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Solving the Basset equation via Chebyshev collocation and LDG methods

Mohammad Izadi[†], **Mehdi Afshar**[‡*]

[†] *Department of Applied Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran*

[‡] *Department of Mathematics and Statistics, Zanjan Branch , Islamic Azad University, Zanjan, Iran.*

Email(s): izadi@uk.ac.ir, mafshar@iauz.ac.ir

Abstract. Two different numerical methods are developed to find approximate solutions of a class of linear fractional differential equations (LFDEs) appearing in the study of the generalized Basset force, when a sphere sinks in a viscous fluid. In the first one, using the Chebyshev bases, the collocation points, and the matrix operations, the given LFDE reduces to a matrix equation while in the second one, we employ the local discontinuous Galerkin (LDG) method, which uses the natural upwind flux yielding a stable discretization. Unlike the first method, in the latter method we are able to solve the problem element by element locally and there is no need to solve a full global matrix. The efficiency of the proposed algorithms are shown via some numerical examples.

Keywords: Basset equation, Caputo fractional derivative, Chebyshev polynomials, collocation method, local discontinuous Galerkin method, numerical stability.

AMS Subject Classification 2010: 34A08, 26A33, 41A10, 65M70, 65L60, 65L07.

1 Introduction

The main objective of this work is to develop a collocation algorithm based on Chebyshev polynomials as well as a discontinuous finite element technique to find an approximate solution of the following fractional differential equation

$$a_2\,\mathcal{D}X(t) + a_1\,\mathcal{D}_*^{(\beta)}X(t) + a_0\,X(t) = f(t), \quad 0 \leqslant t \leqslant T, \tag{1}$$

subject to the initial condition

$$X(0) = X_0, \tag{2}$$

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where $a_1 \neq 0, a_0, a_2$, and X_0 are arbitrary real constants and $f(t)$ is a given function denoting the forcing function. Here, $\mathcal{D}$ denotes the usual integer differential operator $\frac{d}{dt}$, and $\mathcal{D}_*^{(\beta)}$ is the standard Caputo fractional derivative operator of order $\beta \notin \mathbb{N}$ and defined [27]

$$\mathcal{D}_*^{(\beta)} X(t) = \mathcal{J}^{m-\beta}\mathcal{D}^m X(t), \quad m-1 < \beta < m,\ m \in \mathbb{N}, \tag{3}$$

where $\mathcal{J}^\beta$ is called the Riemann-Liouville fractional integral operator of order $\beta > 0$ and is defined as

$$\mathcal{J}^\beta X(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{X(s)}{(t-s)^{1-\beta}} ds, \quad t > 0,$$

and $\Gamma(\cdot)$ is the well-known Gamma function. The following properties of the operator $\mathcal{D}_*^{(\beta)}$ will be used

$$\mathcal{D}_*^{(\beta)}(C) = 0 \quad (C \text{ is a constant}), \tag{4}$$

$$\mathcal{D}_*^{(\beta)} x^\mu = \begin{cases} \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+1-\beta)} x^{\mu-\beta}, & \text{for } \mu \in \mathbb{N}_0 \text{ and } \mu \geqslant \lceil \beta \rceil, \text{ or } \mu \notin \mathbb{N}_0 \text{ and } \mu > \lfloor \beta \rfloor, \\ 0, & \text{for } \mu \in \mathbb{N}_0 \text{ and } \mu < \lceil \beta \rceil. \end{cases} \tag{5}$$

Setting

$$a_2, a_0 = 1, \quad a_1 = \left(\frac{9}{1+2\lambda}\right)^\beta, \quad \lambda \geqslant 0, \quad \beta = 1/2,$$

we recover the Basset equation, which describes the unsteady motion of a sphere immersed in a Stokes fluid [21]. Indeed, the author in [21] modeled the Basset force as fractional differential equation and solved it with some values of β, λ and also compared his solution with asymptotic behavior of the Basset equation. Discussion about uniqueness of an inverse Basset equation is considered in [31] while in [9] the authors derive the stability criteria of the Basset equation by using the duality results of controllability and observability of the linear fractional dynamical systems and the feedback control. Investigation of the behavior of the fractional Basset equations via numerical inverse Laplace transform is reported in [23]. In [6, 7], numerical approximations of the solution of Basset equation is calculated by reduction of the problem to a system of ordinary and fractional differential equations each of order at most unity. The extended Laguerre functions are exploited in [20] to solve the Basset equation.

The subject of fractional calculus and fractional differential equations (FDEs) is quite as old as the classical calculus. However, they have recently proved to be powerful and valuable tools in the modeling of many phenomena in various fields of science and engineering [19, 24, 27]. To model many real world problems, it has turned out the use of fractional-order derivatives are more adequate rather than integer-order ones. That is due to the fact that the fractional derivatives and integrals enable the description of the memory properties of various materials and processes [27]. Therefore, one needs to extend the concept of ordinary differentiation as well as integration to an arbitrary non-integer order. However, most of the resulting FDEs do not have an exact analytical solution, so the approximate and numerical techniques are preferred in identifying the solutions behaviour of such fractional equations. Numerous analytical and numerical methods have been developed for the solution of FDEs. Among other existing methods, we mention some schemes such as the spectral collocation polynomials based methods [10,11], [12–14,17], [28–30], the psuedospectral method [4], B-spline wavelet method [18], the numerical inverse transform method [23], and local discontinuous Galerkin methods [1, 14, 16], to name but a few.

