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Denumerably many positive solutions for singular iterative system of fractional differential equation with R-L fractional integral boundary conditions

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Abstract. In this paper, we establish the existence of denumerably many positive solutions for singular iterative system of fractional order boundary value problem involving Riemann–Liouville integral boundary conditions with increasing homeomorphism and positive homomorphism operator by using Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space.

Keywords: Denumerable, positive solutions, fractional derivative, homeomorphism, homomorphism, fixed point theorem.
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1 Introduction

The study of turbulent flow through porous media is important for a wide range of scientific and engineering applications such as fluidized bed combustion, compact heat exchangers, combustion in an inert porous matrix, high temperature gas-cooled reactors, chemical catalytic reactors and drying of different products such as iron ore [3, 9, 12, 17, 19]. To study this type of problems, Leibenson [8] introduced the p-Laplacian equation,

$$\big(\phi_p(x'(t))\big)' = f\big(t, x(t), x'(t)\big),$$

where $\phi_p(\tau) = |\tau|^{p-2}\tau,\ p > 1$. The operator ϕ_p is invertible and its inverse operator is defined by ϕ_q, where $q > 1$ is a constant such that $q = p/(p-1)$. The recent works on the existence, uniqueness and existence of positive solutions for fractional order boundary value problems, see [1, 5, 10, 11, 13, 14, 16, 18, 20].

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The fractional order p-Laplacian operator arises in many applied fields such as turbulant filtration in porous media, blood flow problems, rheology, modelling of viscoplasticity, material science, and it is worth developing the theory to fractional differential equations with p-Laplacian operator. Moreover research on increasing homeomorphism and positive homomorphism operators has gained momentum recently.

In this paper we define a new operator called increasing homeomorphism and positive homomorphism operator, which improves and generalizes the p-Laplacian operator for some $p > 1$, and ϕ is not necessarily odd. In [21], Zhao and Liu studied the following fractional order boundary value problem,

$$\begin{gathered}(\phi({}^{\mathcal{C}}\mathscr{D}^{v}_{0+}u(t)))' + h(t)\mathsf{g}(t, u(\theta(t))) = 0,\ t \in (0,1),\\ u(0) = au(1),\ u'(1) = bu'(0) + \lambda[u],\\ u^{i}(0) = 0,\ i = 2, \ldots, n-1,\end{gathered}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing and positive homomorphism, $2 \le n-1 < v \le n$ and ${}^{\mathcal{C}}\mathscr{D}^{v}_{0+}$ is the Caputo fractional derivative. In the sense of a monotone homomorphism, they established some sufficient criteria for the existence of at least two monotone positive solutions by employing the fixed point theorem on cone expansion and compression.

In [4], Ege and Topal discussed the existence and multiplicity of positive solutions to the four point fractional order boundary value problem with increasing and positive homomorphism operator by using Krasnoselskiis and Legget–Williams fixed point theorems in a cone,

$$\begin{gathered}{}^{\mathcal{C}}\mathscr{D}^{q}_{0+}\big(\phi({}^{\mathcal{C}}\mathscr{D}^{r}_{0+}x(t))\big) + f\big(t, x(t)\big) = 0,\ t \in (0,1),\\ \alpha_1 x(0) - \beta_1 x'(0) = -\gamma_1 x(\xi_1),\\ \alpha_2 x(1) + \beta_2 x'(1) = -\gamma_2 x(\xi_2),\\ {}^{\mathcal{C}}\mathscr{D}^{r}x(0) = 0\end{gathered}$$

where ${}^{\mathcal{C}}\mathscr{D}^{r}_{0+}$ and ${}^{\mathcal{C}}\mathscr{D}^{q}_{0+}$ are the Caputo fractional derivatives of order r and q respectively with $1 < r \le 2$, $0 < q \le 1$.

Inspired by the aforementioned work, in this article we establish countably infinitely many positive solutions of fractional differential equations with Riemann-Liouville fractional integral boundary conditions with an increasing homeomorphism and positive homomorphism operator,

$$\left.\begin{gathered}\phi\left[{}^{\mathcal{C}}\mathscr{D}^{\sigma}_{0+}\varpi_{\mathtt{j}}(t)\right] + \Psi(t)\mathsf{g}_{\mathtt{j}}(\varpi_{\mathtt{j}+1}(t)) = 0,\ 0 < t < 1,\ \mathtt{j} = 1, 2, \ldots, \ell,\\ \varpi_{\ell+1}(t) = \varpi_1(t),\ 0 < t < 1,\end{gathered}\right\} \tag{1}$$

satisfying integral boundary conditions

$$\left.\begin{aligned}\varpi_{\mathtt{j}}(0) - a\varpi'_{\mathtt{j}}(0) = \mathcal{I}^{\alpha}_{0+}\varpi_{\mathtt{j}}(1),\\ \varpi_{\mathtt{j}}(1) + b\varpi'_{\mathtt{j}}(1) = \mathcal{I}^{\beta}_{0+}\varpi_{\mathtt{j}}(1),\end{aligned}\right\} \tag{2}$$

where ${}^{\mathcal{C}}\mathscr{D}^{\sigma}_{0+}$ denote Caputo fractional derivatives with $1 < \sigma \le 2$, $\mathcal{I}^{\alpha}_{0+}$, $\mathcal{I}^{\beta}_{0+}$ denote Riemann-Liouville fractional integrals, $a, b \in \mathbb{R}$, $\alpha, \beta > 0$, $\Psi(t) = \prod_{i=1}^{n} \psi_i$, and each $\psi_i : [0,1] \to [0, +\infty)$

has a singularity in $(0, \frac{1}{2})$, $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing and positive homomorphism with $\phi(0) = 0$ and $\phi^{-1}(\Psi) \in \mathcal{L}_{\mathtt{p}}[0, 1]$ for some $\mathtt{p} \geq 1$.

A projection $\phi : \mathbb{R} \to \mathbb{R}$ is called an increasing and positive homomorphism, if the following conditions are satisfied:

(a) if $u \leq v$, then $\phi(u) \leq \phi(v)$ for all $u, v \in \mathbb{R}$.

(b) ϕ is a continuous bijection and its inverse is also continuous.

(c) $\phi(uv) = \phi(u)\phi(v)$ for all $u, v \in \mathbb{R}$.

Remark 1. *(i)* $\phi^{-1}(uv) = \phi^{-1}(u)\phi^{-1}(v)$,
(ii)Also $\phi(0) = 0$ *that* $\phi(u) \geq 0$ *if* $u \geq 0$ *and* $\phi(u) \leq 0$ *if* $u \leq 0$.

Remark 2. *It is not difficult to observe that the p-Laplacian operator* $\phi_p(u) = |u|^{p-2}u,\ p > 1$, *is an increasing and positive homomorphism.The operator* ϕ *is regarded as the improvement and generalization of the classical p-Laplacian operator* $\phi_p(u) = |u|^{p-2}u,\ p > 1$.

