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Solution of nonlinear Volterra and Fredholm integro-differential equations by the rational Haar wavelet

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Abstract. We successively apply the rational Haar wavelet to solve the nonlinear Volterra integro-differential equations and nonlinear Fredholm integro-differential equations. Using the Banach fixed point theorem for these equations, we prove the convergence. In this method, no numerical integration is used. Numerical results are presented to show the effectiveness of this method.

Keywords: Fixed point Banach theorem, nonlinear, Volterra, Fredholm, integro-differential, Haar wavelet, convergence.
AMS Subject Classification 2010: 34A34, 65L05.

1 Preliminaries

The integro-differential equations (IDE) have been applied in science, economics, and engineering, such as electromagnetic theory, involving fluid waves, competition between tumor cells, and the immune system; see [3]. Many problems in different fields of science and engineering can be modeled by IDE and nonlinear integro-differential equations (NIDE), such as epidemic models, biological models, physical phenomena, and chemical kinetics. Several numerical methods have been used for approximating the solution of IDE or NIDE such as rationalized Haar functions [8, 11], Galerkin methods [4], Wavelet–Galerkin and cubic B-spline finite element method [12], periodic quasi-wavelets [13], differential transform method [1], spline approximation method, Adomian decomposition method [2], spectral collocation method [14], homotopy perturbation method, variational iterations, interpolation or extrapolation, Laplace transform, and other methods [5, 8, 15].

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2 Introduction

In the general form, a nonlinear Volterra integro-differential equation (NVIDE) of the second kind is defined as

$$\omega'(t) + \beta R_1(t, \omega(t)) = \mu(t) + \alpha \int_0^t R_2(t, s, \omega(s))ds, \tag{1}$$

and the nonlinear Fredholm integro-differential equation (NFIDE) of the second kind is defined as

$$\omega'(t) + \beta R_1(t, \omega(t)) = \mu(t) + \alpha \int_0^1 R_2(t, s, \omega(s))ds, \tag{2}$$

where

$$\alpha, \beta \in \mathbb{R}, \quad R_1(t, \omega(t)) \in C([0,1] \times \mathbb{R}, \mathbb{R}), \quad R_2(s, t, \omega(s)) \in C([0,1]^2 \times \mathbb{R}, \mathbb{R}).$$

in which $\mu(t) : [0,1] \longrightarrow \mathbb{R}$ is a continuous function.
Also, R_1 and R_2 are continuous and Lipschitzian functions as

$$\begin{aligned} |R_1(t, \omega_1(t)) - R_1(t, \omega_2(t))| &\leq q_1 |\omega_1(t) - \omega_2(t)|, \quad \omega_1 \in C([0,1], \mathbb{R}), \\ |R_2(t, s, \omega_1(t)) - R_2(t, s, \omega_2(t))| &\leq q_2 |\omega_1(t) - \omega_2(t)|, \quad \omega_2 \in C([0,1], \mathbb{R}), \end{aligned} \tag{3}$$

in which q_1 and q_2 are Lipschitz constants. By integrating from (1) and (2) from zero to t, we have

$$\omega(t) + \beta \int_0^t R_1(x, \omega(x))dx = \omega(0) + \int_0^t \mu(x)dx + \alpha \int_0^t \int_0^x R_2(x, s, \omega(s))dsdx \tag{4}$$

and

$$\omega(t) + \beta \int_0^t R_1(x, \omega(x))dx = \omega(0) + \int_0^t \mu(x)dx + \alpha \int_0^t \int_0^1 R_2(x, s, \omega(s))dsdx. \tag{5}$$

3 Numerical approximation

The base of our work is the use of the rational Haar wavelet for approaching the NVIDE and NFIDE. In fact, the wavelet and its theory have a new powerful area in mathematical researches, science, and engineering; see [18]. The main character of this method is the elements of solving the system of algebraic equations; see [7, 8]. Haar wavelets are used to solve science and engineering problems [6], such as Painleve equations [10], Darboux problem [9], Emden–Fowler equations [16], and consolidation equation [17]. The RH wavelet has a function defined as follows.

