Journal of Mathematical Modeling
Vol. 9, No. 2, 2021, pp. 173–183

JMM

Unified ball convergence of third and fourth convergence order algorithms under $\omega-$continuity conditions

Gus Argyros[†], **Michael Argyros** [†], **Ioannis K. Argyros**[†], **Santhosh George**[‡*]

[†] *Department of Computing and Technology, Cameron University, Lawton, OK 73505, USA*
[‡] *Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, India-575 025*

Email(s): Gus.Argyros@cameron.edu, Michael.Argyros@cameron.edu, iargyros@cameron.edu, sgeorge@nitk.edu.in

Abstract. There is a plethora of third and fourth convergence order algorithms for solving Banach space valued equations. These orders are shown under conditions on higher than one derivatives not appearing on these algorithms. Moreover, error estimations on the distances involved or uniqueness of the solution results if given at all are also based on the existence of high order derivatives. But these problems limit the applicability of the algorithms. That is why we address all these problems under conditions only on the first derivative that appear in these algorithms. Our analysis includes computable error estimations as well as uniqueness results based on $\omega-$ continuity conditions on the Fréchet derivative of the operator involved.

Keywords: $\omega-$ continuity, ball of convergence, algorithm.
AMS Subject Classification 2010: 65H05, 65J15, 49M15.

1 Introduction

Let X, Y denote Banach spaces [6, 7, 31], $\Omega \subset X$ be an open and convex set, and $F : \Omega \longrightarrow Y$ be a differentiable operator (according to Fréchet [31]). There is a plethora of applications from computational sciences that using mathematical modeling [2, 5, 7–9, 25, 32] can be reduced to finding a solution x^* of equation

$$F(x) = 0. \tag{1}$$

But the solution x^* can be found in closed form only in special cases. That explains why algorithms are used generating sequences approximating x^* under certain conditions on the initial data.

*Corresponding author.
Received: 4 September 2020 / Revised: 18 September 2020 / Accepted: 26 September 2020
DOI: 10.22124/jmm.2020.17556.1513

Recently, due also to the development of new and faster computers there is a greater need for developing high convergence order algorithms. Looking at this direction Noor et al. [28] provided the ball convergence of the third and fourth order classes of algorithms defined for $x_0 \in \Omega$ and all $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - [\sum_{i=1}^{m} q_i F'(x_n + \theta_i(y_n - x_n))]^{-1}F(y_n), \end{aligned} \tag{2}$$

and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - A_n^{-1}F(y_n), \\ x_{n+1} &= z_n - B_n^{-1}F(z_n), \end{aligned} \tag{3}$$

respectively, where m is a fixed natural number, q_i are weights and θ_i suitably chosen real numbers so that $\lim_{n\longrightarrow\infty} x_n = x_*$. By specifying m, q_i, θ_i these algorithms reduce to popular ones studied under a variety of conditions all requiring the existence of higher than one derivatives not appearing on the algorithms limiting their applicability. Operators $F'(x_n), A_n = [\sum_{i=1}^{m} q_i F'(x_n + \theta_i(y_n - x_n))], B_n = [\sum_{i=1}^{m} q_i F'(x_n + \theta_i(z_n - x_n))]$ should be linear and invertible for the iterates to exist in method (2) and (3). We refer the reader to [28] where these methods were originated for further details and to avoid repetitions for these methods.

The assumptions on the fifth or higher order derivative limit the application of these methods, even for scalar examples. Let $X = Y = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define f on Ω by

$$f(s) = s^3 \log s^2 + s^5 - s^4,$$

Then, we have $x_* = 1$, and

$$\begin{aligned} f'(s) &= 3s^2 \log s^2 + 5s^4 - 4s^3 + 2s^2, \\ f''(s) &= 6x \log s^2 + 20s^3 - 12s^2 + 10s, \\ f'''(s) &= 6 \log s^2 + 60s^2 - 24s + 22. \end{aligned}$$

Obviously $f'''(s)$ is not bounded on Ω. So, the convergence of algorithms (2) and (3) are not guaranteed by the analysis in [28].

The error estimations on $\|x_n - x_*\|$ or the uniqueness of x_* if given at all also involve high derivatives. That is why our ball convergence involves only $\omega-$ continuity condition on f' which actually appears on these algorithms. Our work improves as well studies [1–32].

The technique is given in Section 2, the numerical experiments in Section 3 and the conclusions in Section 4.

