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On global existence and Ulam-Hyers stability of $\Psi-$Hilfer fractional integrodifferential equations

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Abstract. In this paper, we propose a fractional integral equation and prove the existence and uniqueness of solutions for the Cauchy-type problem for a nonlinear $\Psi-$Hilfer fractional integrodifferential equations of the type

$$
\begin{aligned}
&{}^{H}D^{\mu,\nu;\Psi}_{a+}y(t) = f\left(t, y(t), \int_a^t K(t,s)y(s)ds\right),\\
&I^{1-\rho;\Psi}_{a^+}y(a) = y_a.
\end{aligned}
$$

In this sense, for this new fractional integrodifferential equation, we study the Ulam-Hyers and Ulam-Hyers-Rassias stability via successive approximation method. Further, we investigate the dependence of solutions on the initial conditions and uniqueness via $\epsilon-$approximated solution.

Keywords: Ulam-Hyers stability, $\Psi-$Hilfer fractional derivative, fractional integrodifferential equations, Banach fixed-point theorem.

AMS Subject Classification 2010: 26A33, 34A08, 34A12, 34K20, 37C25.

1 Introduction

The study of differentiation and integration to a fractional order is important due to its popularity and wide applications to real-world phenomena with the hereditary property. One can refer [10] for more details on fractional calculus theory and interesting applications. The generalization of Riemann-Liouville and Caputo fractional derivatives, introduced in 1999 by R. Hilfer [6]. Hilfer fractional definition facilitated dynamic modeling of non-equilibrium processes

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based on interpolation with respect to the parameter Riemann-Liouville and Caputo type operators. Most of the practical systems are integrodifferential equations in nature and hence the study of integrodifferential system is significant. One of the crucial and interesting areas of research in the theory of functional equations is devoted to the satbility analysis. Stability analysis is the fundamental property of the mathematical analysis which has got paramount importance in many fields of engineering and science. In the existing literature, there are stabilities such as Mittag-Leffler, h-stability, exponential, Lyapunov stability and so on. In the nineteenth-century, Ulam and Hyer presented an interesting type of stability called Ulam-Hyers stability, which, nowadays has been picked up a great deal of consideration due to a wide range of applications in many fields of science such as optimization and mathematical modeling.

The Ulam stability can be considered as a special type of data dependence which was initiated by Ulam [16]. In [12], Rassias extended the concept of Ulam-Hyers stability. Meanwhile, there have been few works considering Ulam-Hyers stability of variety of classes of fractional differential equations [4, 5, 11–20] and the references therein. The most popular techniques that deals with Ulam-Hyers stabilities of different kinds of differential and integral equations includes: fixed point technique, successive approximations method and by applying integral inequalities.

In this paper, we are concerned with the global existence and uniqueness of solution, and Ulam-Hyers stability for the $\Psi-$Hilfer fractional integrodifferential equations (FIDE) of the following type

$$^{H}D_{a+}^{\mu,\nu;\Psi}y(t) = f\left(t, y(t), \int_a^t K(t,s)y(s)ds\right), \tag{1}$$

$$I_{a^+}^{1-\rho;\Psi}y(a) = y_a \tag{2}$$

where $t \in [a,b],\ 0 < \mu < 1,\ 0 \leq \nu \leq 1,\ {}^{H}D_{a+}^{\mu,\nu;\Psi}(.)$ is the (left-sided) $\Psi-$Hilfer fractional derivative of order μ and type $\nu, I_{a^+}^{1-\rho;\Psi}$ is (left-sided) fractional integral of order $1-\rho$ with respect to another function Ψ in Riemann-Liouville sense and $f : [a,b] \times R \times R \to R$ is a given function that will be specified later.

The main objective of this paper is to prove the global existence and uniqueness of solution to $\Psi-$Hilfer fractional integrodifferential equations (1)-(2). Using method of successive approximations we investigate the Ulam-Hyers (HU) and Ulam-Hyers-Rassias (HUR) stability of $\Psi-$Hilfer fractional integrodifferential equation (1). By utilizing generalized Gronwall inequality [17] we obtain an estimations for the difference of two $\epsilon-$approximated solutions of $\Psi-$Hilfer fractional integrodifferential equations (1)-(2), from which we can derive the results pertaining to uniqueness and dependence of solutions on the initial conditions. The obtained results are not only new in the given configuration but also yield several interesting special cases associated with the particular values of the parameters involved in the given problem (for details, refer Conclusion section).

The paper is organized as follows. Some basic definitions and results concerning $\Psi-$Hilfer fractional derivative are introduced in Section 2. Section 3 devoted to discuss global existence and uniqueness of solutions of the problem (1)-(2). Section 4 presents the Ulam-Hyers (HU) and Ulam-Hyers-Rassias (HUR) stability of $\Psi-$Hilfer FIDE (1) via successive approximations. Section 5 deals with $\epsilon-$approximate solution of the $\Psi-$Hilfer FIDE (1).

2 Preliminaries

Here we present some definitions, notations and results from [10,17,18] which are used throughout this paper. Let $0 < a < b < \infty$, $\Delta = [a, b] \subset R_+ = [0, \infty)$, $0 \leq \rho < 1$ and $\Psi \in C^1(\Delta, R)$ be an increasing function such that $\Psi'(x) \neq 0, \forall x \in \Delta$. The weighted spaces $C_{1-\rho;\Psi}(\Delta, R)$, $C^{\rho}_{1-\rho;\Psi}(\Delta, R)$ and , $C^{\mu,\nu}_{1-\rho;\Psi}(\Delta, R)$ of functions are defined as follows:

(i) $C_{1-\rho;\Psi}(\Delta, R) = \{h : (a, b] \to R : (\Psi(t) - \Psi(a))^{1-\rho} h(t) \in C(\Delta, R)\}$, with the norm $\|h\|_{C_{1-\rho;\Psi}} = \max_{t\in\Delta} |(\Psi(t) - \Psi(a))^{1-\rho} h(t)|$,

(ii) $C^{\rho}_{1-\rho;\Psi}(\Delta, R) = \{h \in C_{1-\rho;\Psi}(\Delta, R) : D^{\rho}_{a^+} h(t) \in C_{1-\rho;\Psi}(\Delta, R)\}$,

(iii) $C^{\mu,\nu}_{1-\rho;\Psi}(\Delta, R) = \{h \in C_{1-\rho;\Psi}(\Delta, R) :^H D^{\mu,\nu}_{a^+} h(t) \in C_{1-\rho;\Psi}(\Delta, R)\}$.

Definition 1. *[10, 14] The $\Psi-$Riemann fractional integral of order $\mu > 0$ of the function h is given by*

$$I^{\mu;\Psi}_{a^+} h(t) = \frac{1}{\Gamma(\mu)} \int_a^t \mathcal{L}^{\mu}_{\Psi}(t, \eta) h(\eta) d\eta,$$

where $\mathcal{L}^{\mu}_{\Psi}(t, \eta) = \Psi'(\eta)(\Psi(t) - \Psi(\eta))^{\mu-1}$.

Lemma 1. *[10] Let $\mu > 0$, $\nu > 0$ and $\delta > 0$. Then*

(i) $I^{\mu;\Psi}_{a^+} I^{\nu;\Psi}_{a^+} h(t) = I^{\mu+\nu;\Psi}_{a^+} h(t)$

(ii) If $h(t) = (\Psi(t) - \Psi(\eta))^{\delta-1}$, *then* $I^{\mu;\Psi}_{a^+} h(t) = \frac{\Gamma(\delta)}{\Gamma(\mu+\delta)} (\Psi(t) - \Psi(\eta))^{\mu+\delta-1}$.