Definition 1. The RH wavelet has a function defined by

$$h_r(x) = RH(2^r x - s), \quad s = 0, 1, \ldots, 2^{l-1}, \quad r = 2^l + s,$$

such that

$$RH(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

Definition 2 (see [18])**.** The function $w(t) \in C([0,1])$ can be written as a series expansion by the RH function as

$$w(t) = \sum_{i=0}^{m-1} a_i h_i(t), \quad m = 2^{i+1}, \quad i = 0, 1, \dots, \tag{7}$$

and for $i = 2^j + k$ with $0 \le k \le 2^j$ and $j = 0, 1, \dots, n$, we have

$$a_i = 2^j \int_0^1 w(t) h_i(t) dx = 2^j \langle w, h_i \rangle. \tag{8}$$

For each $x, s \in [0,1]$ and $m \in \mathbb{N}$ with $m = 2^{i+1}$ and $i = 1, 2, \dots,$ we recursively define

$$\omega_i(t) + \beta \int_0^t R_1(x, \omega_{i-1}(x)) dx = \omega_i(0) + \int_0^t \mu(x) dx + \alpha \int_0^t \int_0^x R_2(x, s, \omega_{i-1}(s)) ds dx \tag{9}$$

and

$$\omega_i(t) + \beta \int_0^t R_1(x, \omega_{i-1}(x)) dx = \omega_i(0) + \int_0^t \mu(x) dx + \alpha \int_0^t \int_0^1 R_2(x, s, \omega_{i-1}(s)) ds dx. \tag{10}$$

We assume

$$\begin{aligned} \psi_{i-1}(x) = R_1(x, \omega_{i-1}(x)) &= \sum_{l=1}^{\infty} a_l h_l(x), \\ \varphi_{i-1}(x,s) = R_2(x, s, \omega_{i-1}(s)) &= \sum_{l=1}^{\infty} \sum_{q=1}^{\infty} b_{lq} h_l(x) h_q(s). \end{aligned} \tag{11}$$

Let Q_m have an orthogonal projection. Then

$$Q_m(\psi)(x) = \sum_{l=1}^{m-1} a_l h_l(x), \quad Q_m(\varphi)(x,s) = \sum_{l=1}^{m-1} \sum_{q=1}^{m-1} b_{lq} h_l(x) h_q(s),$$

where a_l is calculated from (8) and $b_{lq} = \langle h_l(x), \langle \varphi(x,s), h_q(s) \rangle \rangle$. Thus for NVIDE, we have

$$\begin{aligned} \Omega_i(t) = {} & \Omega_i(0) + \int_0^t \mu(x) dx \\ & - \beta \int_0^t Q_m(\psi_{i-1}(x)) dx + \alpha \int_0^t \int_0^s Q_m(\varphi_{i-1}(x,s)) ds dx, \quad i = 1, 2, \dots, \end{aligned} \tag{12}$$

and for NFIDE, we have

$$\begin{aligned} \Omega_i(t) = {} & \Omega_i(0) + \int_0^t \mu(x) dx \\ & - \beta \int_0^t Q_m(\psi_{i-1}(x)) dx + \alpha \int_0^t \int_0^1 Q_m(\varphi_{i-1}(x,s)) ds dx, \quad i = 1, 2, \dots. \end{aligned} \tag{13}$$

4 Error analysis

The purpose of this section is to discuss the convergence and to calculate the rate of convergence of the method. To do so, we use the Banach fixed point theorem. Thus, we need to introduce the nonlinear integral operator T in the Banach space as follows

$$T : (C([0,1]), \|\cdot\|_\infty) \longrightarrow (C([0,1]), \|\cdot\|_\infty).$$

The mapping T for NVIDE and NFIDE are respectively defined as

$$T\omega(t) = \omega(0) - \beta \int_0^t R_1(x,\omega(x))dx + \int_0^t \mu(x)dx + \alpha \int_0^t \int_0^x R_2(x,s,\omega(s))dsdx \tag{14}$$

and

$$T\omega(t) = \omega(0) - \beta \int_0^t R_1(x,\omega(x))dx + \int_0^t \mu(x)dx + \alpha \int_0^t \int_0^1 R_2(x,s,\omega(s))dsdx. \tag{15}$$

Similar to (9), (10), and (11), in the ith iteration we have

$$T\omega_i(t) = \omega_i(0) - \beta \int_0^t \psi_{i-1}(x)dx + \int_0^t \mu(x)dx + \alpha \int_0^t \int_0^x \varphi_{i-1}(x,s)dsdx \tag{16}$$

and

$$T\omega_i(t) = \omega_i(0) - \beta \int_0^t \psi_{i-1}(x)dx + \int_0^t \mu(x)dx + \alpha \int_0^t \int_0^1 \varphi_{i-1}(x,s)dsdx. \tag{17}$$

Let us start with an NVIDE case; for NFIDE, it is proved similarly.