Over the past decades, considerable attention has been given to the establishment of techniques for the solution of the fractional differential equations using the orthogonal functions. The main characteristic of this technique is that it reduces the solution of differential equations to the solution of a system of algebraic equations. In most of the presented works, the use of numerical techniques in conjunction with operational matrices for differentiation and integration operators of some orthogonal polynomials, for the solution of fractional differential equations on finite and infinite intervals, produced highly accurate solutions for such equations, see [2] for a recent review. On the other hand, the LDG methods for the fractional ODEs including one-term and multi-terms were first discussed in [5]. The main idea of the LDG scheme is to rewrite a given fractional-order differential equation as a system of first-order classical ODEs and a fractional integral, then apply the discontinuous Galerkin (DG) method on the system and the fractional integral. A key ingredient for the success of LDG schemes is the correct design of interface numerical fluxes.

In this work, we are going to propose two different approximation algorithms as the extensions of the above mentioned papers. Our first approach is based on the generalized fractional order of the Chebyshev orthogonal functions of the first kind to get an approximation solution of (1) accurately on the interval $[0, T]$. The main idea of the proposed technique based on using these (orthogonal) functions along with collocation points is that it converts the differential or integral operator involved in (1)-(2) to an algebraic form, thus greatly reduces the computational effort. Our second approach is based on the LDG approximation along with a numerical upwind flux to solve the model problem element by element.

The content of this note is constructed as follows. In Section 2, the Chebyshev polynomials and some relevant properties are first given. Then, we present the proposed collocation scheme applied to fractional initial value problem. The final part is devoted to the error analysis technique based on the residual function of the present method (1)-(2). Hence, an improvement of the Chebyshev collocation method is introduced by means of the residual error function. The formulation of the LDG scheme for the Basset equation is established in Section 3. The remaining part is devoted to the proof of numerical stability of the scheme. The numerical findings of the Basset fractional equations are reported in detail for various values of involved parameters in Section 4. Moreover, we compare the approximations obtained using our scheme with the solutions obtained using other existing schemes. Finally, Section 5 provides the conclusion.

2 Chebyshev-collocation approach

To proceed, basic definitions and fundamental aspects of (generalized) Chebyshev polynomials and theorems, which are useful for our subsequent sections have been introduced.

2.1 Chebyshev functions

The Chebyshev polynomials play an outstanding role in classical as well as modern numerical computations [8]. It is known that the classical Chebyshev polynomials (of the first kind) were defined on $[-1,1]$. Starting with $\mathcal{T}_0(z)=1$ and $\mathcal{T}_1(z)=1$, these polynomials satisfy the following recurrence relation

$$\mathcal{T}_{n+1}(z) = 2z\,\mathcal{T}_n(z) - \mathcal{T}_{n-1}(z), \quad n = 1, 2, \ldots.$$

By introducing the change of variable $t = 1 - 2(\frac{z}{T})^{\alpha}$, $\alpha > 0$, one obtains the shifted version of the polynomials defined on $[0, T]$ will be denoted by $\mathcal{T}_n^{\alpha}(t) = \mathcal{T}_n(z)$. This transformation was introduced in [26]. The explicit analytical form of $\mathcal{T}_n^{\alpha}(t)$ of degree $(n\alpha)$ is given for $n = 0, 1, \ldots$

$$\mathcal{T}_n^{\alpha}(t) = \sum_{k=0}^{n} c_{n,k}\, t^{\alpha k}, \quad c_{n,k} = (-1)^k \frac{n\, 2^{2k}\,(n+k-1)!}{(n-k)!\, T^{\alpha k}\,(2k)!}, \quad k = 0, 1, \ldots, n, \tag{6}$$

with $c_{0,k} = 1$ for all $k = 0, 1, \ldots, n$. It is proved in [26] that the set of fractional polynomial functions $\{\mathcal{T}_0^{\alpha}, \mathcal{T}_1^{\alpha}, \ldots\}$ are orthogonal on $[0, T]$ with respect to the weight function $w_{T,\alpha}(t) = \frac{t^{\alpha/2-1}}{\sqrt{T^{\alpha}-t^{\alpha}}}$, i.e.,

$$\int_0^T \mathcal{T}_n^{\alpha}(t)\, \mathcal{T}_m^{\alpha}(t)\, w_{T,\alpha}(t) dt = \frac{\pi}{2\alpha} d_n \delta_{mn}, \quad n, m \geqslant 0.$$

Here, δ_{mn} is the Kronecker delta function, $d_0 = 2$ while $d_n = 1$ for $n \geqslant 1$. These polynomials also satisfy the following properties

$$\mathcal{T}_n^{\alpha}(0) = 1, \quad \mathcal{T}_n^{\alpha}(T) = (-1)^n.$$

2.1.1 Approximation of functions

Any square integrable function $g(t)$ in $(0, T)$, may be expanded in terms of shifted Chebyshev polynomials as

$$g(t) = \sum_{0}^{\infty} a_n\, \mathcal{T}_n^{\alpha}(t),$$

where the unknown coefficients a_n are obtained by means of the orthogonality properties of the shifted Chebyshev polynomials as follows

$$a_n = \frac{2\alpha}{\pi d_n} \int_0^T g(t)\, \mathcal{T}_n^{\alpha}(t)\, w_{T,\alpha}(t) dx, \quad n = 0, 1, \ldots.$$