2 Ball convergence

We shall use some real functions and parameters in our convergence analysis. Set

$$\alpha = |\sum_{i=1}^{m} q_i|, \quad \beta = \sum_{i=1}^{m} |q_i|, \quad \gamma = \frac{\beta}{\alpha}$$

and $T=[0,\infty)$.

Suppose that there exists function $w_0 : T \longrightarrow T$ continuous and nondecreasing such that equation

$$w_0(s)-1=0, \tag{1}$$

has a least solution $\rho_1 \in T-\{0\}$. Set $T_1=[0,\rho_0)$.

Suppose that there exists function $w : T_1 \longrightarrow T$ continuous and nondecreasing such that for

$$g_1(s)=\frac{\int_0^1 w((1-\theta)s)d\theta}{1-w_0(s)}, \quad h_1(s)=g_1(s)-1,$$

equation

$$h_1(s)=0, \tag{2}$$

has a least solution $r_1 \in (0,\rho_1)$.

Suppose that equation

$$w_0(\gamma s)-1=0, \tag{3}$$

has a least solution $\rho_2 \in (0,\rho_1]$. Set $T_2=[0,\rho_2)$.

Suppose that there exists a function $w_1 : T_2 \longrightarrow T$ such that for

$$g_2(s)=(1+\frac{\int_0^1 w_1(\theta g_1(s)s)d\theta}{\alpha(1-\gamma w_0(s))})g_1(s),$$

$h_2(s)=g_2(s)-1$,

$$g_3(s)=(1+\frac{\int_0^1 w_1(\theta g_2(s)s)d\theta}{\alpha(1-\gamma w_0(s))})g_2(s),$$

$h_3(s)=g_3(s)-1$, equations

$$h_2(s)=0, \tag{4}$$

and

$$h_3(s)=0, \tag{5}$$

have least solutions $r_2, r_3 \in (0,\rho_2]$. We shall show that r defined by

$$r=\min\{r_j\}, \quad j=1,2,3, \tag{6}$$

is a radius of convergence for algorithm (3). It follows by these definitions that for each $s \in [0,r)$

$$0 \le w_0(s) < 1, \tag{7}$$

$$0 \le w_0(\gamma s) < 1, \tag{8}$$

and

$$0 \le g_j(s) < 1. \tag{9}$$

Let $U(u,\lambda)$ and $\bar{U}(u,\lambda)$ stand for the open and closed balls in X with center $u \in X$ and of radius $\lambda > 0$.

The conditions (A) shall be used in the ball convergence.

(a1) There exists a simple solution x^* of equation $F(x) = 0$, and a real sequence $\{\theta_i\}$ such that $|1 - \theta_i| + |\theta_i| \leq 1$.

(a2) There exists a function $w_0 : T \longrightarrow T$ continuous and nondecreasing such that for all $x \in \Omega$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq w_0(\|x - x^*\|).$$

Set $U_0 = \Omega \cap U(x^*, \rho_1)$, where ρ_1 given in (1) exists.

(a3) There exists a functions $w : T_0 \longrightarrow T$ such that for each $x, y \in U_0$

$$\|F'(x^*)^{-1}(F'(y) - F'(x))\| \leq w(\|y - x\|)$$

Set $U_1 = \Omega \cap U(x^*, \rho_2)$, where ρ_2 exists and is given in (3).

(a4) There exists a function $w_1 : T_1 \longrightarrow T$ such that for each $x \in U_1$

$$\|F'(x^*)^{-1}F(x)\| \leq w_1(\|x - x^*\|).$$

(a5) $\bar{U}(x^*, r) \subset \Omega$, where r is defined by (6).

(a6) There exists $r_* \geq r$ such that

$$\int_0^1 w_0(\theta r_*)d\theta < 1.$$

Set $U_2 = \Omega \cap \bar{U}(x^*, r_*)$.

Next, the ball convergence of algorithm (3) is presented using the conditions (A) and the preceding notation.

Theorem 1. *Suppose the conditions (A) hold. Then, for $x_0 \in U(x^*, r) - \{x^*\}$, sequence $\{x_n\}$ generated by algorithm (3) is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and $\lim_{n\longrightarrow\infty} x_n = x^*$. Moreover, the following error estimations hold for $e_n = \|x_n - x^*\|$ and each $n = 0, 1, 2, \ldots$:*

$$\|y_n - x^*\| \leq g_1(e_n)e_n \leq e_n < r, \tag{10}$$

$$\|z_n - x^*\| \leq g_2(e_n)e_n \leq e_n, \tag{11}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(e_n)e_n \leq e_n, \tag{12}$$

where the functions g_j are given previously and r is defined in (6). Furthermore, x^* is the only solution of equation $F(x) = 0$ in the set U_2 given in (a6).