We will suppose that throughout the paper following conditions hold:

(H_1) $\mathtt{g}_j : [0, +\infty) \to [0, +\infty)$ is continuous,

(H_2) there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that

$$t_{k+1} < t_k,\ t_1 < \frac{1}{2},\ \lim_{k\to\infty} t_k = t^* \geq 0,\ \lim_{t\to t_k} \psi_i(t) = +\infty,\ k \in \mathbb{N}, i = 1, 2, \cdots, m,$$

and $\psi_i(t)$ does not vanish identically on any subinterval of $[0, 1]$.
Moreover,

$$0 < \int_0^1 (1-\tau)^{r-1}\psi_i(\tau)d\tau < +\infty \text{ for } 0 < r \leq 1.$$

(H_3) $\sigma(1+a) - a > 2$, $b \geq 1$, $0 \leq \varsigma_1 + \varsigma_2 < 1$ and $\varsigma = 1 - \varsigma_1 - \varsigma_4 + \varsigma_1\varsigma_4 - \varsigma_2\varsigma_3 > 0$ where

$$\varsigma_1 = \frac{\alpha(1+b)+b}{(1+a+b)\Gamma(\alpha+2)},\ \varsigma_2 = \frac{a(1+\alpha)+1}{(1+a+b)\Gamma(\alpha+2)},$$
$$\varsigma_3 = \frac{\beta(1+b)+b}{(1+a+b)\Gamma(\beta+2)},\ \varsigma_4 = \frac{a(1+\beta)+1}{(1+a+b)\Gamma(\beta+2)}.$$

The rest of the paper is organized in the following fashion. In Section 2, we provide some definitions and lemmas that provide us with some useful information concerning the behavior of solution of the boundary value problem (1)-(2), then we construct the kernel for the homogeneous problem corresponding to (1)-(2), estimate bounds for the kernel, and some lemmas which are needed in establishing our main results. In Section 3, we establish a criterion for the existence of countable number of positive solutions for the boundary value problem (1)-(2) by applying Hölder's inequality and Krasnoselskii's cone fixed point index theorem in a Banach space. Finally, as an application, an example to demonstrate our results is given.

2 Kernel and its bounds

In this section, we list some definitions and lemmas which are useful for our later discussions, and constructed kernel to the homogeneous BVP corresponding to (1)-(2), and establish certain lemmas for the bounds of the kernel.

Definition 1. *[2] The Riemann-Liouville fractional integral of order γ for a function f is defined as*

$$\mathcal{I}_{0+}^{\gamma}\mathtt{g}(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mathtt{g}(s)ds, \ \gamma > 0.$$

In particular, when $t = 1$,

$$\mathcal{I}_{0+}^{\gamma}\mathtt{g}(1) = \frac{1}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}\mathtt{g}(s)ds.$$

Hence, $\mathcal{I}_{0+}^{\gamma}\mathtt{g}(t) \le \mathcal{I}_{0+}^{\gamma}\mathtt{g}(1)$ for $\mathtt{g}(t) \ge 0$ and $0 \le t \le 1$.

Definition 2. *[2] For a function f given on the interval $[0,\infty)$, the Caputo derivative of fractional order γ for the continuous function f on $[0,\infty)$ is defined as*

$${}^{\mathcal{C}}\mathscr{D}_{0+}^{\gamma}\mathtt{g}(t) := \frac{1}{\Gamma(m-\gamma)}\int_0^t (t-s)^{m-\gamma-1}f^{(m)}(s)ds, \ m = [\gamma]+1,$$

where $[\gamma]$ denotes the integer part of γ.

Lemma 1. *[7, 15] Let $\gamma > 0$. Then the differential equation ${}^{\mathcal{C}}\mathscr{D}_{0+}^{\gamma}\varpi(t) = 0$ has solutions*

$$\varpi(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1}t^{m-1}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, m-1$, $m = [\gamma]+1$.

Lemma 2. *[7, 15] Let $\gamma > 0$. Then the differential equation ${}^{\mathcal{C}}\mathscr{D}_{0+}^{\gamma}\varpi(t) = 0$ has solutions*

$$\mathcal{I}_{0+}^{\gamma}({}^{\mathcal{C}}\mathscr{D}_{0+}^{\gamma}y)(t) = \varpi(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1}t^{m-1}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, m$, $m = [\gamma]+1$.

To establish the existence of solution of the boundary value problem (1)-(2), we need the following Lemma 3, which is crucial in changing boundary value problem (1)-(2) into an equivalent integral equation.

Lemma 3. *Suppose (H_3) holds. Let $1 < \sigma \le 2$ and $\mathtt{f} \in \mathcal{C}[0,1]$. Then boundary value problem*

$$\left.\begin{aligned} \phi\big({}^{\mathcal{C}}\mathscr{D}_{0+}^{\sigma}\varpi_1(t)\big) + \mathtt{f}(t) = 0, \ 0 < t < 1,\\ \varpi_1(0) - a\varpi_1'(0) = \mathcal{I}_{0+}^{\alpha}\varpi_1(1),\\ \varpi_1(1) + b\varpi_1'(1) = \mathcal{I}_{0+}^{\beta}\varpi_1(1), \end{aligned}\right\} \tag{1}$$

has a unique solution y and is given by

$$\varpi_1(t) = \int_0^1 \aleph(t,\tau)\phi^{-1}(\mathtt{f}(\tau))d\tau + \int_0^1 \mathscr{G}(t,\tau)\int_0^1 \aleph(\tau,\tau_1)\phi^{-1}(\mathtt{f}(\tau_1))d\tau_1 d\tau,$$

where

$$\aleph(t,\tau) = \frac{1}{\eta}\begin{cases} (a+t)(1-\tau+b(\sigma-1))(1-\tau)^{\sigma-2} - (1+a+b)(t-\tau)^{\sigma-1}, & \tau \le t, \\ (a+t)(1-\tau+b(\sigma-1))(1-\tau)^{\sigma-2}, & t \le \tau, \end{cases}$$

$\eta = (1+a+b)\Gamma(\sigma)$, $\mathscr{G}(t,\tau) = \mathscr{G}_1(\tau)t + \mathscr{G}_2(\tau)$ *in which*

$$\mathscr{G}_1(\tau) = \frac{1}{\varsigma(1+a+b)}\left[\frac{(\varsigma_3+\varsigma_4-1)(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-\varsigma_1-\varsigma_2)(1-\tau)^{\beta-1}}{\Gamma(\beta)}\right]$$

and

$$\mathscr{G}_2(\tau) = \frac{1}{\varsigma(1+a+b)}\left[\frac{\big(a\varsigma_3+(1+b)(1-\varsigma_4)\big)(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\big(a(1-\varsigma_1)+(1+b)\varsigma_2\big)(1-\tau)^{\beta-1}}{\Gamma(\beta)}\right].$$

Proof. From Lemma 2, the general solution of the equation (1) can be written as

$$\varpi_1(t) = -\mathcal{I}_{0^+}^{\sigma}\phi^{-1}(\mathtt{f}(t)) + A + Bt,$$

where $A, B \in \mathbb{R}$ are arbitrary constants. By the boundary conditions, we find that

$$\begin{aligned} A = {} & \frac{ab}{(1+a+b)\Gamma(\sigma-1)}\int_0^1 (1-s)^{\sigma-2}\phi^{-1}(\mathtt{f}(s))ds + \frac{1+b}{1+a+b}\mathcal{I}_{0^+}^{\alpha}\varpi_1(1) \\ & + \frac{a}{(1+a+b)\Gamma(\sigma)}\int_0^1 (1-s)^{\sigma-1}\phi^{-1}(\mathtt{f}(s))ds + \frac{a}{1+a+b}\mathcal{I}_{0^+}^{\beta}\varpi_1(1), \end{aligned}$$

and

$$\begin{aligned} B = {} & \frac{1}{(1+a+b)\Gamma(\sigma)}\int_0^1 (1-s)^{\sigma-1}\phi^{-1}(\mathtt{f}(s))ds - \frac{1}{1+a+b}\mathcal{I}_{0^+}^{\alpha}\varpi_1(1) \\ & + \frac{b}{(1+a+b)\Gamma(\sigma-1)}\int_0^1 (1-s)^{\sigma-2}\phi^{-1}(\mathtt{f}(s))ds + \frac{1}{1+a+b}\mathcal{I}_{0^+}^{\beta}\varpi_1(1). \end{aligned}$$