However, in practice one needs to deal with only the first $(N+1)$-terms shifted Chebyshev polynomials to find an approximate solution of model (1) expressed as

$$g_{N,\alpha}(t) = \sum_{n=0}^{N} a_n\, \mathcal{T}_n^{\alpha}(t), \quad 0 \leqslant t \leqslant T, \tag{7}$$

where the unknown coefficients a_n, $n = 0, 1, \ldots, N$ are sought. To proceed, we write $\mathcal{T}_n^{\alpha}(t)$, $n = 0, 1, \ldots, N$ in the matrix form as follows

$$\mathbb{T}_{\alpha}(t) = \mathbb{B}_{\alpha}(t)\, \mathbb{D}. \tag{8}$$

Here, a superscript t denotes the matrix transpose operation and

$$\mathbb{T}_\alpha(t) = [\mathcal{T}_0^\alpha(t) \quad \mathcal{T}_1^\alpha(t) \quad \dots \quad \mathcal{T}_N^\alpha(t)],$$

and

$$\mathbb{B}_\alpha(t) = \begin{bmatrix}1 & t^\alpha & t^{2\alpha} & \dots & t^{N\alpha}\end{bmatrix}.$$

The upper triangular $(N+1)\times(N+1)$ matrix $\mathbb{D}$ takes the form

$$\mathbb{D} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & c_{1,1} & c_{2,1} & c_{3,1} & \dots & c_{N-1,1} & c_{N,1} \\ 0 & 0 & c_{2,2} & c_{3,2} & \dots & c_{N-1,2} & c_{N,2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & c_{N-1,N-1} & c_{N,N-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & c_{N,N} \end{bmatrix}.$$

By means of (8) one can write the relation (7) in the matrix form

$$u_{N,\alpha}(t) = \mathbb{T}_\alpha(t)\,\mathbb{A} = \mathbb{B}_\alpha(t)\,\mathbb{D}\,\mathbb{A}, \tag{9}$$

where the vector of unknown is

$$\mathbb{A} = [a_0 \quad a_1 \quad \dots \quad a_N]^t.$$

We conclude with discussion about the shifted Chebyshev polynomials by considering their convergence. The following theorem states that the approximation solution $g_{N,\alpha}(t)$ is convergent to $g(t)$ exponentially, if one increases the number of basis functions N [26].

Theorem 1. *Assuming that $\mathcal{D}_*^{(k\alpha)}g(t) \in C[0,T]$ for $k = 0,1,\dots,N$ and let*

$$CT_{N-1}^\alpha = Span\langle \mathcal{T}_0^\alpha(t), \mathcal{T}_1^\alpha(t), \dots, \mathcal{T}_{N-1}^\alpha(t)\rangle.$$

If $g_{N-1,\alpha} = \mathbb{T}_\alpha(t)\mathbb{A}$ is the best approximation to g from CT_{N-1}^α, then the error bound is presented as follows:

$$\|g(t) - g_{N-1,\alpha}(t)\|w \leqslant \frac{T^{N\alpha} M\alpha}{2^N\,\Gamma(N\alpha+1)}\Big(\frac{\pi}{\alpha N!}\Big)^{1/2},$$

where $M_\alpha \geqslant |\mathcal{D}_*^{(N\alpha)}g(t)|$, $t \in [0,T]$.

Ultimately, to obtain a solution in the form (7) of the problem (1) on the interval $0 < t \leqslant T$, we use the spectral collocation points as the roots of generalized fractional order of the Chebyshev functions. According to [26], the following points are used

$$t_k = T\Big(\frac{1-x_k}{2}\Big)^{\frac{1}{\alpha}}, \quad k = 0,1,\dots,N, \tag{10}$$

where $x_k = \cos\big(\frac{2k+1}{N+1}\frac{\pi}{2}\big)$ are the zeros of the usual Chebyshev polynomials of degree $N+1$ on $(-1,1)$.

2.2 The solution procedure

Now, suppose that we approximate the solution $X(t)$ of the linear IVPs (1) in terms of $(N+1)$-terms Chebyshev polynomials series denoted by $X_{N,\alpha}(t)$ on the interval $[0,T]$. As previously stated, in the vector form one may write

$$X(t) \approx X_{N,\alpha}(t) = \mathbb{B}_\alpha(t)\,\mathbb{D}\,\mathbb{A}. \tag{11}$$

By inserting the collocation points (10) into (11), we get a system of matrix equations in the form

$$X_{N,\alpha}(t_k) = \mathbb{B}_\alpha(t_k)\,\mathbb{D}\,\mathbb{A}, \quad k = 0,1,\dots,N.$$

These equations can be expressed in the following compact representations

$$\boldsymbol{X} = \boldsymbol{B}\,\mathbb{D}\,\mathbb{A}, \quad \boldsymbol{X} = \begin{bmatrix} X_{N,\alpha}(t_0) \\ X_{N,\alpha}(t_1) \\ \vdots \\ X_{N,\alpha}(t_N) \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \mathbb{B}_\alpha(t_0) \\ \mathbb{B}_\alpha(t_1) \\ \vdots \\ \mathbb{B}_\alpha(t_N) \end{bmatrix}. \tag{12}$$

To proceed, we take the fractional derivative of order β from both sides of (11) to get

$$\mathcal{D}_*^{(\beta)} X_{N,\alpha}(t) = \mathcal{D}_*^{(\beta)}\,\mathbb{B}_\alpha(t)\,\mathbb{D}\,\mathbb{A}. \tag{13}$$

The computation of $\mathcal{D}_*^{(\beta)}\,\mathbb{B}_\alpha(t)$ can be easily obtained via the properties (4) and (5) as follows

$$\mathbb{B}_\alpha^{(\beta)}(t) = \mathcal{D}_*^{(\beta)}\,\mathbb{B}_\alpha(t) = [0 \quad \mathcal{D}_*^{(\beta)}\,t^\alpha \quad \dots \quad \mathcal{D}_*^{(\beta)}\,t^{\alpha N}].$$