Lemma 1. *Let $R_1(t,\omega(t))$ and $R_2(t,s,\omega(s))$ be defined in (2). If*

$$q = \max\{|\beta|q_1, |\alpha|q_2\} < \frac{1}{2}, \tag{18}$$

and ω is a fixed point of T, then, for all $\omega_0 \in C([0,1],\mathbb{R})$, we have

$$\|\omega - T^i(\omega_0)\|_\infty \le \|T(\omega_0) - \Omega_0\|_\infty \sum_{j=i}^{\infty} q^j. \tag{19}$$

Also, T in (14) has a unique fixed point.

Proof. Applying (14) gives

$$\begin{aligned}\Big|T\omega_1(t)-T\omega_2(t)\Big| &= \Big|\alpha\int_0^t\int_0^x (R_2(x,s,\omega_1(s))-R_2(x,s,\omega_2(s)))dsdx \\ &\quad -\beta\int_0^t (R_1(x,\omega_1(x))-R_1(x,\omega_2(x)))dx\Big| \\ &\le |\alpha|\int_0^t\int_0^x \Big|R_2(x,s,\omega_1(s))-R_2(x,s,\omega_2(s))\Big|dsdx \\ &\quad +|\beta|\int_0^t\Big|R_1(x,\omega_1(x))-R_1(x,\omega_2(x))\Big|dx \\ &\le |\alpha|\int_0^t\int_0^x q_2|\omega_1(s)-\omega_2(s)|dsdx+|\beta|\int_0^t q_1|\omega_1(x)-\omega_2(x)|dx \\ &\le (|\alpha|q_2+|\beta|q_1)\|\omega_1-\omega_2\|_\infty.\end{aligned}$$

Thus

$$\Big|T\omega_1(t)-T\omega_2(t)\Big| \le (|\alpha|q_2+|\beta|q_1)\|\omega_1-\omega_2\|_\infty, \tag{20}$$

Using (18) and (20), we have

$$\Big|T\omega_1(t)-T\omega_2(t)\Big| \le (2q)\|\omega_1-\omega_2\|_\infty. \tag{21}$$

By induction on $i\in\mathbb{N}$, we get $\|T^i\omega_1-T^i\omega_2\|_\infty \le (2q)^i\|\omega_1-\omega_2\|_\infty$. Since $q<1/2$, thus T is a contraction mapping and

$$\sum_{i=1}^{\infty}\|T^i\omega_1-T^i\omega_2\|_\infty<\infty.$$

Consequently, (14) has a unique solution. □

Theorem 1. *Assume that* $\psi_{i-1}\in\mathbb{C}([0,1])$*, that* $\varphi_{i-1}\in\mathbb{C}([0,1]^2)$*, and that* $\{\omega_i\}_{i\ge1}$ *is a subset of* $C([0,1])$*. Then*

$$\|\omega-\Omega_i\|_\infty \le \|T(\omega_0)-\omega_0\|_\infty\sum_{j=i}^{\infty}q^j+\sum_{j=1}^{i}q^{i-j}\varepsilon_j, \tag{22}$$

Proof. From (12) and (16) we have

$$\begin{aligned}\Big\|T(\omega_i)-\Omega_i\Big\|_\infty \le\ & |\beta|\Big\|\int_0^t\psi_{i-1}(x)-Q_m(\psi_{i-1})(x)dx\Big\|_\infty \\ &+|\alpha|\Big\|\int_0^t\int_0^x\varphi_{i-1}(x,s)-Q_m(\varphi_{i-1})(x,s)dxds\Big\|_\infty \\ &\le |\beta|\|\psi_{i-1}-Q_m(\psi_{i-1})\|_\infty+|\alpha|\|\varphi_{i-1}-Q_m(\varphi_{i-1})\|_\infty.\end{aligned} \tag{23}$$