Therefore,

$$\varpi_1(t) = \int_0^1 \aleph(t,\tau)\phi^{-1}(\mathtt{f}(\tau))d\tau + \frac{1+b-t}{1+a+b}\mathcal{I}_{0^+}^{\alpha}\varpi_1(1) + \frac{a+t}{1+a+b}\mathcal{I}_{0^+}^{\beta}\varpi_1(1). \tag{2}$$

By simple calculations, we get

$$\begin{aligned} \mathcal{I}_{0^+}^{\alpha}\varpi_1(1) = \frac{1}{\varsigma}\Big[& (1-\varsigma_4)\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 \aleph(\tau,\tau_1)\phi^{-1}(\mathtt{f}(\tau_1))d\tau_1 d\tau \\ & + \varsigma_2\int_0^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}\int_0^1 \aleph(\tau,\tau_1)\phi^{-1}(\mathtt{f}(\tau_1))d\tau_1 d\tau\Big], \end{aligned}$$

and

$$\mathcal{I}_{0+}^{\beta}\varpi_1(1) = \frac{1}{\varsigma}\Big[\varsigma_3\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 \aleph(\tau,\tau_1)\phi^{-1}(\mathfrak{f}(\tau_1))d\tau_1 d\tau + (1-\varsigma_1)\int_0^1 \frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}\int_0^1 \aleph(\tau,\tau_1)\phi^{-1}(\mathfrak{f}(\tau_1))d\tau_1 d\tau\Big].$$

Substituting above two identities into (2), we get

$$\varpi_1(t) = \int_0^1 \aleph(t,\tau)\phi^{-1}(\mathfrak{f}(\tau))d\tau + \int_0^1 \mathscr{G}(t,\tau)\int_0^1 \aleph(\tau,\tau_1)\phi^{-1}(\mathfrak{f}(\tau_1))d\tau_1 d\tau,$$

which completes the proof. □

Lemma 4. *Suppose* (H_3) *holds. Then the kernel* $\aleph(t,s)$ *has the following properties,*

(i) $\aleph(t,\tau) > 0$ *is continuous on* $[0,1]\times[0,1]$,

(ii) $\aleph(t,\tau) \le \aleph(\tau,\tau)$ *for any* $t,\tau \in [0,1]$,

(iii) *there exists a positive number* ϱ *such that* $\varrho\aleph(\tau,\tau) \le \aleph(t,\tau)$ *for* $t,\tau \in [0,1]$, *where* $\varrho = \min\{\rho, a\rho\}$ *in which*

$$\rho = \frac{4b[a(\sigma-1)+\sigma-2]}{[1-a+b(\sigma-1)]^2+4a[1+b(\sigma-1)]}.$$

The proof of the lemma is similar to that of Lemma 3.2 in [22], so we omit it here.

Lemma 5. *Let* $\mathfrak{z}\in(0,\frac{1}{2})$. *Then* $\min\limits_{t\in[\mathfrak{z},1-\mathfrak{z}]}\mathscr{G}(t,\tau) \ge \mathfrak{z}\max\limits_{t\in[0,1]}\mathscr{G}(t,\tau)$.

Proof. From Lemma 3, we have $\mathscr{G}(t,\tau) = \mathscr{G}_1(\tau)t + \mathscr{G}_2(\tau)$. Then $\frac{\partial}{\partial t}\mathscr{G}(t,\tau) = \mathscr{G}_1(\tau)$. If $\mathscr{G}_1(\tau)\ge 0$, i.e.,

$$\frac{(1-\varsigma_1-\varsigma_2)(1-\tau)^{\beta-1}}{\Gamma(\beta)} \ge \frac{(1-\varsigma_3-\varsigma_4)(1-\tau)^{\alpha-1}}{\Gamma(\alpha)},$$

then $\mathscr{G}(t,\tau)$ increases and the minimum value is $\mathscr{G}(0,\tau) = \mathscr{G}_2(\tau)$. We have

$$\mathscr{G}_2(\tau) > \frac{(1-\varsigma_4)\frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} + \varsigma_2\frac{1-\varsigma_3-\varsigma_4}{1-\varsigma_1-\varsigma_2}\cdot\frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}}{\varsigma(1+a+b)} = \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)(1-\varsigma_1-\varsigma_2)(1+a+b)} > 0.$$

So,

$$\frac{\min_{t\in[\mathfrak{z},1-\mathfrak{z}]}\mathscr{G}(t,\tau)}{\max_{t\in[0,1]}\mathscr{G}(t,\tau)} = \frac{\mathscr{G}(\mathfrak{z},\tau)}{\mathscr{G}(1,\tau)} = \frac{\mathfrak{z}\mathscr{G}_1(\tau)+\mathscr{G}_2(\tau)}{\mathscr{G}_1(\tau)+\mathscr{G}_2(\tau)} \ge \mathfrak{z}.$$

If $\mathscr{G}_1(\tau) < 0$, then $\mathscr{G}(t,\tau)$ decreases so that the minimum value is

$$\mathscr{G}(1,\tau) = \mathscr{G}_1(\tau) + \mathscr{G}_2(\tau),$$

and the maximum value is $\mathscr{G}(0,\tau) = \mathscr{G}_2(\tau)$. Since $\mathscr{G}_1(\tau) < 0$, we have

$$0 < \frac{(1-\varsigma_1-\varsigma_2)(1-\tau)^{\beta-1}}{\Gamma(\beta)} < \frac{(1-\varsigma_3-\varsigma_4)(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}.$$

In this case

$$\mathscr{G}_1(\tau)+\mathscr{G}_2(\tau)=\frac{((a+1)\varsigma_3+b(1-\varsigma_4))\frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}+((a+1)(1-\varsigma_1)+b\varsigma_2)\frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}}{\varsigma(1+a+b)}$$
$$>\frac{(1-\varsigma_4)\frac{1-\varsigma_1-\varsigma_2}{1-\varsigma_3-\varsigma_4}.\frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}+\varsigma_2\frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)}}{\varsigma(1+a+b)}=\frac{(1-\tau)^{\beta-1}}{\Gamma(\beta)(1-\varsigma_3-\varsigma_4)(1+a+b)}>0.$$

Thus,

$$\frac{\min_{t\in[\mathfrak{z},1-\mathfrak{z}]}\mathscr{G}(t,\tau)}{\max_{t\in[0,1]}\mathscr{G}(t,\tau)}=\frac{\mathscr{G}(1-\mathfrak{z},\tau)}{\mathscr{G}(0,\tau)}=\frac{(1-\mathfrak{z})\mathscr{G}_1(\tau)+\mathscr{G}_2(\tau)}{\mathscr{G}_2(\tau)}\geq\frac{(1-\mathfrak{z})\mathscr{G}_1(\tau)}{\mathscr{G}_2(\tau)}+1\geq\mathfrak{z},$$

which completes the proof. □

We note that an $\ell-$tuple $(\varpi_1(t),\varpi_2(t),\varpi_3(t),\ldots,\varpi_\ell(t))$ is a solution of the iterative boundary value problem (1)–(2) if and only if

$$\begin{aligned}\varpi_j(t)&=\int_0^1\aleph(t,\tau)\phi^{-1}\big[\Psi(\tau)\mathrm{g}_j(\varpi_{j+1}(\tau))\big]d\tau\\&\quad+\int_0^1\mathscr{G}(t,\tau)\int_0^1\aleph(\tau,\tau_1)\phi^{-1}\big[\Psi(\tau_1)\mathrm{g}_j(\varpi_{j+1}(\tau_1))\big]d\tau_1 d\tau,\ 1\leq j\leq\ell,\\\varpi_{\ell+1}(t)&=\varpi_1(t),\ 0<t<1,\end{aligned}$$