We need following results [10, 14] which are useful in the subsequent analysis of the paper.

Lemma 2. *[18] If $\mu > 0$ and $0 \leq \rho < 1$, then $I^{\mu;\Psi}_{a^+}$ is bounded from $C_{\rho;\Psi}(\Delta, R)$ to $C_{\rho;\Psi}(\Delta, R)$. Also, if $\rho \leq \mu$, then $I^{\mu;\Psi}_{a^+}$ is bounded from $C_{\rho;\Psi}(\Delta, R)$ to $C(\Delta, R)$.*

Definition 2. *[17] The $\Psi-$Hilfer fractional derivative of a function h of order $0 < \mu < 1$ and type $0 \leq \nu \leq 1$, is defined by*

$${}^H D^{\mu,\nu}_{a^+} h(t) = I^{\nu(1-\mu);\Psi}_{a^+} \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right) I^{(1-\nu)(1-\mu);\Psi}_{a^+} h(t).$$

Lemma 3. *[17] If $h \in C^1(\Delta, R)$, $0 < \mu < 1$ and $0 \leq \nu \leq 1$, then*

(i) $I^{\mu;\Psi}_{a^+} {}^H D^{\mu,\nu}_{a^+} h(t) = h(t) - \Omega^{\rho}_{\Psi}(t, a) I^{(1-\nu)(1-\mu);\Psi}_{a^+} h(a)$ *where* $\Omega^{\rho}_{\Psi}(t, a) = \frac{(\Psi(t)-\Psi(a))^{\rho-1}}{\Gamma(\rho)}$;

(ii) ${}^H D^{\mu,\nu}_{a^+} I^{\mu;\Psi}_{a^+} h(t) = h(t)$.

Definition 3. *([10]) Let $\mu > 0$, $\nu > 0$. The one parameter Mittag-Leffler function is defined as*

$$E_{\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\mu + 1)},$$

and the two parameter Mittag-Leffler function is defined as

$$E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\mu + \nu)}.$$

3 Existence and Uniqueness results

In this section, we will study the existence and uniqueness results of the Cauchy-type problem (1)-(2) by applying the following modified version of contraction principle.

Lemma 4. *[15] Let χ be a Banach space and let T be an operator which maps the elements of χ into itself for which T^r is a contraction, where r is a positive integer then T has a unique fixed point.*

Theorem 1. *Let $0<\mu<1,\ 0\leq\nu\leq 1$ and $\rho=\mu+\nu-\mu\nu$. Let $f:(a,b]\times R\times R\to R$ be a function such that $f\left(t,y(t),\int_a^t K(t,s)y(s)ds\right)\in C_{1-\rho;\Psi}(\Delta,R)$ for any $y\in C_{1-\rho;\Psi}(\Delta,R)$, and let f satisfies the Lipschitz condition*

$$|f(t,y_1,z_1)-f(t,y_2,z_2)|\leq L\left(|y_1-y_2|+|z_1-z_2|\right), \tag{3}$$

for all $t\in(a,b]$ and for all $y_1,y_2,z_1,z_2\in R$, where $L>0$ is Lipschitz constant. Then the Cauchy problem (1)-(2) has unique solution in $C_{1-\rho;\Psi}(\Delta,R)$.

Proof. The equivalent fractional integral to the initial value problem (1)-(2) is given by [17], for $t\in(a,b]$,

$$\begin{aligned} y(t) &= \Omega^{\rho}_{\Psi}(t,a)y_a + I^{\mu:\Psi}_{a^+}f\left(t,y(t),\int_a^t K(t,s)y(s)ds\right) \\ &= \Omega^{\rho}_{\Psi}(t,a)y_a + \frac{1}{\Gamma(\mu)}\int_a^t \mathcal{L}^{\mu}_{\Psi}(t,\eta)f\left(\eta,y(\eta),\int_a^{\eta}K(\eta,\sigma)y(\sigma)d\sigma\right)d\eta. \end{aligned} \tag{4}$$

Our aim is to prove that the fractional integral (4) has a solution in the weighted space $C_{1-\rho;\Psi}(\Delta,R)$.

Consider the operator T defined on $C_{1-\rho;\Psi}(\Delta,R)$ by

$$(Ty)(t)=\Omega^{\rho}_{\Psi}(t,a)y_a + \frac{1}{\Gamma(\mu)}\int_a^t \mathcal{L}^{\mu}_{\Psi}(t,\eta)f\left(\eta,y(\eta),\int_a^{\eta}K(\eta,\sigma)y(\sigma)d\sigma\right)d\eta. \tag{5}$$

By Lemma2.2, it follows that

$$I^{\mu:\Psi}_{a^+}f\left(t,y(t),\int_a^t K(t,s)y(s)ds\right)\in C_{1-\rho;\Psi}(\Delta,R).$$

Clearly, $y_a\Omega^{\rho}_{\Psi}(t,a)\in C_{1-\rho;\Psi}(\Delta,R)$. Therefore, from (5), we have $Ty\in C_{1-\rho;\Psi}(\Delta,R)$ for any $y\in C_{1-\rho;\Psi}(\Delta,R)$. This proves T maps $C_{1-\rho;\Psi}(\Delta,R)$ into itself. Note that the fractional integral equation (5) can be written as fixed point operator equation $y=Ty$, $y\in C_{1-\rho;\Psi}(\Delta,R)$. We prove that the operator T has fixed point which will act as a solution for the problem (1)-(2). For any $t\in(a,b]$, consider the space $C_{t;\Psi}=C_{1-\rho;\Psi}([a,t],R)$ with the norm defined by,

$$\|y\|_{C_{t;\Psi}}=\max_{\omega\in[a,t]}\left|(\Psi(\omega)-\Psi(a))^{1-\rho}y(\omega)\right|.$$

Using mathematical induction for any $y_1, y_2 \in C_{t;\Psi}$ and $t \in (a,b]$, we prove that for $j \in N$,

$$\|T^j y_1 - T^j y_2\|_{C_{t;\Psi}} \leq \Gamma(\rho) \frac{(L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^\mu)^j}{\Gamma(j\mu+\rho)} \|y_1 - y_2\|_{C_{t;\Psi}}, \tag{6}$$

where $k_b = \sup\{|K(t,s)| : a < t, s \leq b\}$.