Set

$$f(t,s):=\varphi_{i-1}-Q_m(\varphi_{i-1}). \tag{24}$$

For variables t_l and s_j with $l, j \leq m-1$, by using the interpolating property, we have

$$t_0 = 0, \quad t_l = \frac{1}{2^{n_1+1}} + \frac{v_1}{2^{n_1}}, \quad l = 2^{n_1} + v_1, \quad n_1 \geq 1,$$
$$s_0 = 0, \quad s_j = \frac{1}{2^{n_2+1}} + \frac{v_2}{2^{n_2}}, \quad j = 2^{n_2} + v_2, \quad n_2 \geq 1.$$

From (24) and applying the mean-value theorem, we get

$$\begin{aligned}
&\|\varphi_{i-1} - Q_m(\varphi_{i-1})\|_\infty \\
&\quad = \left\| f(t_l, s_j) + \frac{\partial f}{\partial t}(\xi,\gamma)(\xi - t_l) + \frac{\partial f}{\partial s}(\xi,\gamma)(\gamma - s_j) \right\|_\infty \\
&\quad = \left\| (I - Q_m)\frac{\partial \varphi_{i-1}}{\partial t}(\xi,\gamma) + (I - Q_m)\frac{\partial \varphi_{i-1}}{\partial s}(\xi,\gamma) \right\|_\infty \max\{\|\xi - t_l\|_\infty, \|\gamma - s_j\|_\infty\} \\
&\quad \leq \frac{2}{2^i}\|(I - Q_m)\|_\infty \left\| \frac{\partial \varphi_{i-1}}{\partial t}(\xi,\gamma) + \frac{\partial \varphi_{i-1}}{\partial s}(\xi,\gamma) \right\|_\infty .
\end{aligned} \tag{25}$$

We assume

$$L_{i-1} = \max\{\|\frac{\partial \psi_{i-1}}{\partial t}\|_\infty, \|\frac{\partial \varphi_{i-1}}{\partial t}\|_\infty, \|\frac{\partial \varphi_{i-1}}{\partial s}\|_\infty\}, \quad i = 1, 2, \ldots. \tag{26}$$

Then in (25) by setting (26), we have

$$\|\varphi_{i-1} - Q_m(\varphi_{i-1})\|_\infty \leq \frac{4L_{i-1}}{2^i}.$$

Setting $f(t) := \psi_{i-1} - Q_m(\psi_{i-1})$, for ψ_{i-1}, we deduce

$$\begin{aligned}
\|\psi_{i-1} - Q_m(\psi_{i-1})\|_\infty &= \left\| f(t_l) + \frac{\partial f}{\partial t}(\xi,\gamma)(\xi - t_l) \right\|_\infty \\
&= \left\| (I - Q_m)\frac{\partial \psi_{i-1}}{\partial t}(\xi,\gamma) \right\|_\infty \|\xi - t_l\|_\infty \\
&\leq \frac{2}{2^i}\|(I - Q_m)\|_\infty \left\| \frac{\partial \psi_{i-1}}{\partial t}(\xi,\gamma) \right\|_\infty \leq \frac{2L_{i-1}}{2^i}.
\end{aligned}$$

Therefore, in inequality (23), we have

$$\|T(\omega_{i-1}) - \Omega_i\|_\infty \leq |\beta|\frac{2L_{i-1}}{2^i} + |\alpha|\frac{4L_{i-1}}{2^i} \leq (|\beta| + |\alpha|)\frac{4L_{i-1}}{2^i}.$$

We choose $\varepsilon_k > 0$ for $k = 1, \ldots, i$ as

$$(|\beta| + |\alpha|)\frac{4L_{k-1}}{2^k} < \varepsilon_k, \quad k = 1, 2, \ldots, i,$$

then

$$\|T(\omega_{i-1}) - \Omega_i\|_\infty < \varepsilon_i. \tag{27}$$

Applying the triangle inequality, we achieve

$$\|\omega - \Omega_i\|_\infty \leq \|\omega - T^i(\omega_0)\|_\infty + \sum_{j=1}^{i} q^j \|T(\omega_{j-1}) - \omega_j\|_\infty.$$