i.e.,

$$\begin{aligned}\varpi_1(t)=&\int_0^1\aleph(t,\tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)\mathrm{g}_1\Bigg(\int_0^1\aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)\mathrm{g}_2\Bigg(\int_0^1\aleph(\tau_2,\tau_3)\\&\times\phi^{-1}\Bigg[\Psi(\tau_3)\mathrm{g}_3\Bigg(\int_0^1\aleph(\tau_3,\tau_4)\cdots\\&\times\mathrm{g}_{\ell-1}\Bigg(\int_0^1\aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)\mathrm{g}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1\\&+\int_0^1\mathscr{G}(t,\tau)\int_0^1\aleph(\tau,\tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)\mathrm{g}_1\Bigg(\int_0^1\aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)\\&\times\mathrm{g}_2\Bigg(\int_0^1\aleph(\tau_2,\tau_3)\phi^{-1}\Bigg[\Psi(\tau_3)\mathrm{g}_3\Bigg(\int_0^1\aleph(\tau_3,\tau_4)\cdots\\&\times\mathrm{g}_{\ell-1}\Bigg(\int_0^1\aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)\mathrm{g}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1 d\tau.\end{aligned}$$

Define the Banach space $\mathscr{E}=\mathcal{C}([0,1],\mathbb{R})$ with norm

$$\|\varpi\|=\sup_{t\in[0,1]}|\varpi(t)|.$$

For a fixed $\mathfrak{z} \in (0, \frac{1}{2})$, define the cone $\mathcal{P}_{\mathfrak{z}} \subset \mathscr{E}$ by

$$\mathcal{P}_{\mathfrak{z}} = \Big\{\varpi \in \mathscr{E} : \varpi(t) \geq 0 \text{ on } [0,1] \text{ and } \min_{t\in[\mathfrak{z},1-\mathfrak{z}]} \varpi(t) \geq \Delta_{\mathfrak{z}}\|\varpi\|\Big\},$$

where $\Delta_{\mathfrak{z}} = \min\{\varrho, \mathfrak{z}\}$.

Define an operator $\Omega : \mathcal{P}_{\mathfrak{z}} \to \mathscr{E}$ by

$$\begin{aligned}(\Omega\varpi_1)(t) = &\int_0^1 \aleph(t,\tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)g_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)g_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\
&\times \phi^{-1}\Bigg[\Psi(\tau_3)g_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\times g_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1\\
&+\int_0^1 \mathscr{G}(t,\tau)\int_0^1 \aleph(\tau,\tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)g_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)\\
&\times g_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\phi^{-1}\Bigg[\Psi(\tau_3)g_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\times g_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1 d\tau.\end{aligned}$$

Lemma 6. *Assume that (H_1)-(H_3) hold. Then $g(\mathcal{P}_{\mathfrak{z}}) \subset \mathcal{P}_{\mathfrak{z}}$ and $\Omega : \mathcal{P}_{\mathfrak{z}} \to \mathcal{P}_{\mathfrak{z}}$ is completely continuous for each $\mathfrak{z} \in (0, \frac{1}{2})$.*

Proof. Fix $\mathfrak{z} \in (0, \frac{1}{2})$. Since $\Psi(\tau)g_j(\varpi_1(\tau)) \geq 0$ for all $\tau \in [0,1]$, $\varpi_1 \in \mathcal{P}_\delta$ and $\aleph(t,\tau) \geq 0$ for all $t, \tau \in [0,1]$, it follows that $(\Omega\varpi_1)(t) \geq 0$ for all $t \in [0,1]$, $\varpi_1 \in \mathcal{P}_{\mathfrak{z}}$. On the other hand, by Lemmas 4 and 5 we obtain

$$\begin{aligned}(\Omega\varpi_1)(t) \leq &\int_0^1 \aleph(\tau_1,\tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)g_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)g_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\
&\times \phi^{-1}\Bigg[\Psi(\tau_3)g_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\times g_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1\\
&+\max_{t\in[0,1]}\int_0^1 \mathscr{G}(t,\tau)\int_0^1 \aleph(\tau,\tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)g_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)\\
&\times g_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\phi^{-1}\Bigg[\Psi(\tau_3)g_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\times g_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1 d\tau\end{aligned}$$

and

$$
\begin{aligned}
&\min_{t\in[\mathfrak{z},1-\mathfrak{z}]}(\Omega\varpi_1)(t)\\
&\quad\ge \varrho\int_0^1 \aleph(\tau_1,\tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)g_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)g_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\
&\qquad\times \phi^{-1}\Bigg[\Psi(\tau_3)g_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\qquad\times g_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1\\
&\qquad+\mathfrak{z}\max_{t\in[0,1]}\int_0^1 \mathscr{G}(t,\tau)\int_0^1 \aleph(\tau,\tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)g_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)\\
&\qquad\times g_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\phi^{-1}\Bigg[\Psi(\tau_3)g_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\qquad\times g_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1 d\tau\\
&\quad\ge \Delta_{\mathfrak{z}}(\Omega\varpi_1)(t),
\end{aligned}
$$

for all $t\in[0,1]$. Thus $\Omega(\mathcal{P}_{\mathfrak{z}})\subset\mathcal{P}_{\mathfrak{z}}$. Next, by standard methods and the Arzela-Ascoli theorem, one can easily prove that the operator Ω is completely continuous. The proof is complete. □

3 Denumerably many positive solutions

In this section, we establish the existence of denumerably many positive solutions for the boundary value problem (1)–(2) by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space.

Theorem 1. *[6] Let $\mathscr{E}$ be a Banach space and let $\mathcal{P}\subset\mathscr{E}$ be a cone in $\mathscr{E}$. Assume that Λ_1,Λ_2 are open with $0\in\Lambda_1,\overline{\Lambda}_1\subset\Lambda_2$, and let $\Omega:\mathcal{P}\cap(\overline{\Lambda}_2\backslash\Lambda_1)\to\mathcal{P}$ be a completely continuous operator such that either*

(i) $\|\Omega\varpi\|\le\|\varpi\|$, $\varpi\in\mathcal{P}\cap\partial\Lambda_1$, *and* $\|\Omega\varpi\|\ge\|\varpi\|$, $\varpi\in\mathcal{P}\cap\partial\Lambda_2$, *or*

(ii) $\|\Omega\varpi\|\ge\|\varpi\|$, $\varpi\in\mathcal{P}\cap\partial\Lambda_1$, *and* $\|\Omega\varpi\|\le\|\varpi\|$, $\varpi\in\mathcal{P}\cap\partial\Lambda_2$.

Then Ω has a fixed point in $\mathcal{P}\cap(\overline{\Lambda}_2\backslash\Lambda_1)$.