Let any $y_1, y_2 \in C_{t;\Psi}$. Then from the definition of operator T given in (5) and using Lipschitz condition on f, we have

$$\begin{aligned}
&\|Ty_1 - Ty_2\|_{C_{t;\Psi}} \\
&\quad \leq \frac{L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^{1-\rho}}{\Gamma(\mu)} \|y_1 - y_2\|_{C_{t;\Psi}} \times \int_a^t \mathcal{L}_\Psi^\mu(t,\eta)\ (\Psi(\eta)-\Psi(a))^{\rho-1} d\eta \\
&\quad \leq L(1+(b-a)k_b)\|y_1 - y_2\|_{C_{t;\Psi}} \left[(\Psi(t)-\Psi(a))^{1-\rho} I_{a^+}^{\mu;\Psi}\ (\Psi(t)-\Psi(a))^{\rho-1}\right] \\
&\quad \leq \frac{L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^\mu \Gamma(\rho)}{\Gamma(\mu+\rho)} \|y_1 - y_2\|_{C_{t;\Psi}}
\end{aligned}$$

Thus the inequality (6) holds for $j = 1$. Let us suppose that the inequality (6) holds for $j = r \in N$, i.e. suppose

$$\|T^r y_1 - T^r y_2\|_{C_{t;\Psi}} \leq \Gamma(\rho) \frac{(L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^\mu)^r}{\Gamma(r\mu+\rho)} \|y_1 - y_2\|_{C_{t;\Psi}}, \tag{7}$$

holds. Next we prove that (6) holds for $j = r+1$. Let $y_1, y_2 \in C_{t;\Psi}$ and denote $y_1^* = T^r y_1$ and $y_2^* = T^r y_2$. Then using the definition of operator T and Lipschitz condition on f, we get

$$\begin{aligned}
&\|T^{r+1} y_1 - T^{r+1} y_2\|_{C_{t;\Psi}} = \|Ty_1^* - Ty_2^*\|_{C_{t;\Psi}} \\
&\quad \leq L(1+(b-a)k_b) \max_{\omega \in [a,t]} \left| (\Psi(\omega)-\Psi(a))^{1-\rho} \frac{1}{\Gamma(\mu)} \int_a^\omega \mathcal{L}_\Psi^\mu(\omega,\eta) |y_1^*(\eta) - y_2^*(\eta)| d\eta \right| \\
&\quad \leq \frac{L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^{1-\rho}}{\Gamma(\mu)} \int_a^t \mathcal{L}_\Psi^\mu(t,\eta)\ (\Psi(\eta)-\Psi(a))^{\rho-1} \|y_1^* - y_2^*\|_{C_{t;\Psi}} d\eta.
\end{aligned}$$

From (7), we have

$$\begin{aligned}
\|y_1^* - y_2^*\|_{C_{t;\Psi}} &= \|T^r y_1 - T^r y_2\|_{C_{t;\Psi}} \\
&\leq \Gamma(\rho) \frac{(L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^\mu)^r}{\Gamma(r\mu+\rho)} \|y_1 - y_2\|_{C_{t;\Psi}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|T^{r+1} y_1 - T^{r+1} y_2\|_{C_{t;\Psi}} \\
&\quad \leq \frac{L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^{1-\rho}}{\Gamma(\mu)} \int_a^t \mathcal{L}_\Psi^\mu(t,\eta)\ (\Psi(\eta)-\Psi(a))^{\rho-1} \\
&\qquad \times \Gamma(\rho) \frac{(L(1+(b-a)k_b)(\Psi(\eta)-\Psi(a))^\mu)^r}{\Gamma(r\mu+\rho)} \|y_1 - y_2\|_{C_{t;\Psi}} d\eta \\
&\quad \leq \frac{(L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^\mu)^{r+1}\Gamma(\rho)}{\Gamma((r+1)\mu+\rho)} \|y_1 - y_2\|_{C_{t;\Psi}}.
\end{aligned}$$

Thus we have

$$\|T^{r+1}y_1 - T^{r+1}y_2\|_{C_{t;\Psi}} \le \frac{(L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^{\mu})^{r+1}\Gamma(\rho)}{\Gamma((r+1)\mu+\rho)}\|y_1 - y_2\|_{C_{t;\Psi}}.$$

Therefore, by principle of mathematical induction the inequality (6) holds for all $j \in N$ and for every t in Δ. As a consequence we find on the fundamental interval Δ,

$$\|T^{j}y_1 - T^{j}y_2\|_{C_{1-\rho;\Psi}(\Delta,R)} \le \Gamma(\rho)\frac{(L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^{\mu})^{j}}{\Gamma(j\mu+\rho)}\|y_1 - y_2\|_{C_{1-\rho;\Psi}(\Delta,R)}. \quad (8)$$

By definition of two parameter Mittag-Leffler function, we have

$$E_{\mu,\rho}(L(1+(b-a)k_b)(\Psi(b)-\Psi(a))^{\mu}) = \sum_{j=0}^{\infty}\frac{(L(1+(b-a)k_b)(\Psi(b)-\Psi(a))^{\mu})^{j}}{\Gamma(j\mu+\rho)}.$$

Note that $\frac{(L(1+(b-a)k_b)(\Psi(b)-\Psi(a))^{\mu})^{j}}{\Gamma(j\mu+\rho)}$ is the j^{th} term of the convergent series of real numbers. Therefore,

$$\lim_{j\to\infty}\frac{(L(1+(b-a)k_b)(\Psi(b)-\Psi(a))^{\mu})^{j}}{\Gamma(j\mu+\rho)} = 0.$$

Thus we can choose $j \in N$ such that

$$\Gamma(\rho)\frac{(L(1+(b-a)k_b)(\Psi(b)-\Psi(a))^{\mu})^{j}}{\Gamma(j\mu+\rho)} < 1,$$

so that T^j is a contraction. Therefore, by Lemma 4, T has a unique fixed point y^* in $C_{1-\rho;\Psi}(\Delta, R)$, which is a unique solution of the Cauchy-type problem (1)-(2). □

Remark 1. *The existence result proved above with no restriction on the interval* $\Delta = [a, b]$, *and hence solution* y^* *of (1)-(2) exists for any* $a, b(0 < a < b < \infty)$. *Thus the Theorem 1 guarantees global unique solution in* $C_{1-\rho;\Psi}(\Delta, R)$.

4 Ulam-Hyers stability

To discuss HU and HUR stability of (1), we adopt the approach of [13, 20]. For $\epsilon > 0$ and continuous function $\phi : \Delta \to [0, \infty)$, we consider the following inequalities :

$$\left|{}^{H}D_{a+}^{\mu,\nu;\Psi}y^*(t) - f\left(t, y^*(t), \int_a^t K(t,s)y^*(s)ds\right)\right| \le \epsilon, \qquad t \in \Delta, \quad (9)$$

$$\left|{}^{H}D_{a+}^{\mu,\nu;\Psi}y^*(t) - f\left(t, y^*(t), \int_a^t K(t,s)y^*(s)ds\right)\right| \le \phi(t), \quad t \in \Delta, \quad (10)$$

$$\left|{}^{H}D_{a+}^{\mu,\nu;\Psi}y^*(t) - f\left(t, y^*(t), \int_a^t K(t,s)y^*(s)ds\right)\right| \le \epsilon\phi(t), \quad t \in \Delta. \quad (11)$$

Definition 4. *The problem (1) has HU stability if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $y^* \in C_{1-\rho;\Psi}(\Delta, R)$ of the inequation (9) there exists a solution $y \in C_{1-\rho;\Psi}(\Delta, R)$ of (1) with $\|y^* - y\|_{C_{1-\rho;\Psi}(\Delta,R)} \le C_f \epsilon$.*

Definition 5. *The problem (1) has generalized HU stability if there exists a function $C_f \in ([0,\infty),[0,\infty))$ with $C_f(0) = 0$ such that for each solution $y^* \in C_{1-\rho;\Psi}(\Delta, R)$ of the inequation (9) there exists a solution $y \in C_{1-\rho;\Psi}(\Delta, R)$ of (1) with $\|y^* - y\|_{C_{1-\rho;\Psi}(\Delta,R)} \le C_f(\epsilon)$.*

Definition 6. *The problem (1) has HUR stability with respect to a function ϕ if there exists a real number $C_{f,\phi} > 0$ such that for each solution $y^* \in C_{1-\rho;\Psi}(\Delta, R)$ of the inequation (11) there exists a solution $y \in C_{1-\rho;\Psi}(\Delta, R)$ of (1) with*

$$|(\Psi(t) - \Psi(a))^{1-\rho}(y^*(t) - y(t))| \le C_{f,\phi}\epsilon\phi(t), \quad t \in (\Delta, R).$$

Definition 7. *The problem (1) has generalized HUR stability with respect to a function ϕ if there exists a real number $C_{f,\phi} > 0$ such that for each solution $y^* \in C_{1-\rho;\Psi}(\Delta, R)$ of the inequation (10) there exists a solution $y \in C_{1-\rho;\Psi}(\Delta, R)$ of (1) with*

$$|(\Psi(t) - \Psi(a))^{1-\rho}(y^*(t) - y(t))| \le C_{f,\phi}\phi(t), \quad t \in \Delta.$$

In the next theorem we will make use of the successive approximation method to prove that the Ψ−Hilfer FDE (1) is HU stable.