To obtain a system of matrix equations for the fractional derivative, we substitute the collocation points (10) into (13) to get

$$\mathcal{D}_*^{(\beta)} X_{N,\alpha}(t_k) = \mathbb{B}_\alpha^{(\beta)}(t_k)\,\mathbb{D}\,\mathbb{A}, \quad k = 0,1\dots,N,$$

which can also be expressed in the matrix form

$$\boldsymbol{X}^{(\beta)} = \boldsymbol{B}^{(\beta)}\,\mathbb{D}\,\mathbb{A}, \quad \boldsymbol{X}^{(\beta)} = \begin{bmatrix} \mathcal{D}_*^{(\beta)} X_{N,\alpha}(t_0) \\ \mathcal{D}_*^{(\beta)} X_{N,\alpha}(t_1) \\ \vdots \\ \mathcal{D}_*^{(\beta)} X_{N,\alpha}(t_N) \end{bmatrix}, \quad \boldsymbol{B}^{(\beta)} = \begin{bmatrix} \mathbb{B}_\alpha^{(\beta)}(t_0) \\ \mathbb{B}_\alpha^{(\beta)}(t_1) \\ \vdots \\ \mathbb{B}_\alpha^{(\beta)}(t_N) \end{bmatrix}. \tag{14}$$

Our next aim is to find a relationship between $X_{N,\alpha}(t)$ and its first derivation. To end this, it suffices to compute $\frac{d}{dt}\mathbb{B}_\alpha(t)$. Evidently, the calculation of the integer-order derivations of $\mathbb{B}_\alpha(t)$ strictly depends on the values of α and N. These tasks also are obtainable by means of properties (4)-(5) using integer value of $\beta = 1$. For instance, by choosing $\alpha = 1/2$ and $N = 7$ we get

$$\mathbb{B}_{\frac{1}{2}}(t) = \begin{bmatrix} 1 & t^{1/2} & t & t^{3/2} & t^2 & t^{5/2} & t^3 & t^{7/2} & t^4 & t^{9/2} & t^5 \end{bmatrix}.$$

Differentiation with respect to t gives

$$\frac{d}{dt}\mathbb{B}_{\frac{1}{2}}(t) = \begin{bmatrix} 0 & 0 & 1 & \frac{3}{2}t^{1/2} & 2t & \frac{5}{2}t^{3/2} & 3t^2 & \frac{7}{2}t^{5/2} & 4t^3 & \frac{9}{2}t^{7/2} & 5t^4 \end{bmatrix}.$$

Now, by defining

$$\dot{\mathbb{B}}_\alpha(t) := \frac{d}{dt}\mathbb{B}_\alpha(t),$$

and using the relation (6) one obtains that

$$\mathcal{D}X_{N,\alpha}(t) = \dot{\mathbb{B}}_\alpha(t)\,\mathbb{D}\,\mathbb{A}. \tag{15}$$

By placing the collocation points (10) into (15), we arrive at the following matrix expression

$$\dot{\boldsymbol{X}} = \dot{\boldsymbol{B}}\,\mathbb{D}\,\mathbb{A}, \quad \dot{\boldsymbol{X}} = \begin{bmatrix} \mathcal{D}X_{N,\alpha}(t_0) \\ \mathcal{D}X_{N,\alpha}(t_1) \\ \vdots \\ \mathcal{D}X_{N,\alpha}(t_N) \end{bmatrix}, \quad \dot{\boldsymbol{B}} = \begin{bmatrix} \dot{\mathbb{B}}_\alpha(t_0) \\ \dot{\mathbb{B}}_\alpha(t_1) \\ \vdots \\ \dot{\mathbb{B}}_\alpha(t_N) \end{bmatrix}. \tag{16}$$

Now, we are in place to calculate the Chebyshev solutions of (1). The collocation procedure is based on computing these polynomial coefficients by the aid of collocation points defined in (10). This can be done by inserting the collocation points into the fractional IVPs (1) to get the system

$$a_2\,\mathcal{D}X(t_k) + a_1\,\mathcal{D}_*^{(\beta)}\,X(t_k) + a_0\,X(t_k) = f(t_k), \quad k = 0, 1, \ldots, N.$$

In the matrix form we may write the above equations as

$$\boldsymbol{M}_2\,\dot{\boldsymbol{X}} + \boldsymbol{M}_1\,\boldsymbol{X}^{(\beta)} + \boldsymbol{M}_0\,\boldsymbol{X} = \boldsymbol{F}, \tag{17}$$

where the coefficient diagonal matrices $\boldsymbol{M}_j$ for $j = 0, 1, 2$ and the right-hand side vector $\boldsymbol{F}$ take the forms

$$\boldsymbol{M}_j = \begin{bmatrix} a_j & 0 & \ldots & 0 \\ 0 & a_j & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_j \end{bmatrix}_{(N+1)\times(N+1)}, \quad \boldsymbol{F} = \begin{bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_N) \end{bmatrix}_{(N+1)\times 1}.$$

Substituting the relations (14) and (16) into (17), the fundamental matrix equation is obtained

$$\boldsymbol{W}\,\mathbb{A} = \boldsymbol{F}, \tag{18}$$

where

$$\boldsymbol{W} := \left(\boldsymbol{M}_2\,\dot{\boldsymbol{B}} + \boldsymbol{M}_1\,\boldsymbol{B}^{(\beta)} + \boldsymbol{M}_0\,\boldsymbol{B}\right)\mathbb{D}.$$

Obviously, Eq. (18) is a linear matrix equation with a_n, $n = 0, 1, \ldots, N$, being the unknown Chebyshev coefficients to be sought.