In order to establish some of the norm inequalities in Theorem 1 we need Hölder's inequality. We use standard notation of $\mathcal{L}^p[0,1]$ for the space of measurable functions such that

$$\int_0^1 |f(\tau)|^p d\tau<\infty,$$

where the integral is understood in the Lebesgue sense. The norm on $\mathcal{L}^{\mathtt{p}}[0,1]$, $\|\cdot\|_{\mathtt{p}}$, is defined by

$$\|\mathtt{f}\|_{\mathtt{p}} = \begin{cases} \left(\displaystyle\int_0^1 |\mathtt{f}(\tau)|^{\mathtt{p}} d\tau\right)^{\frac{1}{\mathtt{p}}}, & \mathtt{p} \in \mathbb{R}, \\ inf\Big\{M \in \mathbb{R} \,/\, |\mathtt{f}| \le M \ a.e. \ on \ [0,1]\Big\}, & \mathtt{p} = \infty. \end{cases}$$

Theorem 2 (Hölder's Inequality). *Let* $\mathtt{f} \in \mathcal{L}^{\mathtt{p}_i}[0,1]$ *with* $\mathtt{p}_i > 1$, *for* $i = 1, 2, \dots, n$ *and* $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} = 1$. *Then* $\prod_{i=1}^{n} \mathtt{f}_i \in \mathcal{L}^1[0,1]$ *and* $\|\prod_{i=1}^{n} \mathtt{f}_i\|_1 \le \prod_{i=1}^{n} \|\mathtt{f}_i\|_{\mathtt{p}_i}$. *Further, if* $\mathtt{f} \in \mathcal{L}^1[0,1]$ *and* $\mathtt{g} \in \mathcal{L}^{\infty}[0,1]$, *then* $\mathtt{fg} \in \mathcal{L}^1[0,1]$ *and* $\|\mathtt{fg}\|_1 \le \|\mathtt{f}\|_1 \|\mathtt{g}\|_{\infty}$.

Consider the following three possible cases for $\phi^{-1}(\Psi) \in \mathcal{L}^{\mathtt{p}_i}[0,1]$:

$(i)\ \displaystyle\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} < 1, \quad (ii)\ \displaystyle\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} = 1, \quad (iii)\ \displaystyle\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} > 1.$

Firstly, we seek denumerably infinitely many positive solutions for the case $\displaystyle\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} < 1$.

Theorem 3. *Assume that* $(H_1) - (H_3)$ *hold and let* $\{\mathfrak{z}_k\}_{k=1}^{\infty}$ *be such that* $t_{k+1} < \mathfrak{z}_k < t_k$, $k = 1, 2, 3, \dots$. *Let* $\{R_k\}_{k=1}^{\infty}$ *and* $\{r_k\}_{k=1}^{\infty}$ *be such that*

$$R_{k+1} < \Delta_{\mathfrak{z}_k} r_k < \theta r_k < R_k, \ k \in \mathbb{N},$$

where

$$\theta = \max\left\{ \left[\Delta_{\mathfrak{z}_1} \prod_{i=1}^{n} \lambda_i \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph(\tau_\ell, \tau_\ell) d\tau_\ell \right]^{-1}, \ 1 \right\}.$$

Assume that g_j *satisfies*

(A_1) $\mathtt{g}_j(\varpi(t)) \le \phi(\frac{M_1 R_k}{1+\mathfrak{K}})$ *for all* $t \in [0,1]$, $0 \le \varpi \le R_k$,

where $M_1 < \left[\|\aleph\|_{\mathtt{q}} \displaystyle\prod_{i=1}^{n} \left\|\phi^{-1}(\psi_i)\right\|_{\mathtt{p}_i} \right]^{-1}$, $\mathfrak{K} = \displaystyle\max_{t\in[0,1]} \left\{ \int_0^1 \mathscr{G}(t,\tau) d\tau \right\}$.

(A_2) $\mathtt{g}_j(\varpi(t)) \ge \phi(\theta r_k)$ *for all* $t \in [\mathfrak{z}_k, 1-\mathfrak{z}_k]$, $\Delta_{\mathfrak{z}_k} r_k \le \varpi \le r_k$.

Then the iterative boundary value problem (1)-(2) *has denumerably many positive solutions* $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \dots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ *such that* $\varpi_j^{[r]}(t) > 0$ *on* $(0,1)$, $j = 1, 2, \dots, \ell$ *and* $r \in \mathbb{N}$.

Proof. Consider the sequences $\{\Lambda_{1,k}\}_{k=1}^{\infty}$ and $\{\Lambda_{2,k}\}_{k=1}^{\infty}$ of open subsets of $\mathscr{E}$ defined by

$$\Lambda_{1,k} = \{\varpi \in \mathscr{E} : \|\varpi\| < R_k\}, \quad \Lambda_{2,k} = \{\varpi \in \mathscr{E} : \|\varpi\| < r_k\}.$$

Let $\{\mathfrak{z}_k\}_{k=1}^{\infty}$ be as in the hypothesis and note that $t^* < t_{k+1} < \mathfrak{z}_k < t_k < \frac{1}{2}$, for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, define the cone $\mathcal{P}_{\mathfrak{z}_k}$ by

$$\mathcal{P}_{\mathfrak{z}_k} = \Big\{ \varpi \in \mathscr{E} : \varpi(t) \ge 0 \ \text{ and } \min_{t\in[\mathfrak{z}_k,\, 1-\mathfrak{z}_k]} \varpi(t) \ge \Delta_{\mathfrak{z}_k} \|\varpi(t)\| \Big\}.$$

Let $\varpi_1 \in \mathcal{P}_{\mathfrak{z}_k} \cap \partial\Lambda_{1,k}$. Then, $\varpi_1(\tau) \leq R_k = \|\varpi_1\|$ for all $\tau \in (0,1)$. By (A_1) and $0 < \tau_{\ell-1} < 1$, we have

$$\begin{aligned}\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)\mathrm{g}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell &\leq \int_0^1 \aleph(\tau_\ell,\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)\mathrm{g}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\\ &\leq \frac{M_1R_k}{1+\mathfrak{K}}\int_0^1 \aleph(\tau_\ell,\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)\big]d\tau_\ell\\ &\leq \frac{M_1R_k}{1+\mathfrak{K}}\int_0^1 \aleph(\tau_\ell,\tau_\ell)\phi^{-1}\Big[\prod_{i=1}^n \psi_i(\tau_\ell)\Big]d\tau_\ell\\ &\leq \frac{M_1R_k}{1+\mathfrak{K}}\int_0^1 \aleph(\tau_\ell,\tau_\ell)\prod_{i=1}^n \phi^{-1}(\psi_i(\tau_\ell))d\tau_\ell.\end{aligned}$$

There exists a $\mathtt{q} > 1$ such that $\dfrac{1}{\mathtt{q}} + \displaystyle\sum_{i=1}^n \frac{1}{\mathtt{p}_i} = 1$. So,

$$\begin{aligned}\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)\mathrm{g}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell &\leq \frac{M_1R_k}{1+\mathfrak{K}}\Big\|\aleph\Big\|_{\mathtt{q}}\Big\|\prod_{i=1}^n \phi^{-1}(\psi_i)\Big\|_{\mathtt{p}_i}\\ &\leq \frac{M_1R_k}{1+\mathfrak{K}}\Big\|\aleph\Big\|_{\mathtt{q}}\prod_{i=1}^n \big\|\phi^{-1}(\psi_i)\big\|_{\mathtt{p}_i}\\ &\leq \frac{R_k}{1+\mathfrak{K}} \leq R_k.\end{aligned}$$