Theorem 2. *Let $f : (a, b] \times R \times R \to R$ be a function such that $f\left(t, y(t), \int_a^t K(t,s)y(s)ds\right) \in C_{1-\rho;\Psi}(\Delta, R)$ for any $y \in C_{1-\rho;\Psi}(\Delta, R)$, and that satisfies the Lipschitz condition*

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \le L(|y_1 - y_2| + |z_1 - z_2|),$$

where $t \in (a, b]$, $y_1, y_2, z_1, z_2 \in R$ and $L > 0$ is Lipschitz constant. For every $\epsilon > 0$, if $y^* \in C_{1-\rho;\Psi}(\Delta, R)$ satisfies

$$\left| {}^H D_{a+}^{\mu,\nu;\Psi} y^*(t) - f\left(t, y^*(t), \int_a^t K(t,s)y^*(s)ds\right)\right| \le \epsilon, \quad t \in \Delta,$$

then there exists a solution y of equation (1) in $C_{1-\rho;\Psi}(\Delta, R)$ with $I_{a^+}^{1-\rho;\Psi} y^*(a) = I_{a^+}^{1-\rho;\Psi} y(a)$ such that for $t \in \Delta$,

$$\|y^* - y\|_{C_{1-\rho;\Psi}(\Delta,R)} \le \left[\frac{(E_\mu(L(1 + (b-a)k_b)(\Psi(b) - \Psi(a))^\mu) - 1)}{L(1 + (b-a)k_b)}(\Psi(b) - \Psi(a))^{1-\rho}\right]\epsilon.$$

Proof. Fix any $\epsilon > 0$, let $y^* \in C_{1-\rho;\Psi}(\Delta, R)$ satisfies

$$\left| {}^H D_{a+}^{\mu,\nu;\Psi} y^*(t) - f\left(t, y^*(t), \int_a^t K(t,s)y^*(s)ds\right)\right| \le \epsilon, \quad t \in \Delta. \tag{12}$$

Then there exists a function $\sigma_{y^*} \in C_{1-\rho;\Psi}(\Delta, R)$ (depending on y^*) such that $|\sigma_{y^*}(t)| \leq \epsilon,\ t \in \Delta$ and

$$^H D^{\mu,\nu;\Psi}_{a+} y^*(t) = f\left(t, y^*(t), \int_a^t K(t,s)y^*(s)ds\right) + \sigma_{y^*}(t), \quad t \in \Delta. \tag{13}$$

If $y^*(t)$ satisfies (13) then it satisfies equivalent fractional integral equation

$$\begin{aligned} y^*(t) = \Omega^\rho_\Psi(t,a) I^{1-\rho}_{a+} y^*(a) + \frac{1}{\Gamma(\mu)} \int_a^t \mathcal{L}^\mu_\Psi(t,\eta) \\ \times f\left(\eta, y^*(\eta), \int_a^\eta K(\eta,\sigma)y^*(\sigma)d\sigma\right) d\eta + \frac{1}{\Gamma(\mu)} \int_a^t \mathcal{L}^\mu_\Psi(t,\eta)\sigma_{y^*}(\eta)d\eta. \end{aligned} \tag{14}$$

Define

$$y_0(t) = y^*(t), \quad t \in \Delta, \tag{15}$$

and consider the sequence $\{y_n\}_{n=1}^\infty \subseteq C_{1-\rho;\Psi}(\Delta, R)$ defined by

$$\begin{aligned} y_n(t) &= \Omega^\rho_\Psi(t,a) I^{1-\rho}_{a+} y^*(a) \\ &+ \frac{1}{\Gamma(\mu)} \int_a^t \mathcal{L}^\mu_\Psi(t,\eta) f\left(\eta, y_{n-1}(\eta), \int_a^\eta K(\eta,\sigma)y_{n-1}(\sigma)d\sigma\right) d\eta, \quad t \in \Delta. \end{aligned} \tag{16}$$

Using mathematical induction firstly we prove that for every $t \in \Delta$ and $y_j \in C_{1-\rho;\Psi}[a,t] = C_{t;\Psi}$

$$\begin{aligned} \|y_j - y_{j-1}\|_{C_{t;\Psi}} \leq & \frac{\epsilon}{L(1+(b-a)k_b)} \frac{(L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^\mu)^j}{\Gamma(j\mu+1)} \\ & \times\ (\Psi(t)-\Psi(a))^{1-\rho}, \quad j \in N. \end{aligned} \tag{17}$$

By definition of successive approximations and using (14) we have

$$\begin{aligned} \|y_1 - y_0\|_{C_{t;\Psi}} &= \max_{\omega \in [a,t]} \left| (\Psi(\omega)-\Psi(a))^{1-\rho} \frac{1}{\Gamma(\mu)} \int_a^\omega \mathcal{L}^\mu_\Psi(\omega,\eta)\sigma_{y_1}(\eta)d\eta \right| \\ &\leq \epsilon \max_{\omega \in [a,t]} \left[(\Psi(\omega)-\Psi(a))^{1-\rho} \frac{1}{\Gamma(\mu)} \int_a^\omega \mathcal{L}^\mu_\Psi(\omega,\eta)d\eta \right] \\ &\leq \frac{\epsilon}{L(1+(b-a)k_b)} \frac{(L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^\mu)}{\Gamma(\mu+1)} (\Psi(t)-\Psi(a))^{1-\rho}. \end{aligned}$$

Therefore,

$$\|y_1 - y_0\|_{C_{t;\Psi}} \leq \frac{\epsilon}{L(1+(b-a)k_b)} \frac{(L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^\mu)}{\Gamma(\mu+1)}\ (\Psi(t)-\Psi(a))^{1-\rho},$$

which proves the inequality (17) for $j = 1$. Let us suppose that the inequality (17) holds for $j = r \in N$, we prove it for $j = r+1$. By definition of successive approximations and Lipschitz condition on f, we obtain

$$\begin{aligned}
&\|y_{r+1}-y_r\|_{C_{t;\Psi}}\\
&\quad=\max_{\omega\in[0,t]}\left|(\Psi(\omega)-\Psi(a))^{1-\rho}\{y_{r+1}(\omega)-y_r(\omega)\}\right|\\
&\quad\le L(1+(b-a)k_b)\max_{\omega\in[a,t]}\left[(\Psi(\omega)-\Psi(a))^{1-\rho}\frac{1}{\Gamma(\mu)}\int_a^{\omega}\mathcal{L}_{\Psi}^{\mu}(\omega,\eta)|y_r(\eta)-y_{r-1}(\eta)|d\eta\right]\\
&\quad\le\frac{L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^{1-\rho}}{\Gamma(\mu)}\int_a^t\mathcal{L}_{\Psi}^{\mu}(t,\eta)(\Psi(\eta)-\Psi(a))^{\rho-1}\|y_r-y_{r-1}\|_{C_{\eta;\Psi}}d\eta
\end{aligned}$$