Proof. Let $u \in U(x^*, r) - \{x^*\}$. Using (1) and (6), we obtain

$$\|F'(x^*)^{-1}(F'(u) - F'(x^*))\| \leq w_0(\|u - x^*\|) \leq w_0(r) < 1,$$

imply by the Banach lemma on invertible operators [31], that $F'(u)$ is invertible and

$$\|F'(u)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|u - x^*\|)}. \tag{13}$$

By setting $u = x_0$, we see that y_0 is well defined by the first substep of algorithm (3) for $n = 0$. Then, we can also write

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0). \tag{14}$$

In view of (a1)-(a3), (6), (9) (for $j = 1$) and (14), we get in turn that

$$\begin{aligned}
\|y_0 - x^*\| &= \|(F'(x_0)^{-1}F'(x^*))(\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0)(x_0 - x^*)d\theta\| \\
&\leq \|F'(x_0)^{-1}F'(x^*)\|\|\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0)d\theta\|\|x_0 - x^*\| \\
&\leq \frac{[\int_0^1 w((1-\theta)e_0)d\theta e_0}{1 - w_0(e_0)} \\
&\leq g_1(e_0)e_0 \leq e_0 < r, \qquad (15)
\end{aligned}$$

so $y_0 \in U(x^*, r)$ and (10) is verified for $n = 0$.

Next, set for convenience

$$M_n = \sum_{i=1}^{m} q_i F'(x_n + \theta_i(y_n - x_n)).$$

Then, by (a1), (6), (8), (15) and (a2), we have

$$\begin{aligned}
&\|(\sum_{i=1}^{m} q_i F'(x^*))^{-1}(\sum_{i=1}^{m} q_i(F'(x_0 + \theta_i(y_0 - x_0)) - F'(x^*))\| \\
&\qquad \leq \frac{1}{\alpha}\|\sum_{i=1}^{m} q_i F'(x^*)^{-1}(F'(x_0 + \theta_i(y_0 - x_0)) - F'(x^*))\| \\
&\qquad \leq \frac{1}{\alpha}\|\sum_{i=1}^{m} |q_i|\|F'(x^*)^{-1}(F'(x_0 + \theta_i(y_0 - x_0)) - F'(x^*))\| \\
&\qquad \leq \frac{1}{\alpha}\|\sum_{i=1}^{m} |q_i| w_0(|1 - \theta_i|e_0 + |\theta_i|e_0)\| \\
&\qquad \leq \frac{1}{\alpha}\|\sum_{i=1}^{m} |q_i| w_0((|1 - \theta_i| + |\theta_i|)e_0)\| \\
&\qquad \leq \frac{1}{\alpha}\|\sum_{i=1}^{m} |q_i| w_0(e_0)\| = \gamma w_0(e_0) \\
&\qquad \leq \gamma w_0(e_0) < 1,
\end{aligned}$$

so

$$\|M_0^{-1}F'(x^*)\| \leq \frac{1}{\alpha(1 - \gamma w_0(e_0))}, \tag{16}$$

and z_0 is well defined by the second substep of algorithm (3) for $n = 0$. Moreover, we can write

$$z_0 - x^* = y_0 - x^* - M_0^{-1}F(y_0).$$

Then, by (6), (9) (for $j = 2$), and (15)-(16), we have in turn that

$$\begin{aligned}\|z_0 - x^*\| &\leq \|y_0 - x^*\| + \|M_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(y_0)\|[1 + \frac{\int_0^1 w_1(\theta\|y_0 - x^*\|)d\theta}{\alpha(1-\gamma w_0(e_0))}]\|y_0 - x^*\| \\ &\leq g_2(e_0)e_0 \leq e_0 < r, \end{aligned} \tag{17}$$

so $z_0 \in U(x^*, r)$, (11) is verified for $n = 0$. Set

$$N_n = \sum_{i=1}^{m} q_i F'(x_n + \theta_i(z_n - x_n)).$$

Then, as in the calculations for (16) but with z_n replacing y_n, we also get

$$\|N_0^{-1}F'(x^*)\| \leq \frac{1}{\alpha(1-\gamma w_0(e_0))}, \tag{18}$$

so x_1 is well defined too by the last substep of algorithm (3) for $n = 0$. Furthermore, we can write