We are left with the task of entering the boundary conditions (2) into the former matrix equation. To take into account the first condition $X(0) = X_0$, we tend $t \to 0$ in (11) to get the following matrix representation

$$\overline{\boldsymbol{W}}_0\, \mathbb{A} = X_0, \qquad \overline{\boldsymbol{W}}_0 := \mathbb{B}_\alpha(0)\, \mathbb{D} = [1 \quad 1 \quad \dots \quad 1].$$

For convenience, by replacing the first row of the augmented matrix $[\boldsymbol{W}; \boldsymbol{F}]$ by the row matrix $[\overline{\boldsymbol{W}}_0; X_0]$ we arrive at the new augmented system

$$[\overline{\boldsymbol{W}}; \bar{\boldsymbol{F}}] = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & ; & X_0 \\ w_{1,0} & w_{1,1} & w_{1,2} & w_{1,3} & \dots & w_{1,N} & ; & f(t_1) \\ w_{2,0} & w_{2,1} & w_{2,2} & w_{2,3} & \dots & w_{2,N} & ; & f(t_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & ; & \vdots \\ w_{N-1,0} & w_{N-1,1} & w_{N-1,2} & w_{N-1,3} & \dots & w_{N-1,N} & ; & f(t_{N-1}) \\ w_{N,0} & w_{N,1} & w_{N,2} & w_{N,3} & \dots & w_{N,N} & ; & f(t_N) \end{bmatrix}. \tag{19}$$

Thus, the unknown Chebyshev coefficients in (11) will be calculated via solving this linear system of equations. This task can be easily performed by means of linear solvers.

2.3 Error estimation based on residual functions and improvement of solutions

In this section, the error estimation based on the residual function is introduced for the method and thus the approximate solution (1) is corrected by the residual correction technique. This technique was previously used in [3, 25] and recently in [32]. This error estimation is useful, in particular, when the exact solution of the boundary value problems is not yet known and one requires some tools to measure the accuracy of the proposed collocation scheme. Briefly speaking, our goal is to construct an approximate solution based on the already calculated Chebyshev solution $X_{N,\alpha}(t)$ in the form

$$X_{N,M,\alpha}(t) = X_{N,\alpha}(t) + \widehat{\mathcal{E}_{N,M,\alpha}}(t), \tag{20}$$

where $\widehat{\mathcal{E}_{N,M,\alpha}}(t)$ is the Chebyshev solution of the error problem obtained by using the residual error function as described below. Here, the positive constant M is selected such that $M > N$.

To continue, let us define the residual function for the present method as

$$\mathcal{R}_{N,\alpha}(t) := \mathcal{L}[X_{N,\alpha}](t) - f(t) = a_2\, \mathcal{D} X_{N,\alpha}(t) + a_1\, \mathcal{D}_*^{(\beta)} X_{N,\alpha}(t) + a_0\, X_{N,\alpha}(t) - f(t). \tag{21}$$

Clearly, the approximate solution $X_{N,\alpha}(t)$ is satisfied with the following problem

$$\mathcal{L}[X_{N,\alpha}](t) = f(t) + \mathcal{R}_{N,\alpha}(t), \quad X_{N,\alpha}(0) = X_0. \tag{22}$$

Assuming that the function $X(t)$ is the exact solution of (1), we define the error function $\mathcal{E}_{N,\alpha}(t)$ as

$$\mathcal{E}_{N,\alpha}(t) = X(t) - X_{N,\alpha}(t). \tag{23}$$

By putting (23) into (1) and (2) while exploiting (21)-(22), we arrive at the error differential equation with the homogeneous boundary conditions

$$a_2\,\mathcal{D}\mathcal{E}_{N,\alpha}(t) + a_1\,\mathcal{D}_*^{(\beta)}\mathcal{E}_{N,\alpha}(t) + a_0\,\mathcal{E}_{N,\alpha}(t) = -\mathcal{R}_{N,\alpha}(t), \quad \mathcal{E}_{N,\alpha}(0) = 0. \tag{24}$$

Now, we solve the error differential equation (24) by means of the Chebyshev-collocation scheme already described in the last section to get the approximation

$$\widehat{\mathcal{E}_{N,M,\alpha}}(t) = \sum_{m=0}^{M} c_m\,\mathcal{T}_m^{\alpha}(t), \tag{25}$$

for the error function $\mathcal{E}_{N,\alpha}(t)$ for $M > N$. Once the approximate solution $\widehat{\mathcal{E}_{N,M,\alpha}}(t)$ is obtained, the corrected solution $X_{N,M,\alpha}(t)$ defined in (20) will be known.

3 LDG approach

In this part, we shall formulate the LDG methods for the Basset equation in (1). To do this we first introduce some basic notation, see also [5, 15, 16, 22].