Using the inequality (17) for $j=r$, we have

$$\begin{aligned}
\|y_{r+1}-y_r\|_{C_{t;\Psi}}&\le\frac{\epsilon}{L(1+(b-a)k_b)}\,\frac{(L(1+(b-a)k_b))^{r+1}}{\Gamma(r\mu+1)}(\Psi(t)-\Psi(a))^{1-\rho}I_{a^+}^{\mu;\Psi}(\Psi(t)-\Psi(a))^{r\mu}\\
&\le\frac{\epsilon}{L(1+(b-a)k_b)}\,\frac{(L(1+(b-a)k_b))^{r+1}}{\Gamma(r\mu+1)}\,(\Psi(t)-\Psi(a))^{1-\rho}\\
&\quad\times\frac{\Gamma(r\mu+1)}{\Gamma((r+1)\mu+1)}(\Psi(t)-\Psi(a))^{(r+1)\mu}.
\end{aligned}$$

Therefore,

$$\|y_{r+1}-y_r\|_{C_{t;\Psi}}\le\frac{\epsilon}{L(1+(b-a)k_b)}\,\frac{(L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^{\mu})^{r+1}}{\Gamma((r+1)\mu+1)}(\Psi(t)-\Psi(a))^{1-\rho},$$

which is the inequality (17) for $j=r+1$. Using the principle of mathematical induction the inequality (17) holds for every $j\in N$ and every $t\in\Delta$. Therefore,

$$\|y_j-y_{j-1}\|_{C_{1-\rho;\Psi}(\Delta,R)}\le\frac{\epsilon}{L(1+(b-a)k_b)}\,\frac{(L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^{\mu})^{j}}{\Gamma(j\mu+1)}(\Psi(t)-\Psi(a))^{1-\rho}.$$

Now using this estimation we have

$$\begin{aligned}
\sum_{j=1}^{\infty}\|y_j-y_{j-1}\|_{C_{1-\rho;\Psi}(\Delta,R)}&\le\frac{\epsilon}{L(1+(b-a)k_b)}\,(\Psi(t)-\Psi(a))^{1-\rho}\\
&\quad\times\sum_{j=1}^{\infty}\frac{(L(1+(b-a)k_b)(\Psi(t)-\Psi(a))^{\mu})^{j}}{\Gamma(j\mu+1)}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\sum_{j=1}^{\infty}\|y_j-y_{j-1}\|_{C_{1-\rho;\Psi}(\Delta,R)}&\le\frac{\epsilon}{L(1+(b-a)k_b)}\,(\Psi(t)-\Psi(a))^{1-\rho}\\
&\quad\times(E_{\mu}(L(1+(b-a)k_b)(\Psi(b)-\Psi(a))^{\mu})-1).
\end{aligned}\tag{18}$$

Hence the series

$$y_0+\sum_{j=1}^{\infty}(y_j-y_{j-1})\tag{19}$$

converges in the weighted space $C_{1-\rho;\Psi}(\Delta, R)$. Let $y \in C_{1-\rho;\Psi}(\Delta, R)$ such that

$$y = y_0 + \sum_{j=1}^{\infty}(y_j - y_{j-1}), \tag{20}$$

Noting that $y_n = y_0 + \sum_{j=1}^{n}(y_j - y_{j-1})$, is the n^{th} partial sum of the series (19), we have $\|y_n - y\|_{C_{1-\rho;\Psi}(\Delta,R)} \to 0$ as $n \to \infty$.

Next, we prove that this limit function y is the solution of fractional integral equation with $I_{a+}^{1-\rho;\Psi}y^*(a) = I_{a+}^{1-\rho;\Psi}y(a)$. Therefore, by the definition of successive approximation, for any $t \in \Delta$, we have

$$\begin{aligned}
&\Big|(\Psi(t) - \Psi(a))^{1-\rho}\Big(y(t) - \Omega_{\Psi}^{\rho}(t,a)I_{a^+}^{1-\rho}y(a) \\
&-\frac{1}{\Gamma(\mu)}\int_a^t \mathcal{L}_{\Psi}^{\mu}(t,\eta)f\Big(\eta, y(\eta), \int_a^{\eta} K(\eta,\sigma)y(\sigma)d\sigma\Big)d\eta\Big)\Big| \\
&\le \|y - y_n\|_{C_{1-\rho;\Psi}[a,b]} + L(1 + (b-a)k_b) \\
&\quad \times \Big[(\Psi(t) - \Psi(a))^{1-\rho}\frac{1}{\Gamma(\mu)}\int_a^t \mathcal{L}_{\Psi}^{\mu}(t,\eta)|y_{n-1}(\eta) - y(\eta)|d\eta\Big] \\
&\le \|y - y_n\|_{C_{1-\rho;\Psi}[a,b]} + \Big(\frac{L(1 + (b-a)k_b)\Gamma(\rho)}{\Gamma(\mu+\rho)}\,(\Psi(t) - \Psi(a))^{\mu}\Big) \\
&\quad \times \|y_{n-1} - y\|_{C_{1-\rho;\Psi}[a,b]}, \quad \forall n \in N.
\end{aligned}$$

By taking limit as $n \to \infty$ in the above inequality, for all $t \in [a, b]$, we obtain

$$\begin{aligned}
&\Big|(\Psi(t) - \Psi(a))^{1-\rho}\Big(y(t) - \Omega_{\Psi}^{\rho}(t,a)I_{a^+}^{1-\rho}y(a) \\
&\quad -\frac{1}{\Gamma(\mu)}\int_a^t \mathcal{L}_{\Psi}^{\mu}(t,\eta)f\Big(\eta, y(\eta), \int_a^{\eta} K(\eta,\sigma)y(\sigma)d\sigma\Big)d\eta\Big)\Big| = 0.
\end{aligned}$$

Since, $(\Psi(t) - \Psi(a))^{1-\rho} \neq 0$ for all $t \in \Delta$, we have

$$y(t) = \Omega_{\Psi}^{\rho}(t,a)I_{a^+}^{1-\rho}y(a) + \frac{1}{\Gamma(\mu)}\int_a^t \mathcal{L}_{\Psi}^{\mu}(t,\eta)f\Big(\eta, y(\eta), \int_a^{\eta} K(\eta,\sigma)y(\sigma)d\sigma\Big)d\eta. \tag{21}$$

This proves that y is the solution of (1)-(2) in $C_{1-\rho;\Psi}(\Delta, R)$. Further, for the solution y^* of inequation (12) and the solution y of the equation (1), using (15) and (20), for any $t \in \Delta$, we have

$$\begin{aligned}
&\left|(\Psi(t) - \Psi(a))^{1-\rho}(y^*(t) - y(t))\right| \\
&\le \sum_{j=1}^{\infty}\left|(\Psi(t) - \Psi(a))^{1-\rho}\,(y_j(t) - y_{j-1}(t))\right| \le \sum_{j=1}^{\infty}\|y_j - y_{j-1}\|_{C_{1-\rho}[a,b]} \\
&\le \frac{\epsilon}{L(1 + (b-a)k_b)}\,(\Psi(b) - \Psi(a))^{1-\rho}(E_{\mu}(L(1 + (b-a)k_b)(\Psi(b) - \Psi(a))^{\mu}) - 1).
\end{aligned}$$

