1 y(\tau - \frac{1}{3})d\tau + \int_0^1 (t\,\tau)y(\tau - \frac{1}{3})d\tau + u(t),$$

with boundary conditions $y(t) = 0$, for $-\frac{1}{3} \leq t < 0$, and $y(0) = 1$.

Numerical results of E_M^* by applying the proposed method for this problem leads to Table 3. The accuracy of these solutions for different choices of M and considering $\alpha = 0.2$ are reported in Table 4. These results show that the accuracy errors have been improved by increasing the number of M. The graphs of $y(t)$ and $u(t)$ for $M = 6$ and $\alpha = 0, 0.1$ has been shown in Figure 2.

Example 3. In this example we solved the following problem:

$$\min J(y, u) = \frac{1}{2} \int_0^1 (y^2(t) + u^2(t))dt,$$

subject to

$$y'(t) = y^3(t - \frac{1}{2}) + \int_0^1 y^2(\tau - \frac{1}{2})d\tau + u(t),$$

with boundary condition $y(t) = 1$, for $-\frac{1}{2} \leq t \leq 0$.

Table 4: Accuracy errors with $\alpha = 0.2$ for Example 2.

t	0	0.2	0.4	0.6	0.8	1
E_4	5.081×10^{-1}	8.20517×10^{-3}	3.89347×10^{-5}	1.23039×10^{-3}	2.21951×10^{-3}	5.6062×10^{-5}
E_6	8.61139×10^{-10}	1.90308×10^{-10}	9.76343×10^{-11}	6.14881×10^{-10}	8.13797×10^{-10}	7.29182×10^{-10}

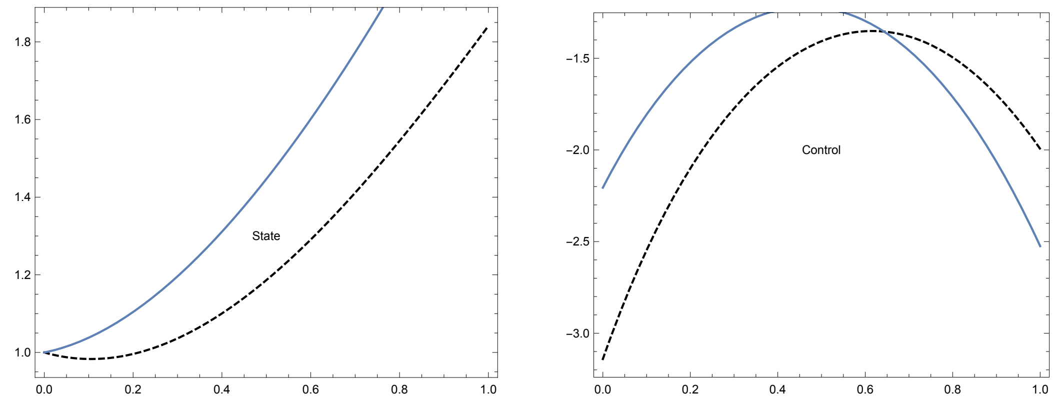

Figure 2: The black dash-line and blue curves denote the approximate solutions with $\alpha = 0$ and $\alpha = 0.1$, respectively, for Example 2.

Table 5: Numerical results of J_M^* for Example 3.

Itr	$\alpha = -1$	$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.2$	Marzban et al. [18]
$M = 2$	2.16591	2.16595	2.16596	2.16594	-
$M = 4$	2.07812	1.4788	1.37714	1.38865	-
$M = 6$	2.2367	1.44071	1.75525	1.32294	1.201779606134
$M = 8$	2.70686	14.363	1.86363	1.30674	1.201779606106786

Numerical results by applying the offered method for J_M^* have been reported in Table 5 and compared with the method of [18]. The effect of parameter M on these approximations with $\alpha = 0.2$ has been demonstrated in Table 6. The results show that the accuracy of the approximate solutions improved by increasing the number of M. Also, Table 7 shows the values of $y(t)$ and $u(t)$ at different time t with $M = 8$.