Finally, based on (19) in Lemma 1 and (27), we conclude that

$$\|\omega - \Omega_i\|_\infty \leq \|T(\omega_0) - \Omega_0\|_\infty \sum_{j=i}^{\infty} q^j + \sum_{j=1}^{i} q^{i-j}\varepsilon_j. \tag{28}$$

Using the geometric series $\sum_{j=i}^{\infty} q^j = q^i/(1-q)$, we set

$$q = \frac{1}{2} - \frac{1}{2^{l+1}} < \frac{1}{2}, \qquad l \in \mathbb{N}. \tag{29}$$

By using (28) and (29), we have

$$\|\omega - \Omega_i\|_\infty \leq \|T(\omega_0) - \Omega_0\|_\infty \frac{q^i}{1-q} + \sum_{j=1}^{i} (\frac{1}{2} - \frac{1}{2^{l+1}})^{i-j} \frac{4(|\alpha| + |\beta|)L_{j-1}}{2^j}. \tag{30}$$

So the sequence L_{j-1} is uniformly bounded for any $j \in \mathbb{N}$. Therefore for every $1 \leq j \leq i$, there exists $N < \infty$ such that $|L_{j-1}| \leq N$. Thus from (30), we have

$$\begin{aligned} \|\omega - \Omega_i\|_\infty &\leq \|T(\omega_0) - \Omega_0\|_\infty \frac{q^i}{1-q} + \sum_{j=1}^{i} (\frac{1}{2} - \frac{1}{2^{l+1}})^{i-j} \frac{4(|\alpha| + |\beta|)N}{2^j} \\ &= \|T(\omega_0) - \Omega_0\|_\infty \frac{q^i}{1-q} + 4N(|\alpha| + |\beta|)(\frac{1}{2} - \frac{1}{2^{l+1}})^i \sum_{j=1}^{i} (\frac{1}{2} - \frac{1}{2^{l+1}})^{-j} \frac{1}{2^j} \\ &= \|T(\omega_0) - \Omega_0\|_\infty \frac{q^i}{1-q} + 4N(|\alpha| + |\beta|)q^i \sum_{j=1}^{i} (1 + \frac{1}{2^l - 1})^j. \end{aligned} \tag{31}$$

Since $(1 + \frac{1}{2^l - 1}) \leq 2$, for any $l \in \mathbb{N}$, the inequality (31) implies

$$\begin{aligned} \|\omega - \Omega_i\|_\infty &\leq \|T(\omega_0) - \Omega_0\|_\infty \frac{q^i}{1-q} + q^i 4N(|\alpha| + |\beta|) \sum_{j=1}^{i} 2^j \\ &\leq \|T(\omega_0) - \Omega_0\|_\infty \frac{q^i}{1-q} + 4N(|\alpha| + |\beta|)q^i i 2^i. \end{aligned} \tag{32}$$

Since $q < \frac{1}{2}$, if $i \to \infty$, then the first summand of the right-hand side of (32) tends to zero, so we have

$$\|\omega - \Omega_i\|_\infty \leq 4N(|\alpha| + |\beta|) i (2q)^i.$$

Thus the rate of convergence will be $O(i(2q)^i)$. □

5 Illustrative examples

In this section, we give some numerical examples. In our examples, we use the method discussed in (12) and (13).

Example 1. Consider the NFIDE equation

$$\omega'(x)-\frac{1}{68}x^3\cos^3(2x)\omega(x)=\mu(x)+\frac{1}{30}\int_0^1 x^5\sin^2(t)t^4\omega(t)dt.$$

Then the exact solution is $\omega(x)=\cos^3(2x)-1+\sin^2(2x)$. In Table 1, the absolute error for $n=2$ and $n=3$ is calculated. Moreover, their running times are 0.047 and 0.125 seconds, respectively. We compare numerical solutions with the exact solutions in Figure 1. Evolutions of absolute errors for $n=3$ or $m=2^4$ are shown in Figure 2.