Therefore,

$$\|y^* - y\|_{C_{1-\rho;\Psi}[a,b]} \le \left(\frac{(E_\mu(L(1+(b-a)k_b)(\Psi(b)-\Psi(a))^\mu)-1)}{L(1+(b-a)k_b)} (\Psi(b)-\Psi(a))^{1-\rho} \right) \epsilon.$$

This proves the equation (1) is HU stable. □

Corollary 3. *Suppose that the function f satisfies the assumptions of Theorem 2. Then the problem (1) is generalized HU stable.*

Proof. Set

$$\Psi_f(\epsilon) = \left(\frac{(E_\mu(L(1+(b-a)k_b)(\Psi(b)-\Psi(a))^\mu)-1)}{L(1+(b-a)k_b)} (\Psi(b)-\Psi(a))^{1-\rho} \right) \epsilon,$$

in the proof of Theorem 2. Then $\Psi_f(0) = 0$ and for each $y^* \in C_{1-\rho;\Psi}(\Delta, R)$ that satisfies the inequality

$$\left| {}^H D_{a+}^{\mu,\nu;\Psi} y^*(t) - f\left(t, y^*(t), \int_a^t K(t,s) y^*(s) ds\right) \right| \le \epsilon, \quad t \in \Delta,$$

there exists a solution y of equation (1) in $C_{1-\rho;\Psi}(\Delta, R)$ with $I_{a^+}^{1-\rho;\Psi} y^*(a) = I_{a^+}^{1-\rho;\Psi} y(a)$ such that

$$\|y^* - y\|_{C_{1-\rho;\Psi}[a,b]} \le \Psi_f(\epsilon), \quad t \in \Delta.$$

Hence fractional integrodifferential equation (1) is generalized HU stable. □

Theorem 4. *Let $f : (a,b] \times R \times R \to R$ be a function such that*

$$f\left(t, y(t), \int_a^t K(t,s) y(s) ds\right) \in C_{1-\rho;\Psi}(\Delta, R),$$

for any $y \in C_{1-\rho;\Psi}(\Delta, R)$, and that satisfies the Lipschitz condition

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \le L(|y_1 - y_2| + |z_1 - z_2|),$$

where $t \in (a,b]$, $y_1, y_2, z_1, z_2 \in R$ and $L > 0$ is Lipschitz constant. For every $\epsilon > 0$, if $y^* \in C_{1-\rho;\Psi}(\Delta, R)$ satisfies

$$\left| {}^H D_{a^+}^{\mu,\nu;\Psi} y^*(t) - f\left(t, y^*(t), \int_a^t K(t,s) y^*(s) ds\right) \right| \le \epsilon\phi(t), \quad t \in \Delta,$$

where $\phi \in C(\Delta, R_+)$ is a non-decreasing function such that

$$|I_{a^+}^{\mu;\Psi} \phi(t)| \le \lambda \phi(t), \quad t \in \Delta$$

and $\lambda > 0$ is a constant satisfying $0 < \lambda L(1+(b-a)k_b) < 1$. Then, there exists a solution $y \in C_{1-\rho;\Psi}(\Delta, R)$ of equation (1) with $I_{a^+}^{1-\rho;\Psi} y^*(a) = I_{a^+}^{1-\rho;\Psi} y(a)$ such that

$$\left|(\Psi(t)-\Psi(a))^{1-\rho}(y^*(t) - y(t))\right| \le \left[\frac{\lambda}{1 - \lambda L(1+(b-a)k_b)} (\Psi(b)-\Psi(a))^{1-\rho} \right] \epsilon\phi(t), \; t \in \Delta.$$

Proof. For every $\epsilon > 0$, let $y^* \in C_{1-\rho;\Psi}(\Delta, R)$ satisfies

$$\left|{}^H D_{a^+}^{\mu,\nu;\Psi} y^*(t) - f\left(t, y^*(t), \int_a^t K(t,s) y^*(s) ds\right)\right| \le \epsilon\phi(t), \quad t \in \Delta,$$

Proceeding as in the proof of Theorem 2 there exists a function $\sigma y^* \in C_{1-\rho;\Psi}(\Delta, R)$ (depending on y^*) such that

$$y^*(t) = \Omega_\Psi^\rho(t,a) I_{a^+}^{1-\rho;\Psi} y^*(a) + I_{a^+}^{\mu;\Psi} f\left(t, y^*(t), \int_a^t K(t,\sigma) y^*(\sigma) d\sigma\right) + I_{a^+}^{\mu;\Psi} \sigma_{y^*}(t), \quad t \in \Delta,$$

Further, using mathematical induction, one can prove that the sequence of successive approximations $\{y_n\}_{n=1}^\infty \subset C_{1-\rho;\Psi}(\Delta, R)$ defined by

$$y_n(t) = \Omega_\Psi^\rho(t,a) I_{a^+}^{1-\rho} y^*(a) \frac{1}{\Gamma(\mu)} \int_a^t \mathcal{L}_\Psi^\mu(t,\eta) f\left(\eta, y_{n-1}(\eta), \int_a^\eta K(\eta,\sigma) y_{n-1}(\sigma) d\sigma\right) d\eta. \quad (22)$$

satisfy the inequality

$$\|y_j - y_{j-1}\|_{C_{t;\Psi}} \le \frac{\epsilon}{L(1+(b-a)k_b)} (\lambda L(1+(b-a)k_b))^j \times (\Psi(t) - \Psi(a))^{1-\rho} \phi(t), \quad j \in N. \quad (23)$$

Using the inequation (23), we obtain

$$\sum_{j=1}^\infty \|y_j - y_{j-1}\|_{C_{t;\Psi}} \le \frac{\epsilon}{L(1+(b-a)k_b)} \left(\sum_{j=1}^\infty (\lambda L(1+(b-a)k_b))^j\right) (\Psi(t) - \Psi(a))^{1-\rho} \phi(t).$$

Thus

$$\sum_{j=1}^\infty \|y_j - y_{j-1}\|_{C_{t;\Psi}} \le \epsilon \left(\frac{\lambda}{1 - \lambda L(1+(b-a)k_b)}\right) (\Psi(t) - \Psi(a))^{1-\rho} \phi(t). \quad (24)$$

Following the steps as in the proof of the Theorem 2 there exists $y \in C_{1-\rho;\Psi}(\Delta, R)$ such that $\|y_n - y\|_{C_{1-\rho;\Psi}(\Delta,R)} \to 0$ as $n \to \infty$. This y is the solution of the problem (1)-(2) with $I_{a^+}^{1-\rho;\Psi} y(a) = I_{a^+}^{1-\rho;\Psi} y^*(a)$, and we have $y = y_0 + \sum_{j=1}^\infty (y_j - y_{j-1})$. Further, for the solution y^* and the solution y of the equation (1), for any $t \in \Delta$,

$$\begin{aligned}
\left|(\Psi(t) - \Psi(a))^{1-\rho}(y^*(t) - y(t))\right| &\le \sum_{j=1}^\infty \left|(\Psi(t) - \Psi(a))^{1-\rho} \ (y_j(t) - y_{j-1}(t))\right| \\
&\le \sum_{j=1}^\infty \|y_j - y_{j-1}\|_{C_{t,\Psi}} \\
&\le \epsilon \left(\frac{\lambda}{1 - \lambda L(1+(b-a)k_b)}\right) (\Psi(t) - \Psi(a))^{1-\rho} \phi(t).
\end{aligned}$$