It follows in a similar manner (for $0 < \tau_{\ell-2} < 1$) that

$$\begin{aligned}&\int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\phi^{-1}\Big[\Psi(\tau_{\ell-1})\mathrm{g}_{\ell-1}\Big(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)\mathrm{g}_\ell(\vartheta_1(\tau_\ell))\big]d\tau_\ell\Big)\Big]d\tau_{\ell-1}\\ &\quad\leq \int_0^1 \aleph(\tau_{\ell-2},\tau_{\ell-1})\phi^{-1}\big[\Psi(\tau_{\ell-1})\mathrm{g}_{\ell-1}(R_k)\big]d\tau_{\ell-1}\\ &\quad\leq \int_0^1 \aleph(\tau_{\ell-1},\tau_{\ell-1})\phi^{-1}\big[\Psi(\tau_{\ell-1})\mathrm{g}_{\ell-1}(R_k)\big]d\tau_{\ell-1}\\ &\quad\leq \frac{M_1R_k}{1+\mathfrak{K}}\int_0^1 \aleph(\tau_{\ell-1},\tau_{\ell-1})\phi^{-1}\big[\Psi(\tau_{\ell-1})\big]d\tau_{\ell-1}\\ &\quad\leq \frac{M_1R_k}{1+\mathfrak{K}}\int_0^1 \aleph(\tau_{\ell-1},\tau_{\ell-1})\phi^{-1}\Big[\prod_{i=1}^n \psi_i(\tau_{\ell-1})\Big]d\tau_{\ell-1}\\ &\quad\leq \frac{M_1R_k}{1+\mathfrak{K}}\int_0^1 \aleph(\tau_{\ell-1},\tau_{\ell-1})\prod_{i=1}^n \phi^{-1}(\psi_i(\tau_{\ell-1}))d\tau_{\ell-1}\\ &\quad\leq \frac{M_1R_k}{1+\mathfrak{K}}\|\aleph\|_{\mathtt{q}}\prod_{i=1}^n \big\|\phi^{-1}(\psi_i)\big\|_{\mathtt{p}_i}\\ &\quad\leq \frac{R_k}{1+\mathfrak{K}} \leq R_k.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
(\Omega\varpi_1)(t) = &\int_0^1 \aleph(t,\tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)g_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)g_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\
&\times \phi^{-1}\Bigg[\Psi(\tau_3)g_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\times g_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1\\
&+\int_0^1 \mathscr{G}(t,\tau)\int_0^1 \aleph(\tau,\tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)g_1\Bigg(\int_0^1 \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)\\
&\times g_2\Bigg(\int_0^1 \aleph(\tau_2,\tau_3)\phi^{-1}\Bigg[\Psi(\tau_3)g_3\Bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\times g_{\ell-1}\Bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\Bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1 d\tau\\
\leq &\frac{R_k}{1+\mathfrak{K}} + \max_{t\in[0,1]}\left\{\int_0^1 \mathscr{G}(t,\tau)d\tau\right\}\frac{R_k}{1+\mathfrak{K}}\\
\leq & R_k.
\end{aligned}$$

Since $R_k = \|\varpi_1\|$ for $\varpi_1 \in \mathcal{P}_{\mathfrak{z}_k} \cap \partial\Lambda_{1,k}$, we get

$$\|\Omega\varpi_1\| \leq \|\varpi_1\|. \tag{1}$$

Let $t \in [\mathfrak{z}_k, 1-\mathfrak{z}_k]$. Then,

$$r_k = \|\varpi_1\| \geq \varpi_1(t) \geq \min_{t\in[\mathfrak{z}_k,1-\mathfrak{z}_k]} \varpi_1(t) \geq \Delta_{\mathfrak{z}_k}\|\varpi_1\| \geq \Delta_{\mathfrak{z}_k} r_k.$$

By (A_2) and for $\tau_{\ell-1} \in [\mathfrak{z}_k, 1-\mathfrak{z}_k]$, we have

$$\begin{aligned}
&\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\\
&\qquad\geq \Delta_{\mathfrak{z}_k}\int_{\mathfrak{z}_k}^{1-\mathfrak{z}_k} \aleph(\tau_\ell,\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\\
&\qquad\geq \Delta_{\mathfrak{z}_k}\theta r_k\int_{\mathfrak{z}_k}^{1-\mathfrak{z}_k} \aleph(\tau_\ell,\tau_\ell)\phi^{-1}(\Psi(\tau_\ell))d\tau_\ell\\
&\qquad\geq \Delta_{\mathfrak{z}_k}\theta r_k\int_{\mathfrak{z}_k}^{1-\mathfrak{z}_k} \aleph(\tau_\ell,\tau_\ell)\prod_{i=1}^n \phi^{-1}(\psi_i(\tau_\ell))d\tau_\ell\\
&\qquad\geq \Delta_{\mathfrak{z}_k}\theta r_k\prod_{i=1}^n \lambda_i\int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph(\tau_\ell,\tau_\ell)d\tau_\ell\\
&\qquad\geq r_k.
\end{aligned}$$

Continuing with bootstrapping argument, we get

$$
\begin{aligned}
(\Omega\varpi_1)(t) = &\int_0^1 \aleph(t,\tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)\mathrm{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)\mathrm{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\\
&\times \phi^{-1}\Bigg[\Psi(\tau_3)\mathrm{g}_3\bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\times \mathrm{g}_{\ell-1}\bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)\mathrm{g}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1\\
&+\int_0^1 \mathscr{G}(t,\tau)\int_0^1 \aleph(\tau,\tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)\mathrm{g}_1\bigg(\int_0^1 \aleph(\tau_1,\tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)\\
&\times \mathrm{g}_2\bigg(\int_0^1 \aleph(\tau_2,\tau_3)\phi^{-1}\Bigg[\Psi(\tau_3)\mathrm{g}_3\bigg(\int_0^1 \aleph(\tau_3,\tau_4)\cdots\\
&\times \mathrm{g}_{\ell-1}\bigg(\int_0^1 \aleph(\tau_{\ell-1},\tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)\mathrm{g}_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1 d\tau\\
\geq\, & r_k + r_k\int_0^1 \mathscr{G}(t,\tau)d\tau\\
\geq\, & r_k, \qquad \text{(since } \mathscr{G}(t,\tau) \text{ is positive).}
\end{aligned}
$$

Thus, if $\varpi_1 \in \mathcal{P}_{\mathfrak{z}_k} \cap \partial\Lambda_{2,k}$, then

$$\|\Omega\varpi_1\| \geq \|\varpi_1\|. \tag{2}$$

It is evident that $0 \in \Lambda_{2,k} \subset \overline{\Lambda}_{2,k} \subset \Lambda_{1,k}$. From (1) and (2), it follows from Theorem 1 that the operator Ω has a fixed point $\varpi_1^{[k]} \in \mathcal{P}_{\mathfrak{z}_k} \cap (\overline{\Lambda}_{1,k} \backslash \Lambda_{2,k})$ such that $\varpi_1^{[k]}(t) \geq 0$ on $(0,1)$, and $k \in \mathbb{N}$. Next setting $\varpi_{\ell+1} = \varpi_1$, we obtain denumerably many positive solutions $\{(\varpi_1^{[k]}, \varpi_2^{[k]}, \ldots, \varpi_\ell^{[k]})\}_{k=1}^{\infty}$ of (1)-(2) given iteratively by

$$\varpi_j(t) = \int_0^1 \aleph(t,\tau)\phi^{-1}\big[\Upsilon(\tau)f_j(\varpi_{j+1}(\tau))\big]d\tau,\ t\in(0,1),\ j=\ell,\ell-1,\ldots,1.$$

The proof is completed. □

For $\displaystyle\sum_{i=1}^{n}\frac{1}{p_i} = 1$, we have the following theorem.