Table 1: Absolute errors of Example 1.

x_i	$m=2^3$	$m=2^4$
0.1	$1.24E{-}6$	$4.64E{-}7$
0.2	$1.21E{-}6$	$5.53E{-}7$
0.3	$1.21E{-}6$	$1.15E{-}6$
0.4	$1.81E{-}6$	$2.63E{-}6$
0.5	$3.29E{-}6$	$4.35E{-}6$
0.6	$5.01E{-}6$	$5.07E{-}6$
0.7	$5.72E{-}6$	$3.73E{-}6$
0.8	$4.37E{-}6$	$2.61E{-}6$
0.9	$1.98E{-}6$	$4.07E{-}6$
CPU-Time (s)	0.047	0.125

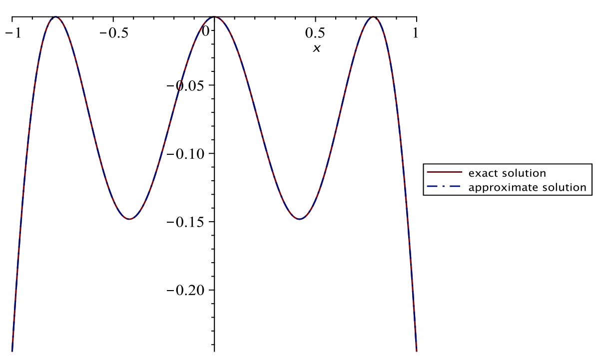

Figure 1: The plot of exact and numerical solution for $m=2^4$ in Example 1.

Example 2. Consider the NVIDE equation

$$\omega'(x)-\frac{1}{45}x^3\ln^2(x+2)\omega^2(x)=\mu(x)+\frac{1}{15}\int_0^x xt\ln^2(t+1)\omega^3(t)dt.$$

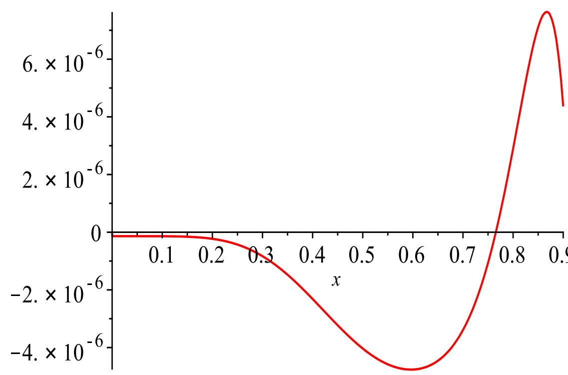

Figure 2: Absolute errors of Example 1.

Then the exact solution is $\omega(x) = \frac{1}{5}x^3 + \frac{1}{7}x^5$. In Table 2, the absolute error for $n = 3$ and $n = 4$ is calculated. Moreover, their running times are 0.078 and 1.046 seconds, respectively. We compare numerical solutions with the exact solutions in Figure 3. The values of absolute errors for $n = 4$ are shown in Figure 4.

Table 2: Absolute errors of Example 2.

x_i	$m = 2^5$	$m = 2^4$
0.1	2.00E−8	2.00E−8
0.2	0.000	3.57E−18
0.3	0.000	9.99E−9
0.4	3.00E−8	2.99E−8
0.5	2.40E−7	2.39E−7
0.6	7.60E−7	1.00E−6
0.7	6.60E−6	5.39E−6
0.8	5.35E−5	2.13E−5
0.9	6.92E−5	9.12E−5
CPU-Time (s)	1.046	0.078

Example 3. Consider the NVIDE equation

$$\omega'(x) - \frac{1}{20}(2 + x^2)\sin^2(x)\sin^3(\omega(x)) = \mu(x) + \frac{1}{40}\int_0^x x^2t^2\sin^2(t)\sin^2(\omega(t))dt.$$

The exact solution is $\omega(x) = \frac{1}{8}(x)^2 - \frac{1}{6}x^3$. In Table 3, the absolute error for $n = 2$ and $n = 3$ is calculated. Moreover, their running times are 0.156 and 0.562 seconds, respectively. Also, in Figure 5, we plot the comparison of the exact and approximate solutions. One can see in Figure 6 the absolute errors for $n = 3$.