Thus, we have

$$\left|(\Psi(t) - \Psi(a))^{1-\rho}(y^*(t) - y(t))\right| \le \left(\frac{\lambda}{1 - \lambda L(1+(b-a)k_b)} (\Psi(b) - \Psi(a))^{1-\rho}\right) \epsilon \ \phi(t).$$

This proves the equation (1) is HUR stable. □

Corollary 5. *Suppose that the function f satisfies the assumptions of Theorem 4. Then, the problem (1) is generalized HUR stable.*

Proof. Set $\epsilon = 1$ and

$$C_{f,\phi} = \left(\frac{\lambda}{1 - \lambda L(1 + (b-a)k_b)} \left(\Psi(b) - \Psi(a) \right)^{1-\rho} \right),$$

in the proof of Theorem 4. Then for each solution $y^* \in C_{1-\rho;\Psi}(\Delta, R)$ that satisfies the inequality

$$\left| {}^H D_{a^+}^{\mu,\nu;\Psi} y^*(t) - f\left(t, y^*(t), \int_a^t K(t,s) y^*(s) ds \right) \right| \le \phi(t), \quad t \in \Delta,$$

there exists a solution y of equation (1) in $C_{1-\rho;\Psi}(\Delta, R)$ with $I_{a^+}^{1-\rho;\Psi} y^*(a) = I_{a^+}^{1-\rho;\Psi} y(a)$ such that

$$\left| (\Psi(t) - \Psi(a))^{1-\rho} (y^*(t) - y(t)) \right| \le C_{f,\phi}\phi(t), \quad t \in \Delta.$$

Hence the fractional integrodifferential equation (1) is generalized HUR stable. □

5 $\epsilon-$Approximate solutions to Hilfer FIDE

Definition 8. *A function $y^* \in C_{1-\rho;\Psi}(\Delta, R)$ that satisfy the fractional integrodifferential inequality*

$$\left| {}^H D_{a^+}^{\mu,\nu;\Psi} y^*(t) - f\left(t, y^*(t), \int_a^t K(t,s) y^*(s) ds \right) \right| \le \epsilon, \quad t \in \Delta,$$

is called an $\epsilon-$approximate solutions of $\Psi-$Hilfer FIDE (1).

Theorem 6. *([17]) Let u, v be two integrable, non negative functions and g be a continuous, nonnegative, nondecreasing function with domain Δ. If*

$$u(t) \le v(t) + g(t) \int_a^t \mathcal{L}_\Psi^\mu(\tau, s) u(\tau) d\tau,$$

then

$$u(t) \le v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\mu)]^k}{\Gamma(\mu k)} \mathcal{L}_\Psi^{\mu k}(\tau, s) v(\tau) d\tau, \quad \forall t \in \Delta. \tag{25}$$

Theorem 7. *Let $f : (a, b] \times R \times R \to R$ be a function which satisfies the Lipschitz condition $|f(t, y_1, z_1) - f(t, y_2, z_2)| \le L(|y_1 - y_2| + |z_1 - z_2|)$, where $t \in (a, b],\ y_1, y_2, z_1, z_2 \in R$ and $L > 0$ is Lipschitz constant.*

Let $y_i^* \in C_{1-\rho;\Psi}(\Delta, R)$, $(i = 1, 2)$ be an ϵ_i-approximate solutions of FIDE (1) corresponding to $I_{a^+}^{1-\rho;\Psi} y_i^*(a) = y_a^{(i)} \in R$, respectively. Then

$$\begin{aligned}
&\|y_1^* - y_2^*\|_{C_{1-\rho;\Psi}(\Delta,R)} \\
&\quad \le (\epsilon_1 + \epsilon_2)\left(\frac{(\Psi(b) - \Psi(a))^{\mu-\rho+1}}{\Gamma(\mu+1)} + \sum_{k=1}^{\infty} \frac{(L(1 + (b-a)k_b))^k}{\Gamma((k+1)\mu - \rho + 1)} (\Psi(b) - \Psi(a))^{(k+1)\mu}\right) \\
&\quad + |y_a^{(1)} - y_a^{(2)}| \left(\frac{1}{\Gamma(\rho)} + \sum_{k=1}^{\infty} \frac{(L(1 + (b-a)k_b))^k}{\Gamma(\rho + k\mu)} (\Psi(b) - \Psi(a))^{k\mu}\right).
\end{aligned} \tag{26}$$

Proof. Let $y_i^* \in C_{1-\rho;\Psi}(\Delta, R)$, $(i = 1, 2)$ be an ϵ_i-approximate solutions of FIDE (1) that satisfy the initial condition $I_{a^+}^{1-\rho;\Psi} y_i^*(a) = y_a^{(i)} \in R$. Then

$$\left|{}^H D_{a^+}^{\mu,\nu;\Psi} y_i^*(t) - f\left(t, y_i^*(t), \int_a^t K(t,s) y_i^*(s) ds\right)\right| \le \epsilon_i, \quad t \in \Delta. \tag{27}$$

Operating $I_{a^+}^{\mu;\Psi}$ on both the sides of the above inequation and using the Lemma3, we get

$$\begin{aligned}
I_{a^+}^{\mu;\Psi} \epsilon_i &\ge I_{a^+}^{\mu;\Psi} \left|{}^H D_{a^+}^{\mu,\nu;\Psi} y_i^*(t) - f\left(t, y_i^*(t), \int_a^t K(t,s) y_i^*(s) ds\right)\right| \\
&\ge \left|I_{a^+}^{\mu;\Psi\, H} D_{a^+}^{\mu,\nu;\Psi} y_i^*(t) - I_{a^+}^{\mu;\Psi} f\left(t, y_i^*(t), \int_a^t K(t,s) y_i^*(s) ds\right)\right| \\
&\ge \left|y_i^*(t) - I_{a^+}^{1-\rho;\Psi} y_i^*(a) \Omega_\Psi^\rho(t,a) - I_{a^+}^{\mu;\Psi} f\left(t, y_i^*(t), \int_a^t K(t,s) y_i^*(s) ds\right)\right|.
\end{aligned}$$

Therefore,

$$\frac{\epsilon_i}{\Gamma(\mu+1)} (\Psi(t) - \Psi(a))^\mu \ge \left|y_i^*(t) - y_a^{(i)} \Omega_\Psi^\rho(t,a) - I_{a^+}^{\mu;\Psi} f\left(t, y_i^*(t), \int_a^t K(t,s) y_i^*(s) ds\right)\right|, \tag{28}$$

for $i = 1, 2$. Using the following inequalities

$$|x - y| \le |x| + |y| \text{ and } |x| - |y| \le |x - y|, \; x, y \in \mathbb{R},$$

from the equation (28), for any $t \in \Delta$, we have

$$\begin{aligned}
&\frac{\epsilon_1 + \epsilon_2}{\Gamma(\mu+1)} (\Psi(t) - \Psi(a))^\mu \\
&\quad \ge \left|\left(y_1^*(t) - y_a^{(1)} \Omega_\Psi^\rho(t,a) - I_{a^+}^{\mu;\Psi} f\left(t, y_1^*(t), \int_a^t K(t,s) y_1^*(s) ds\right)\right)\right. \\
&\qquad \left. - \left(y_2^*(t) - y_a^{(2)} \Omega_\Psi^\rho(t,a) - I_{a^+}^{\mu;\Psi} f\left(t, y_2^*(t), \int_a^t K(t,s) y_2^*(s) ds\right)\right)\right| \\
&\quad \ge |(y_1^*(t) - y_2^*(t))| - \left|(y_a^{(1)} - y_a^{(2)}) \Omega_\Psi^\rho(t,a)\right| \\
&\qquad - \left|I_{a^+}^{\mu}\left[f\left(t, y_1^*(t), \int_a^t K(t,s) y_1^*(s) ds\right) - f\left(t, y_2^*(t), \int_a^t K(t,s) y_2^*(s) ds\right)\right]\right|.
\end{aligned}$$