$$x_1 - x^* = z_0 - x^* - N_0^{-1}F(z_0). \tag{19}$$

Then, by (6), (9) (for $j = 3$), (a3) and (17)- (19), we get in turn that

$$\begin{aligned} e_1 &\leq \|z_0 - x^*\| + \|N_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(z_0)\| \\ &\leq [1 + \frac{\int_0^1 w_1(\theta\|z_0 - x^*\|)d\theta}{\alpha(1-\gamma w_0(e_0))}]\|z_0 - x^*\| \\ &\leq g_3(e_0)e_0 \leq e_0 < r, \end{aligned} \tag{20}$$

so (12) is verified for $n = 0$ and $x_1 \in U(x^*, r)$. Hence, the induction for estimation (10)-(12) is completed for $n = 0$. Suppose these estimations hold for all $p = 0, 1, 2, \ldots, n$. Then, by repeating the preceding calculations with x_0, y_0, z_0, x_1 by x_p, y_p, z_p, x_{p+1}, we complete the induction for estimations (13)-(15). Then, from the estimation

$$e_{p+1} \leq c e_p < r, \tag{21}$$

where $c = g_3(e_0) \in [0, 1)$, we conclude that $\lim_{p\longrightarrow\infty} x_p = x^*$, and $x_{p+1} \in U(x^*, r)$. Finally, consider $x^{**} \in U_2$ with $F(x^{**}) = 0$. Then, by (a2) and (a6) for

$$T = \int_0^1 F'(x^{**} + \theta(x^* - x^{**}))d\theta,$$

we obtain in turn that

$$\|F'(x^*)^{-1}(T - F'(x^*))\| \leq \int_0^1 w_0(\theta\|x^* - x^{**}\|)d\theta \leq \int_0^1 w_0(\theta r_*)d\theta < 1,$$

so $x^* = x^{**}$ follows from the invertability of T and the identity $0 = F(x^*) - F(x^{**}) = T(x^* - x^{**})$. $\square$

Remark 1.

1. *By (a2), and the estimate*

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x)-F'(x^*))+I\|\\ &\leq 1+\|F'(x^*)^{-1}(F'(x)-F'(x^*))\|\\ &\leq 1+w_0(\|x-x^*\|)\end{aligned}$$

second condition in (a3) can be dropped, and w_1 *be defined as*

$$w_1(t)=1+w_0(t).$$

Notice that, if $w_1(t)<1+w_0(t)$*, then* R_1 *can be large (see Example 1).*

2. *The results obtained here can be used for operators* G *satisfying autonomous differential equations [5–10] of the form*

$$F'(x)=T(F(x))$$

where T *is a continuous operator. Then, since* $F'(x^*)=T(F(x^*))=T(0)$*, we can apply the results without actually knowing* x^**. For example, let* $F(x)=e^x-1$*. Then, we can choose:* $T(x)=x+1$*.*

3. *The local results obtained here can be used for projection algorithms such as the Arnoldi's algorithm, the generalized minimum residual algorithm (GMRES), the generalized conjugate algorithm(GCR) for combined Newton/finite projection algorithms and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [5–10].*

4. *Let* $w_0(t)=L_0t$*, and* $w(t)=Lt$*. The parameter* $r_A=2/(2L_0+L)$ *was shown by us to be the convergence radius of Newton's algorithm [10]*

$$x_{n+1}=x_n-F'(x_n)^{-1}F(x_n)\ \text{ for each }\ n=0,1,2,\ldots,$$

under the conditions (a1)-(a3) (w_1 *is not used). It follows that the convergence radius* R *of algorithm (2) cannot be larger than the convergence radius* r_A *of the second order Newton's algorithm. As already noted in [5–10]* r_A *is at least as large as the convergence ball given by Rheinboldt [31]*

$$r_{TR}=\frac{2}{3L_1},$$

where L_1 *is the Lipschitz constant on* $\Omega, L_0\leq L_1$ *and* $L\leq L_1$*. In particular, for* $L_0<L_1$ *or* $L<L_1$*, we have that*

$$r_{TR}<r_A$$

and

$$\frac{r_{TR}}{r_A}\to\frac{1}{3}\ \ as\ \ \frac{L_0}{L_1}\to 0.$$

That is our convergence ball r_A *is at most three times larger than Rheinboldt's. The same value for* r_{TR} *was given by Traub [32].*

5. *It is worth noticing that algorithm (2) is not changing, when we use the conditions (A) of Theorem 1 instead of the stronger conditions used in [28]. Moreover, we can compute the computational order of convergence (COC) defined by*

$$\mu = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\mu_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds given in [28] involving estimates up to the seventh Fréchet derivative of operator F.