Let $J = (0, T)$ is given and consider (1) on J. To proceed, we introduce two variables $x_0(t) = X(t)$ and $x_1(t) = \mathcal{D}X(t)$. By means of (3) we may rewrite (1) as the following first-order system

$$\begin{cases} x_1(t) - \dfrac{dx_0(t)}{dt} = 0, \\ a_2\,x_1(t) + a_1\,{}_0I_t^{(1-\beta)}x_1(t) + a_0\,x_0(t) = f(t), \\ x_0(0) - X_0 = 0, \end{cases} \tag{26}$$

where $\beta \in (0, 1]$ and $t \in J$. Now, let $\mathcal{M}$ be a subdivision of the time interval J into into N subintervals $\{I_n\}_{n=1}^{N}$ given by $I_n = (t_{n-1}, t_n)$. The nodes of $\mathcal{M}$ are given as $0 =: t_0 < t_1 < \ldots < t_{\mathcal{N}-1} < t_{\mathcal{N}} := T$. By k_n we denote the length of each I_n, i.e., $k_n = t_n - t_{n-1}$ for $n = 1, 2, \ldots, \mathcal{N}$. We further set $k := \max_{n=1}^{\mathcal{N}} k_n$. To the mesh $\mathcal{M}$, we associate the broken Sobolev spaces

$$C(J, \mathcal{M}) = \{v : J \to \mathbb{R} \mid v|_{I_n} \in L_2(I_n),\ n = 1, 2, \ldots, \mathcal{N}\},$$

and

$$H^1(J, \mathcal{M}) = \{v : J \to \mathbb{R} \mid v|_{I_n} \in H^1(I_n),\ n = 1, 2, \ldots, \mathcal{N}\}.$$

Having defined these function spaces, we assume that the solutions belong to corresponding spaces

$$\Big(x_0(t), x_1(t)\Big) \in H^1(J, \mathcal{M}) \times C(J, \mathcal{M}).$$

We emphasize that a function $v \in H^1(J, \mathcal{M})$ may be discontinuous in t at time level t_n. Thus, at the nodes of $\mathcal{M}$ the left-sided as well as the right-sided limits of piecewise continuous functions $v : J \to \mathbb{R}$ will be important. We let v_n^- and v_n^+ being the left- and right-sided limits of v at t_n

$$v_n^+ = v^+(t_n) = v(t_n^+) := \lim_{s\to 0^+} v(t_n + s), \quad v_n^- = v^-(t_n) = v(t_n^-) := \lim_{t\to 0^-} v(t_n + s).$$

In Fig. 1, an illustration of the possible jumps in $v \in H^1(J, \mathcal{M})$ at element interfaces is presented.

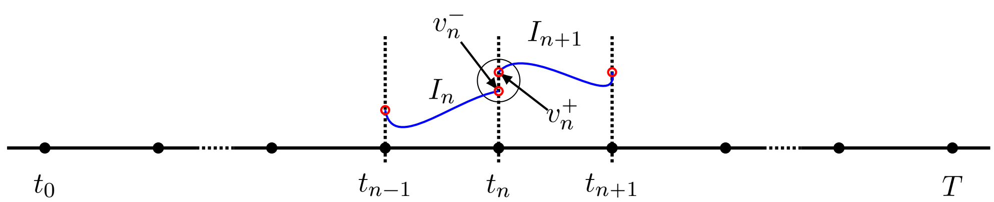

Figure 1: Partition of the time domain J into elements I_n with width k_n and interfaces t_n. The open circles indicate the left and right limits of function v at interfaces.

Let now q be a positive integer. By $P_q(I_n)$ we denote the space of polynomials of degree $\leqslant q$ on $I_n \in \mathcal{M}$. Restricting our approximate solutions to be in a local finite dimensional subspace $\mathcal{V}^{(q)} \subset H^1(J, \mathcal{M})$. Next, we choose $\mathcal{V}^{(q)}$ being the space of discontinuous and piecewise polynomial functions defined by

$$\mathcal{V}^{(q)} = \{v : J \to \mathbb{R} \mid v|_{I_n} \in P_q(I_n),\ n = 1, 2, \ldots, \mathcal{N}\}.$$

On the element I_n we use the quantities $\mathcal{X}_0(t)$ and $\mathcal{X}_1(t)$ belong to $\mathcal{V}^{(q)}$ represent the computed DG approximations to the exact solutions $x_0(t)$ and $x_1(t)$ of he system (26). The following L_2-inner products shall be used throughout the paper

$$(u,\ v)_n := \int_{I_n} u\, v\, dt, \quad (u,\ v)_n := \int_0^{t_n} u\, v\, dt, \qquad \|u\|_n^2 := \int_0^{t_n} |u|^2\, dt,$$

We define the weak DG formulation for (26) by first multiplying the first equation by a test function $v_0 \in \mathcal{V}^{(q)}$, integrating over I_n and integrating by parts to obtain that

$$\Big(\mathcal{X}_1(t),\ v_0\Big)_n + \Big(\mathcal{X}_0(t),\ \frac{dv_0}{dt}\Big)_n - \mathcal{X}_0(t_n^-)\, v_0(t_n^-) + \mathcal{X}_0(t_{n-1}^+)\, v_0(t_{n-1}^+) = 0. \tag{27}$$

Similarly, we multiply the second integral equation in (26) by a test function $v_1 \in \mathcal{V}^{(q)}$ and integrate over I_n. Utilizing the upwind flux $\mathcal{X}_0(t_{n-1}^-)$ instead of $\mathcal{X}_0(t_{n-1}^+)$ in (27), the discrete formulation consists of determining $\mathcal{X}_0, \mathcal{X}_1 \in \mathcal{V}^{(q)}$ such that for all $v_0, v_1 \in \mathcal{V}^{(q)}$, and $n = 1, 2, \ldots, \mathcal{N}$

$$\begin{cases} \Big(\mathcal{X}_1(t),\ v_0(t)\Big)_n + \Big(\mathcal{X}_0(t),\ \frac{dv_0(t)}{dt}\Big)_n - \mathcal{X}_0(t_n^-)\, v_0(t_n^-) + \mathcal{X}_0(t_{n-1}^-)\, v_0(t_{n-1}^+) = 0, \\ a_2\Big(\mathcal{X}_1(t),\ v_1(t)\Big)_n + a_1\Big({}_0I_t^{(1-\beta)}\mathcal{X}_1(t),\ v_1(t)\Big)_n + a_0\Big(\mathcal{X}_0(t),\ v_1(t)\Big)_n - \Big(f(t),\ v_1(t)\Big)_n = 0, \\ \mathcal{X}_0(t_0^- = 0) - X_0 = 0, \end{cases} \tag{28}$$