Therefore,

$$\left|(y_1^*(t) - y_2^*(t))\right| \leq \frac{\epsilon_1 + \epsilon_2}{\Gamma(\mu+1)} \, (\Psi(t) - \Psi(a))^{\mu} + \left|(y_a^{(1)} - y_a^{(2)})\Omega_{\Psi}^{\rho}(t,a)\right| + \frac{L(1+(b-a)k_b)}{\Gamma(\mu)} \int_0^t \mathcal{L}_{\Psi}^{\mu}(t,\eta) \left|(y_1^*(\eta) - y_2^*(\eta))\right| d\eta.$$

Applying Theorem 6 with

$$\begin{aligned} u(t) &= \left|(y_1^*(t) - y_2^*(t))\right|, \\ v(t) &= \frac{\epsilon_1 + \epsilon_2}{\Gamma(\mu+1)} \, (\Psi(t) - \Psi(a))^{\mu} + \left|(y_a^{(1)} - y_a^{(2)})\Omega_{\Psi}^{\rho}(t,a)\right|, \\ g(t) &= \frac{L(1+(b-a)k_b)}{\Gamma(\mu)}, \end{aligned}$$

we obtain

$$\begin{aligned} &\left|(y_1^*(t) - y_2^*(t))\right| \\ &\quad \leq (\epsilon_1 + \epsilon_2) \left(\frac{(\Psi(t) - \Psi(a))^{\mu}}{\Gamma(\mu+1)} + \sum_{k=1}^{\infty} \frac{(L(1+(b-a)k_b))^k}{\Gamma((k+1)\mu - \rho + 1)} (\Psi(t) - \Psi(a))^{(k+1)\mu} \right) \\ &\qquad + \left|y_a^{(1)} - y_a^{(2)}\right| \left(\frac{(\Psi(t) - \Psi(a))^{\rho-1}}{\Gamma(\rho)} + \sum_{k=1}^{\infty} \frac{(L(1+(b-a)k_b))^k}{\Gamma(\rho + k\mu)} (\Psi(t) - \Psi(a))^{k\mu+\rho-1} \right) \end{aligned}$$

Thus for every $t \in \Delta$, we have

$$\begin{aligned} &(\Psi(t) - \Psi(a))^{1-\rho} \left|(y_1^*(t) - y_2^*(t))\right| \\ &\quad \leq (\epsilon_1 + \epsilon_2) \left(\frac{(\Psi(t) - \Psi(a))^{\mu-\rho+1}}{\Gamma(\mu+1)} + \sum_{k=1}^{\infty} \frac{(L(1+(b-a)k_b))^k}{\Gamma((k+1)\mu - \rho + 1)} (\Psi(t) - \Psi(a))^{(k+1)\mu} \right) \\ &\qquad + \left|y_a^{(1)} - y_a^{(2)}\right| \left(\frac{1}{\Gamma(\rho)} + \sum_{k=1}^{\infty} \frac{(L(1+(b-a)k_b))^k}{\Gamma(\rho + k\mu)} (\Psi(t) - \Psi(a))^{k\mu} \right) \\ &\quad \leq (\epsilon_1 + \epsilon_2) \left(\frac{(\Psi(b) - \Psi(a))^{\mu-\rho+1}}{\Gamma(\mu+1)} + \sum_{k=1}^{\infty} \frac{(L(1+(b-a)k_b))^k}{\Gamma((k+1)\mu - \rho + 1)} (\Psi(b) - \Psi(a))^{(k+1)\mu} \right) \\ &\qquad + \left|y_a^{(1)} - y_a^{(2)}\right| \left(\frac{1}{\Gamma(\rho)} + \sum_{k=1}^{\infty} \frac{(L(1+(b-a)k_b))^k}{\Gamma(\rho + k\mu)} (\Psi(b) - \Psi(a))^{k\mu} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} &\|y_1^* - y_2^*\|_{C_{1-\rho;\Psi}(\Delta,R)} \\ &\quad \leq (\epsilon_1 + \epsilon_2) \left(\frac{(\Psi(b) - \Psi(a))^{\mu-\rho+1}}{\Gamma(\mu+1)} + \sum_{k=1}^{\infty} \frac{(L(1+(b-a)k_b))^k}{\Gamma((k+1)\mu - \rho + 1)} (\Psi(b) - \Psi(a))^{(k+1)\mu} \right) \\ &\qquad + \left|y_a^{(1)} - y_a^{(2)}\right| \left(\frac{1}{\Gamma(\rho)} + \sum_{k=1}^{\infty} \frac{(L(1+(b-a)k_b))^k}{\Gamma(\rho + k\mu)} (\Psi(b) - \Psi(a))^{k\mu} \right) \end{aligned}$$

which is the desired inequality. □

Remark 2. *If $\epsilon_1 = \epsilon_2 = 0$ in the inequality (27) then y_1^* and y_2^* are the solutions of Cauchy problem (1)-(2) in the space $C_{1-\rho;\Psi}[a,b]$. Further, for $\epsilon_1 = \epsilon_2 = 0$ the inequality takes the form*

$$\|y_1^* - y_2^*\|_{C_{1-\rho;\Psi}(\Delta,R)} \leq \left| y_a^{(1)} - y_a^{(2)} \right| \left(\frac{1}{\Gamma(\rho)} + \sum_{k=1}^{\infty} \frac{(L(1+(b-a)k_b))^k}{\Gamma(\rho+k\mu)} (\Psi(b) - \Psi(a))^{k\mu} \right),$$

which provides the information regarding continuous dependance of the solution of the problem (1)-(2) on initial condition. In addition, if $y_a^{(1)} = y_a^{(2)}$ we have $\|y_1^* - y_2^*\|_{C_{1-\rho;\Psi}(\Delta,R)} = 0$, which gives the uniqueness of solution of the problem (1)-(2).

6 Conclusion

In this paper, we have presented the existence-criteria for solutions of $\Psi-$Hilfer FIDE on initial condition. HU stability is obtaining via successive approximation method. Also, continuous dependence and uniqueness is studied through $\epsilon-$approximated solution. Our results are not only new in the given setting, but also yield some special cases. For example: if $\int_a^t K(t,s)y(s)ds = 0$, then problem (1)-(2) reduces to the problem (1.1)-(1.2) in [11]. We can obtain the results for different fractional integrodifferential equations by fixing different values of the parameters like for $\nu \to 1$ problem (1) reduces to $\Psi-$Caputo FIDE, for $\nu \to 0$ problem (1) reduces to $\Psi-$Riemann-Liouville FIDE, for $\Psi(x) = x^\rho$ and $\nu \to 0$ problem (1) reduces to Katugampola FIDE, for $\Psi(x) = x$; $\nu \to 1$ problem (1) reduces to Caputo FIDE, also for $\Psi(x) = x$; $\nu \to 0$ problem (1) reduces to Riemann-Liouville FIDE. We belive that the reported results will have a positive impact on the development of further applications in engineering and applied sciences.

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