6. *Clearly, if g_3 is dropped, z_n is x_{n+1} and*

$$\bar{r} = \min\{r_1, r_2\}, \tag{22}$$

we obtain from Theorem 1, the corresponding results for algorithm (2).

3 Numerical examples

In all the examples, we have taken $m = 2$ and $q_i = \frac{1}{2}$.

Example 1. Let $X = Y = \mathbb{R}$. Define $F(x) = \sin x$. Then, we get that $x^* = 0,\ w_0(s) = w(s) = s$ and $w_1(s) = 1$. The parameters are given in Table 1.

Table 1: Radius for Example 1.

Radius	$\omega_1(s) = e^{\frac{1}{e-1}}$	$\omega_1(s) = 1 + \omega_0(s)$
r_1	0.6667	0.6667
r_2	0.4226	0.4140
r_3	0.2643	0.2581

Example 2. Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$ with the max norm. Let $\Omega = \overline{U}(0,1)$. Define function F on Ω by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in \Omega.$$

Then, we get that $x^* = 0,\ w_0(s) = w_1(s) = \frac{15}{2}s, w_1(s) = 15$. The parameters are given in Table 2.

Table 2: Radius for Example 2.

Radius	$\omega_1(s) = e^{\frac{1}{e-1}}$	$\omega_1(s) = 1 + \omega_0(s)$
r_1	0.0889	0.0889
r_2	0.0543	0.0552
r_3	0.0069	0.0344

Example 3. Let $X = Y = \mathbb{R}^3$, $\Omega = U(0,1), x_* = (0,0,0)^T$ and define F on Ω by

$$F(x) = F(u_1, u_2, u_3) = (e^{u_1} - 1, \frac{e-1}{2}{u_2}^2 + u_2, u_3)^T. \tag{2}$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$F'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows and since $G'(x^*) = diag(1,1,1)$, we get by conditions (A) $w_0(s) = (e-1)s$, $w(s) = e^{\frac{1}{e-1}}s$, and $w_1(s) = e^{\frac{1}{e-1}}$. The parameters are given in Table 3.

Table 3: Radius for Example 3.

Radius	$\omega_1(s) = e^{\frac{1}{e-1}}$	$\omega_1(s) = 1 + \omega_0(s)$
r_1	0.3827	0.3827
r_2	0.2797	0.2364
r_3	0.1433	0.1468

Example 4. Returning back to the motivational example at the introduction of this study, we have $w_0(s) = w(s) = 96.662907s$, $w_1(s) = 1.0631$. The parameters are given in Table 4.

Table 4: Radius for Example 4.

Radius	$\omega_1(s) = e^{\frac{1}{e-1}}$	$\omega_1(s) = 1 + \omega_0(s)$
r_1	0.0069	0.0069
r_2	0.0069	0.0043
r_3	0.0048	0.0027

4 Conclusions

We present a new technique for comparing competing algorithms based only on the first derivative that actually appears on these algorithms in contrast to earlier ones using higher than one derivatives. Our technique is so general that it can be used to extend the applicability of other algorithms along the same lines. Our technique also provides computable error estimations and uniqueness results based on $\omega-$ continuity conditions on F' not possible in earlier works [1–32].

References

[1] S. Abbasbandy, *Extended Newtons method for a system of nonlinear equations by modified Adomian decomposition method*, Appl. Math. Comput. **170** (2005) 648-656.

[2] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Dordrecht, 1994.

[3] S. Amat, C. Bermudez, M. A. Hernandez, E. Martinez, *On an efficient $k-$step iterative method for nonlinear equations*, J. Comput. Appl. Math. **302** (2016) 258-271.

[4] S. Amat, I.K Argyros, S. Busquier, M.A. Hernández, *On two high-order families of frozen Newton-type methods*, Numer. Linear. Algebra Appl. **25** (2018) e2126.

[5] I.K. Argyros, *Computational Theory of Iterative Solvers*, Series: Studies in Computational Mathematics, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[6] I.K. Argyros, S. Hilout, *Weaker conditions for the convergence of Newton's method*, J. Complexity **28** (2012) 364–387.

[7] I.K.Argyros, A.A. Magréñan, *A Contemporary Study of Iterative Methods*, Elsevier (Academic Press), New York, 2018.

[8] I.K.Argyros, A.A. Magréñan, *Iterative Methods and Their Dynamics with Applications*, CRC Press, New York, USA, 2017.