Note that, on the initial step $I_1 = (t_0, t_1)$ we use $\mathcal{X}_0(t_0^-) = X_0$. We also emphasize that using the upwind flux as natural choice enables us to solve the equation interval by interval on each subinterval I_n for $n = 1, 2, \ldots, \mathcal{N}$. Thus, we require to invert a local $(k+1) \times (k+1)$ low-order matrix instead of a global full matrix. For the implementation details using the Legendre basis functions, we refer readers to [16, 22].

3.1 Numerical stability

In this section, we investigate the stability of the proposed LDG scheme for the Basset equation in (1). By rewriting the LDG scheme (28) we have

$$\begin{cases} \mathcal{X}_0(t_n^-)\, v_0(t_n^-) - \mathcal{X}_0(t_{n-1}^-)\, v_0(t_{n-1}^+) - \Big(\mathcal{X}_1(t),\ v_0(t)\Big)_n - \Big(\mathcal{X}_0(t),\ \frac{dv_0(t)}{dt}\Big)_n = 0, \\ a_2\Big(\mathcal{X}_1(t),\ v_1(t)\Big)_n + a_1\Big({}_0I_t^{(1-\beta)}\mathcal{X}_1(t),\ v_1(t)\Big)_n + a_0\Big(\mathcal{X}_0(t),\ v_1(t)\Big)_n - \Big(f(t),\ v_1(t)\Big)_n = 0, \\ \mathcal{X}_0(t_0^-) - X_0 = 0, \end{cases} \tag{29}$$

which hold for all $v_0, v_1 \in \mathcal{V}^{(q)}$, and $n = 1, 2, \ldots, \mathcal{N}$. To proceed, we need the following lemma, which is based on the semigroup properties of fractional integral operators

Lemma 1. *[5] For any $\beta \in (0,1)$ we have*

$$\Big({}_0I_t^{(1-\beta)}w,\ w\Big)_n = \Big({}_0I_t^{(\frac{1-\beta}{2})}w,\ {}_t\mathcal{I}_{t_n}^{(\frac{1-\beta}{2})}w\Big)_n = \cos\Big(\frac{(1-\beta)\pi}{2}\Big)\|w\|^2_{H^{\frac{1-\nu}{2}}([0,t_n])}. \tag{30}$$

Denoting by $\widetilde{\mathcal{X}}_0, \widetilde{\mathcal{X}}_1 \in \mathcal{V}^{(q)}$ as the approximate solution of $\mathcal{X}_0, \mathcal{X}_1$ and define the numerical errors as $\Xi_i := \widetilde{\mathcal{X}}_i - \mathcal{X}_i$ for $i = 0, 1$. Observe that $\widetilde{\mathcal{X}}_0$ and $\widetilde{\mathcal{X}}_1$ both satisfy (29). Subtracting equations (29) from the same equations with $\widetilde{\mathcal{X}}_0$ and $\widetilde{\mathcal{X}}_1$ give us the following expression for the error equations

$$\begin{cases} \Xi_0(t_n^-)\, v_0(t_n^-) - \Xi_0(t_{n-1}^-)\, v_0(t_{n-1}^+) - \Big(\Xi_1(t),\ v_0(t)\Big)_n - \Big(\Xi_0(t),\ \frac{dv_0(t)}{dt}\Big)_n = 0, \\ a_2\Big(\Xi_1(t),\ v_1(t)\Big)_n + a_1\Big({}_0I_t^{(1-\beta)}\Xi_1(t),\ v_1(t)\Big)_n + a_0\Big(\Xi_0(t),\ v_1(t)\Big)_n = 0, \end{cases} \tag{31}$$

for all $v_0, v_1 \in \mathcal{V}^{(q)}$. For convenience we set $a_2, a_0 = 1$. Taking $v_0 = \Xi_0$, $v_1 = \Xi_1$ in (32) and by adding these two equations together we arrive at

$$\begin{aligned} &\Xi_0^2(t_n^-) - \Xi_0(t_{n-1}^-)\,\Xi_0(t_{n-1}^+) - \Big(\Xi_0(t),\ \frac{d}{dt}\Xi_0(t)\Big)_n + \Big(\Xi_1(t),\ \Xi_1(t)\Big)_n \\ &\quad + a_1\Big({}_0I_t^{(1-\beta)}\Xi_1(t),\ \Xi_1(t)\Big)_n = 0. \end{aligned}$$

Using the fact that $\big(w,\ \frac{dw}{dt}\big)_n = (w^2(t_n^-) - w^2(t_{n-1}^+))/2$, one can replace the third term with $w = \Xi_0$. After multiplying by two, adding $\pm\Xi_0^2(t_{n-1}^-)$ to the last equation and rearranging the terms to get

$$\Big(\Xi_0(t_{n-1}^+) - \Xi_0(t_{n-1}^-)\Big)^2 + \Xi_0^2(t_n^-) - \Xi_0^2(t_{n-1}^-) + 2\|\Xi_1\|_n^2 + 2a_1\Big({}_0I_t^{(1-\beta)}\Xi_1(t),\ \Xi_1(t)\Big)_n = 0.$$