Example 4. In the last example we consider the nonlinear DIOCP presented by

$$\min J(y, u) = \frac{1}{2} \int_0^1 u^2(t)dt,$$

subject to

$$y'(t) = y^2(t - \frac{1}{2}) + \int_0^1 y^2(\tau - \frac{1}{2})d\tau + u(t),$$

Table 6: Accuracy errors with $\alpha = 0.2$ for Example 3.

t	0	0.2	0.4	0.6	0.8	1
E_4	1.14611×10^{-2}	2.48427×10^{-4}	1.69239×10^{-5}	3.60955×10^{-5}	1.00276×10^{-4}	2.68722×10^{-6}
E_6	1.78809×10^{-3}	4.3076×10^{-5}	1.63087×10^{-5}	4.10421×10^{-5}	1.0721×10^{-4}	1.91353×10^{-5}
E_8	1.09349×10^{-5}	2.89252×10^{-7}	2.15438×10^{-7}	2.45622×10^{-7}	1.17473×10^{-6}	4.71367×10^{-6}

Table 7: Numerical results of $y(t)$ and $u(t)$ with $\alpha = 0.2$ and $M = 8$ for Example 3.

t	$y(t)$	$u(t)$
0.0	1.00000	-0.100302
0.2	1.09078	-0.819379
0.4	1.12108	-1.428560
0.6	1.19270	-1.355930
0.8	1.43005	0.393385
1.0	1.936450	0.915803

Table 8: Numerical results of J_M^* for Example 4.

Itr	$\alpha = -1$	$\alpha = -0.9$	$\alpha = -0.88$	$\alpha = 1$
$M = 2$	5.58721×10^{-18}	8.41906×10^{-21}	5.56755×10^{-23}	2.75666×10^{-7}

with boundary condition $y(t) = t$, for $t \leq 0$.

The exact optimal control function is $u(t) = 0$ and optimal trajectory $y(t)$ is

$$y(t) = \begin{cases} \frac{7}{24}t - \frac{1}{2}t^2 + \frac{1}{3}t^3, & 0 \leq t < \frac{1}{2}, \\ \frac{8083}{161280} + \frac{373}{3780}t - \frac{125}{384}t^2 + \frac{985}{1128}t^3 - \frac{55}{96}t^4 + \frac{61}{180}t^5 - \frac{1}{9}t^6 + \frac{1}{63}t^7, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Numerical results by applying the our approach for J_M^* have been reported in Table 8. In addition, the accuracy of the approximate solution for $M = 2$ and $\alpha = -0.88$ have been obtained in Table 9. High accuracy for the numerical results of this example occurs in the low iteration. In Table 10 it is shown that our method provides very accurate numerical solution for this problem.

6 Conclusion

We have presented Dickson polynomials with a collocation method to solve a class of DIOCPs. Our design uses the control variables and the state via a linear combination of Dickson polynomials as basic functions. The properties of these functions, allows us to reduce the time-delay optimal control problem to a system of nonlinear algebraic equations for choosing the unknown coefficients optimally. Using Dickson polynomials via a collocation method bears some advantages such as simply evaluation of high order derivatives and integral terms of given differential

Table 9: Accuracy errors with $\alpha = -0.88$ for Example 4.

t	0	0.2	0.4	0.6	0.8	1
E_2	1.33924×10^{-18}	7.81568×10^{-19}	4.33901×10^{-19}	2.96197×10^{-19}	3.68421×10^{-19}	6.50551×10^{-19}

Table 10: Numerical results of $y(t)$ and $u(t)$ with $\alpha = -0.88$ and $M = 2$ for Example 4.

t	$y(t)$	$u(t)$
0.0	0.00000	2.07023×10^{-11}
0.2	3.44836×10^{-12}	1.37813×10^{-11}
0.4	5.51251×10^{-12}	6.86024×10^{-12}
0.6	6.19245×10^{-12}	-6.07916×10^{-14}
0.8	5.48819×10^{-12}	-6.98182×10^{-12}
1.0	3.39972×10^{-12}	-1.39028×10^{-11}

equation and less expensive of computational costs. Four examples are solved to illustrate the efficiency, applicability and high performance of this approach. As can be seen in these examples, the parameter α plays an important role in the Dickson polynomials in a way that can change the behavior of the solution. The accuracy of the Dickson collocation method can be easily concluded from the improved results by our introduced method. Moreover, this approach is applicable to both optimal control problems with time-delay governed by Volterra integro-differential equations and Volterra- Fredholm integro-differential equations. All calculations were carried out using Mathematica software.

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