[9] I.K. Argyros, S. George, *Mathematical Modeling for the Solution of Equations and Systems of Equations with Applications*, Volume-IV, Nova Publishes, NY, 2020.

[10] I.K. Argyros, S. George, *On the complexity of extending the convergence region for Traub's method*, J. Complexity **56** (2020) 101423.

[11] D.K.R. Babajee, M.Z. Dauhoo, M.T. Darvishi, A. Barati, *A note on the local convergence of iterative methods based on Adomian decomposition method and 3-node quadrature rule*, Appl. Math. Comput. **200** (2008) 452-458.

[12] E. Babolian, J. Biazar, A.R. Vahidi, *Solution of a system of nonlinear equations by Adomian decomposition method*, Appl. Math. Comput. **150** (2004) 847-854.

[13] A. Cordero, J.R. Torregrosa, *Variants of Newtons method for functions of several variables*, Appl. Math. Comput. **183** (2006) 199-208.

[14] A. Cordero, J.R. Torregrosa, *Variants of Newtons method using fifth-order quadrature formulas*, Appl. Math. Comput. **190** (2007) 686-698.

[15] A. Cordero, E. Martnez, J.R. Torregrosa, *Iterative methods of order four and five for systems of nonlinear equations*, J. Comput. Appl. Math. **231** (2009) 541-551.

[16] M. Frontini, E. Sormani, *Third-order methods from quadrature formulae for solving systems of nonlinear equations*, Appl. Math. Comput. **149** (2004) 771-782.

[17] A. Golbabai, M. Javidi, *A new family of iterative methods for solving system of nonlinear algebraic equations*, Appl. Math. Comput. **190** (2007) 1717-1722.

[18] A. Golbabai, M. Javidi, *Newton-like iterative methods for solving system of nonlinear equations*, Appl. Math. Comput. **192** (2007) 546-551.

[19] J.H. He, *A new iteration method for solving algebraic equations*, Appl. Math. Comput. **135** (2003) 81-84.

[20] J.L. Hueso, E. Martinez, J.R. Torregrosa, *Third and fourth order iterative methods free from second derivative for nonlinear systems*, Appl. Math. Comput. **211** (2009) 190-197.

[21] J.L. Hueso, E. Martinez, J.R. Torregrosa, *Third order iterative methods free from second derivative for nonlinear systems*, Appl. Math. Comput. **215** (2009) 58-65.

[22] H. Jafari, V. Daftardar-Gejji, *Revised Adomian decomposition method for solving a system of nonlinear equations*, Appl. Math. Comput. **175** (2006) 1-7.

[23] D. Kaya, S.M. El-Sayed, *Adomians decomposition method applied to systems of nonlinear algebraic equations*, Appl. Math. Comput. **154** (2004) 487-493.

[24] A.A. Magreñán, Cordero, M.J. Gutiérrez, J.R. Torregrosa, *Real qualitative behavior of a fourth-order family of iterative methods by using the convergence plane*, Math. Comput. Simul. **105** (214) 49-61.

[25] A.A. Magreñán, I.K. Argyros, *Two-step Newton methods*, J. Complexity **30** (2014) 533-553.

[26] M.A. Noor, K.I. Noor, M. Waseem, *Decomposition method for solving system of linear equations*, Engin. Math. Lett. **2** (2013) 34-41.

[27] M.A. Noor, M. Waseem, K.I. Noor, E. Al-Said, *Variational iteration technique for solving a system of nonlinear equations*, Optim. Lett. **7** (2013) 991-1007.

[28] M.A. Noor, M. Waseem, K.I. Noor, *New iterative technique for solving a system of equation*, Appl. Math. Comput. **271** (2015) 446-466.

[29] M. Özel, *A new decomposition method for solving system of nonlinear equations*, Math. Comput. Appl. **15** (2010) 89-95.

[30] M. Podisuk, U. Chundang, W. Sanprasert, *Single-step formulas and multi-step formulas of the integration method for solving the initial value problem of ordinary differential equation*, Appl. Math. Comput. **190** (2007) 1438-1444.

[31] W.C. Rheinboldt, *An adaptive continuation process for solving systems of nonlinear equations*, In: Mathematical models and numerical solvers (A.N. Tikhonov et al. eds.) pub.3, (1977), 129-142 Banach Center, Warsaw Poland.

[32] J.F. Traub, *Iterative Solvers for the Solution of Equations*, AMS Chelsea Publishing, 1982.