Theorem 4. *Assume that* $(H_1)-(H_3)$ *hold, let* $\{\mathfrak{z}_k\}_{k=1}^{\infty}$ *be such that* $t_{k+1} < \mathfrak{z}_k < t_k$, $k = 1,2,3,\ldots$ *. Let* $\{R_k\}_{k=1}^{\infty}$ *and* $\{r_k\}_{k=1}^{\infty}$ *be such that*

$$R_{k+1} < \Delta_{\mathfrak{z}_k} r_k < \theta r_k < R_k,\ k\in\mathbb{N},$$

where θ *is defined in Theorem 3. Further, assume that* g_j *satisfies* (A_2) *and*

(A_3) $\quad \mathrm{g}_j(\varpi) \leq \phi(\frac{M_2R_k}{1+\mathfrak{K}})$ *for all* $t\in[0,1]$, $0\leq\varpi\leq R_k$, *where*

$$M_2 < \left\{\left[\|\aleph\|_\infty \prod_{i=1}^{n}\left\|\phi^{-1}(\psi_i)\right\|_{\mathrm{p}_i}\right]^{-1}, \theta\right\}.$$

Then the iterative boundary value problem (1)-(2) *has denumerably many positive solutions* $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \dots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ *such that* $\varpi_j^{[r]}(t) > 0$ *on* $(0,1)$, $j = 1, 2, \dots, \ell$ *and* $r \in \mathbb{N}$.

Proof. For a fixed k, let $\Lambda_{1,k}$ be as in the proof of Theorem 3 and let $\varpi_1 \in \mathcal{P}_{\mathfrak{z}_k} \cap \partial\Lambda_{2,k}$. Again $\varpi_1(\tau) \le R_k = \|\varpi_1\|$, for all $\tau \in (0,1)$. By (A_3) and for $\tau_{\ell-1} \in (0,1)$, we have

$$\begin{aligned}
\int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell &\le \int_0^1 \aleph(\tau_\ell, \tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell \\
&\le \frac{M_2 R_k}{1+\mathfrak{K}} \int_0^1 \aleph(\tau_\ell, \tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)\big]d\tau_\ell \\
&\le \frac{M_2 R_k}{1+\mathfrak{K}} \int_0^1 \aleph(\tau_\ell, \tau_\ell)\phi^{-1}\Big[\prod_{i=1}^{n}\psi_i(\tau_\ell)\Big]d\tau_\ell \\
&\le \frac{M21 R_k}{1+\mathfrak{K}} \int_0^1 \aleph(\tau_\ell, \tau_\ell)\prod_{i=1}^{n}\phi^{-1}(\psi_i(\tau_\ell))d\tau_\ell \\
&\le \frac{M_2 R_k}{1+\mathfrak{K}} \big\|\aleph\big\|_\infty \prod_{i=1}^{n}\big\|\phi^{-1}(\psi_i)\big\|_{\mathrm{p}_i} \le \frac{R_k}{1+\mathfrak{K}} \le R_k.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
(\Omega\varpi_1)(t) = &\int_0^1 \aleph(t, \tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)g_1\bigg(\int_0^1 \aleph(\tau_1, \tau_2)\phi^{-1}\Bigg[\Psi(\tau_2)g_2\bigg(\int_0^1 \aleph(\tau_2, \tau_3) \\
&\times \phi^{-1}\Bigg[\Psi(\tau_3)g_3\bigg(\int_0^1 \aleph(\tau_3, \tau_4)\cdots \\
&\times g_{\ell-1}\bigg(\int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1 \\
&+ \int_0^1 \mathscr{G}(t, \tau)\int_0^1 \aleph(\tau, \tau_1)\phi^{-1}\Bigg[\Psi(\tau_1)g_1\bigg(\int_0^1 \aleph(\tau_1, \tau_2)\phi^{-1}\Bigg[\Psi(\tau_2) \\
&\times g_2\bigg(\int_0^1 \aleph(\tau_2, \tau_3)\phi^{-1}\Bigg[\Psi(\tau_3)g_3\bigg(\int_0^1 \aleph(\tau_3, \tau_4)\cdots \\
&\times g_{\ell-1}\bigg(\int_0^1 \aleph(\tau_{\ell-1}, \tau_\ell)\phi^{-1}\big[\Psi(\tau_\ell)g_\ell(\varpi_1(\tau_\ell))\big]d\tau_\ell\bigg)\cdots d\tau_3\Bigg]d\tau_2\Bigg]d\tau_1 d\tau \\
\le &\frac{R_k}{1+\mathfrak{K}} + \max_{t\in[0,1]}\left\{\int_0^1 \mathscr{G}(t,\tau)d\tau\right\}\frac{R_k}{1+\mathfrak{K}} \\
\le &R_k.
\end{aligned}$$

Since $R_r = \|\varpi_1\|$ for $\varpi_1 \in \mathcal{P}_{\mathfrak{z}_k} \cap \partial\Lambda_{1,k}$, we get

$$\|\Omega\varpi_1\| \le \|\varpi_1\|. \tag{3}$$

Now define $\Lambda_{2,k} = \{\varpi_1 \in \mathscr{E} : \|\varpi_1\| < O_r\}$. Let $\varpi_1 \in \mathcal{P}_{\mathfrak{z}_k} \cap \partial\Lambda_{2,k}$ and let $\tau \in [\mathfrak{z}_k,\ 1-\mathfrak{z}_k]$. Then, the argument leading to (2) can be done to the present case. Hence, the proof is complete. $\square$

Lastly, the case $\sum_{i=1}^{n} \frac{1}{p_i} > 1$.

Theorem 5. *Assume that $(H_1) - (H_3)$ hold, let $\{\mathfrak{z}_k\}_{k=1}^{\infty}$ be such that $t_{k+1} < \mathfrak{z}_k < t_k$, $k = 1, 2, 3, \ldots$. Let $\{R_k\}_{k=1}^{\infty}$ and $\{r_k\}_{k=1}^{\infty}$ be such that*

$$R_{k+1} < \Delta_{\mathfrak{z}_k} r_k < \theta r_k < R_k, \ k \in \mathbb{N},$$

where θ is defined in Theorem 3. Further, assume that $\mathtt{f}_j$ satisfies (A_2) and

(A_4) $\quad \mathtt{g}_j(\varpi) \leq \phi(\frac{M_3 R_k}{1+\mathfrak{K}})$ *for all $t \in [0,1]$, $0 \leq \varpi \leq R_k$, where*

$$M_3 < \left\{ \left[\|\aleph\|_{\infty} \prod_{i=1}^{n} \left\| \phi^{-1}(\psi_i) \right\|_1 \right]^{-1}, \theta \right\}.$$

Then the iterative boundary value problem (1)-(2) has denumerably many positive solutions $\{(\varpi_1^{[r]}, \varpi_2^{[r]}, \ldots, \varpi_\ell^{[r]})\}_{r=1}^{\infty}$ such that $\varpi_j^{[r]}(t) > 0$ on $(0,1)$, $j = 1, 2, \ldots, \ell$ and $r \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 3. So, we omit details here. □

4 Example

In this section, we present an example to check validity of our main results.