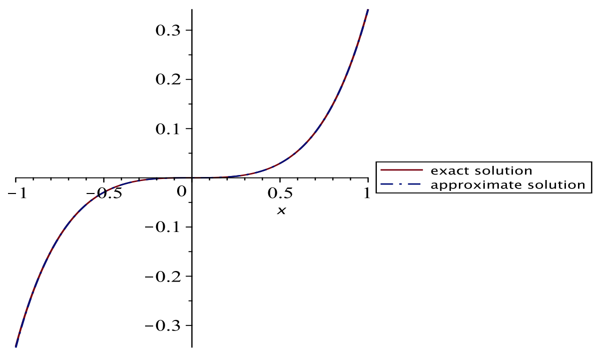

Figure 3: The plot of exact and numerical solutions for $m = 2^5$ in Example 2.

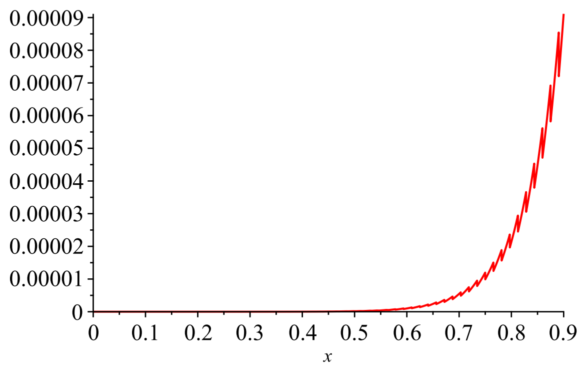

Figure 4: Absolute errors of Example 2.

Table 3: Absolute errors of Example 3.

x_i	$m = 2^3$	$m = 2^4$
0.1	$2.2E{-}10$	$2.2E{-}10$
0.2	$1.37E{-}11$	$1.37E{-}11$
0.3	$1.43E{-}9$	$1.33E{-}9$
0.4	$4.82E{-}9$	$4.74E{-}9$
0.5	$1.05E{-}8$	$1.05E{-}8$
0.6	$1.09E{-}8$	$1.15E{-}8$
0.7	$1.41E{-}8$	$1.41E{-}8$
0.8	$3.69E{-}8$	$3.67E{-}8$
0.9	$4.14E{-}7$	$4.106E{-}7$
CPU-Time (s)	0.156	0.562

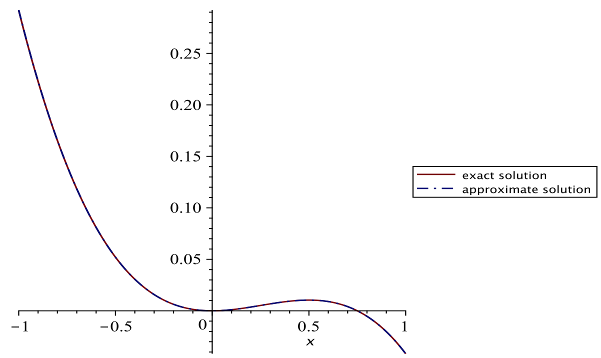

Figure 5: The plot of exact and numerical solutions for $m = 2^4$ in Example 3.

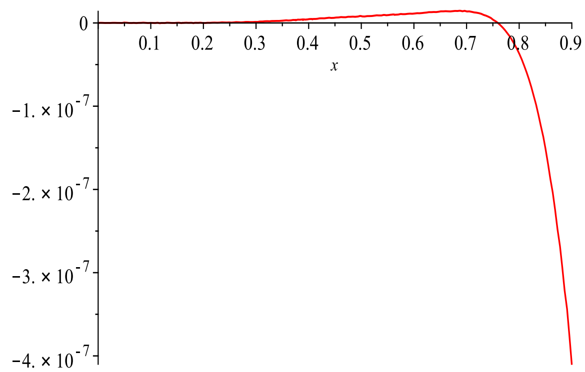

Figure 6: Absolute errors of Example 3.

6 Conclusion

In this paper, we presented a new method based on the RH wavelet basis for solving the NVIDE, and NFIDE. In this method, we did not use numerical integrations, and the method dose not require the solve of the algebraic systems. It means that by using a small number of basis functions we obtain high accuracy. We proved the convergence and computed the rate of convergence of method that is $O(i(2q)^i)$. Moreover, we used the successive method for approximations of (1) and (2). Also, an advantage of this method is its easy implementing for a computer.

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