Example 1. Consider the following fractional order boundary value problem,

$$\left.\begin{array}{r} \phi\left[{}^{\mathcal{C}}\mathscr{D}_{0^+}^{1.8}\varpi_{\mathtt{j}}(t)\right] + \Psi(t)\mathtt{g}_{\mathtt{j}}(\varpi_{\mathtt{j+1}}(t)) = 0, \ 0 < t < 1, \ \mathtt{j} = 1, 2, \\ \varpi_{\mathtt{j+1}}(t) = \varpi_1(t), \ 0 < t < 1, \\ \varpi_{\mathtt{j}}(0) - \varpi_{\mathtt{j}}'(0) = \mathcal{I}_{0^+}^{2}\varpi_{\mathtt{j}}(1), \\ \varpi_{\mathtt{j}}(1) + \varpi_{\mathtt{j}}'(1) = \mathcal{I}_{0^+}^{2}\varpi_{\mathtt{j}}(1), \end{array}\right\} \tag{1}$$

where

$$\phi(\varpi) = \begin{cases} \dfrac{\varpi^3}{1+\varpi^2}, & \varpi \leq 0, \\ \varpi^2, & \varpi > 0, \end{cases}$$

and

$$\Psi(t) = \psi_1(t) \cdot \psi_2(t),$$

in which

$$\psi_1(t) = \frac{1}{|t - \frac{1}{4}|^{\frac{1}{2}}} \quad \text{and} \quad \psi_2(t) = \frac{1}{|t - \frac{1}{3}|^{\frac{1}{2}}},$$

$$\mathfrak{g}_j(\varpi) = \begin{cases} 0.07 \times 10^{-16}, & \varpi \in (10^{-16}, +\infty), \\ \frac{77468 \times 10^{-(16k+8)} - 0.07 \times 10^{-16k}}{10^{-(16k+8)} - 10^{-16k}} (\varpi - 10^{-16k}) + 0.07 \times 10^{-16k}, & \\ & \varpi \in \left[10^{-(16k+8)}, 10^{-16k}\right], \\ 77468 \times 10^{-(16k+8)}, \quad \varpi \in \left(\frac{1}{5} \times 10^{-(16k+8)}, 10^{-(16k+8)}\right), & \\ \frac{77468 \times 10^{-(16k+8)} - 0.07 \times 10^{-(16k+16)}}{\frac{1}{5} \times 10^{-(16k+8)} - 10^{-(16k+16)}} (\varpi - 10^{-(16k+16)}) + 0.07 \times 10^{-(16k+16)}, & \\ & \varpi \in \left(10^{-(16k+16)}, \frac{1}{5} \times 10^{-(16k+8)}\right], \end{cases}$$

for $j = 1, 2$. Let

$$t_j = \frac{31}{64} - \sum_{r=1}^{j} \frac{1}{4(r+1)^4}, \; \mathfrak{z}_j = \frac{1}{2}(t_j + t_{j+1}), \; j = 1, 2, 3, \ldots.$$

Then

$$\mathfrak{z}_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32},$$

and

$$t_{j+1} < \mathfrak{z}_j < t_j, \, \mathfrak{z}_j > \frac{1}{5}.$$

Also,

$$\varsigma_1 = \frac{5}{18}, \; \varsigma_2 = \frac{2}{9}, \; \varsigma_3 = \frac{5}{18}, \; \varsigma_4 = \frac{2}{9}, \; \varsigma = \frac{1}{2}, \; \mathscr{G}(t, \tau) = 2(1 - \tau),$$

$$\rho = \frac{4b[a(\sigma - 1) + \sigma - 2]}{[1 - a + b(\sigma - 1)]^2 + 4a[1 + b(\sigma - 1)]} = 0.1224489796.$$

Therefore,

$$\mathfrak{K} = 1, \; \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph(\tau, \tau) d\tau = 0.05389278403, \; \Delta_{\mathfrak{z}_j} = \max\{\varrho, \mathfrak{z}_j\} > \frac{1}{5}, \quad j = 1, 2, 3, \ldots.$$

It is not difficult to see that

$$t_1 = \frac{15}{32} < \frac{1}{2}, \; t_j - t_{j+1} = \frac{1}{4(j+2)^4}, \quad j = 1, 2, 3, \ldots.$$

Since $\sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{\pi^4}{90}$ and $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$, it follows that

$$t^* = \lim_{j \to \infty} t_j = \frac{31}{64} - \sum_{i=1}^{\infty} \frac{1}{4(i+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} > \frac{1}{5},$$

$\psi_1, \psi_2 \in \mathcal{L}^{\mathtt{p}}[0,1]$ for all $0 < \mathtt{p} < 2$. So

$$\psi_1 = \psi_2 = \frac{1}{\sqrt{3}},$$

and

$$\Delta_{\mathfrak{z}_1} \prod_{i=1}^{n} \psi_i \int_{\mathfrak{z}_1}^{1-\mathfrak{z}_1} \aleph(\tau_\ell, \tau_\ell) d\tau_\ell \approx 0.003592852269.$$

So, $\theta = \max \left\{ \frac{1}{0.003592852269}, 1 \right\} = 278.3303974$, and

$$\|\aleph\|_q = \left[\int_0^1 |\aleph(\tau, \tau)|^q d\tau \right]^{\frac{1}{q}} \approx 0.7456267277 \quad \text{for} \quad q = 2.$$

Next, let $0 < \varepsilon < 1$ be fixed. Then $\psi_1, \psi_2 \in \mathcal{L}^{1+\varepsilon}[0,1]$. It follows that

$$\|\phi^{-1}(\psi_1)\|_{1+\varepsilon} = \left[\frac{1}{3-\varepsilon} \left(3^{\frac{3-\varepsilon}{4}} + 1 \right) 2^{\frac{1+\varepsilon}{2}} \right]^{\frac{1}{1+\varepsilon}},$$

$$\|\phi^{-1}(\psi_2)\|_{1+\varepsilon} = \left[\frac{4}{3-\varepsilon} \left(2^{\frac{3-\varepsilon}{4}} + 1 \right) (1/3)^{\frac{3-\varepsilon}{4}} \right]^{\frac{1}{1+\varepsilon}}.$$

So, for $0 < \varepsilon < 1$, we have

$$0.5679904165 \leq \left[\|\aleph\|_{\mathtt{q}} \prod_{i=1}^{n} \left\| \phi^{-1}(\psi_i) \right\|_{\mathtt{p}_i} \right]^{-1} \leq 0.7830857747.$$

Take $M_1 = 0.56$. In addition if we take

$$R_k = 10^{-8k},\ r_k = 10^{-(8k+4)},$$

then

$$\begin{aligned} R_{k+1} = 10^{-(8k+8)} &< \frac{1}{5} \times 10^{-(8k+4)} < \Delta_{\mathfrak{z}_k} r_k \\ &< r_k = 10^{-(8k+4)} < R_k = 10^{-8k}, \end{aligned}$$

$\theta r_k = 278.3303974 \times 10^{-(8k+4)} < \frac{0.56}{2} \times 10^{-8k} = \frac{\mathtt{M}_1 R_k}{1+\mathfrak{K}}$, $k = 1, 2, 3, \ldots$, and $\mathtt{g}_j$ satisfies the following growth conditions:

$$\begin{aligned} \mathtt{g}_j(\varpi) &\leq \phi\left(\frac{M_1 R_k}{1+\mathfrak{K}}\right) = \frac{\mathtt{M}_1^2 R_k^2}{(1+\mathfrak{K})^2} = 0.0784 \times 10^{-16k}, \quad \varpi \in \left[0, 10^{-16k}\right] \\ \mathtt{g}_j(\varpi) &\geq \phi(\theta r_k) = \theta^2 r_k^2 \\ &= 77467.81012 \times 10^{-(16k+8)}, \quad \varpi \in \left[\frac{1}{5} \times 10^{-(16k+8)}, 10^{-(16k+8)}\right]. \end{aligned}$$

Then all the conditions of Theorem 3 are satisfied. Therefore, by Theorem 3, the boundary value problem (1) has countably many positive solutions $\{\varpi_j^{[k]}\}_{k=1}^{\infty}$ such that $10^{-(8k+4)} \leq \|\varpi_j^{[k]}\| \leq 10^{-8k}$ for each $j = 1, 2, 3, \ldots, \ell$, $k = 1, 2, 3, \ldots$.

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