Summing over $n = 1, \ldots, \mathcal{N}$ yields

$$\Xi_0^2(t_{\mathcal{N}}^-) - \Xi_0^2(t_0^-) + \sum_{n=1}^{\mathcal{N}}\Big(\Xi_0(t_{n-1}^+) - \Xi_0(t_{n-1}^-)\Big)^2 + 2\|\Xi_1\|_{\mathcal{N}}^2 + 2a_1\Big({}_0I_t^{(1-\beta)}\Xi_1(t),\ \Xi_1(t)\Big)_{\mathcal{N}} = 0.$$

Utilizing Lemma 1 and assuming that $a_1 > 0$, we have proved the following L_∞ stability of (29) (see also [5, 16]:

Lemma 2. *The LDG scheme* (29) *is L_∞ stable and the numerical errors satisfy*

$$\begin{aligned}\Xi_0^2(t_{\mathcal{N}}^-) = \Xi_0^2(t_0^-) - \sum_{n=1}^{\mathcal{N}} \Big(\Xi_0(t_{n-1}^+) - \Xi_0(t_{n-1}^-)\Big)^2 - 2\|\Xi_1\|_{\mathcal{N}}^2 \\ - 2a_1 \cos\Big(\frac{(1-\beta)\pi}{2}\Big)\|\Xi_1\|^2_{H^{\frac{1-\beta}{2}}([0,t_{\mathcal{N}}])}.\end{aligned} \tag{32}$$

4 Numerical applications

In this section, to describe the efficiency of the proposed Chebyshev collocation as well as LDG methods, some numerical experiments are performed. To test the validity of the presented schemes and to compare our results with methods available in the standard literature, the following two schemes are employed

a) The traditional fractional finite difference method (FFDM) based on the Grünwald-Letnikov formula [23, 27]

$$a_2 \frac{X_j - X_{j-1}}{\Delta x} + \frac{a_1}{\sqrt{\Delta x}} \sum_{i=0}^{j} w_i^{(\frac{1}{2})} X_{j-i} + a_0 X_j = 1, \quad j = 1, 2, \ldots, m,$$

where the time step Δx is taken as $\Delta x = T/m$ and procedure starts with zero condition $X_0 = 0$.

b) Inverse Laplace transform (ILT) [23].

All numerical computations have been done by using Matlab R2017a.

We consider the Basset equation, which corresponds to $\beta = 1/2$ and $f(t) = 1$. We solve this equation on the computational domain $[0, T]$ with the initial condition given by $X_0 = 0$ and parameters $\lambda = 0.25, 2, 10, 100$. Various final times $T = 5, 10, 15$, and $T = 20$ are considered in both schemes.

First, we consider the Chebyshev collocation method and take $\alpha = 1$, $N = 10$. In this case, we are looking for an approximate solution in the form $X_{10,1}(t) = \sum_{n=0}^{10} a_n \mathcal{T}_n^1(t)$, on the interval $[0, T]$. Using $T = 5$, $T = 20$ and setting the parameter $\lambda = 100$, the approximate solution $X_{10,1}(t)$ of this model problem using Chebyshev basis functions on the interval $0 \leqslant t \leqslant T$ with $T = 5, 20$ are obtained as follows respectively:

$$\begin{aligned} X_{10,1}(t) =& -8.8383 \times 10^{-7}\, t^{10} + 2.7245 \times 10^{-5}\, t^9 - 3.7014 \times 10^{-4}\, t^8 \\ &+ 0.002927775866\, t^7 - 0.01507095807\, t^6 + 0.05399542273\, t^5 - 0.1437831673\, t^4 \\ &+ 0.3107734634\, t^3 - 0.5911335485\, t^2 + 0.9482283051\, t + 1.8163 \times 10^{-108}, \\ X_{10,1}(t) =& -4.8127 \times 10^{-11}\, t^{10} + 5.7236 \times 10^{-9}\, t^9 - 2.9744 \times 10^{-7}\, t^8 \\ &+ 8.8740 \times 10^{-6}\, t^7 - 1.6801 \times 10^{-4}\, t^6 + 0.002107882581\, t^5 - 0.01778296924\, t^4 \\ &+ 0.1003783155\, t^3 - 0.3697532231\, t^2 + 0.8369021913\, t + 1.2840 \times 10^{-108}. \end{aligned}$$

Using the second approach, i.e. the LDG scheme with $q = 10$, $\mathcal{N} = 1$, the corresponding approximative solutions for $t \in [0, T]$, $T = 5, 20$ take the forms

$$\begin{aligned}\mathcal{X}_0(t) = &- 1.7057 \times 10^{-6}\, t^{10} + 7.7421 \times 10^{-5}\, t^9 - 9.6527 \times 10^{-4}\, t^8 \\ &+ 0.006902075652\, t^7 - 0.03146521583\, t^6 + 0.09710725216\, t^5 - 0.2157254116\, t^4 \\ &+ 0.384055245\, t^3 - 0.6326395462\, t^2 + 0.9581583011\, t + 0.0006770401163, \\ \mathcal{X}_0(t) = &- 9.2446 \times 10^{-11}\, t^{10} + 1.0478 \times 10^{-8}\, t^9 - 5.1636 \times 10^{-7}\, t^8 \\ &+ 1.4525 \times 10^{-5}\, t^7 - 2.5759 \times 10^{-4}\, t^6 + 0.003005128908\, t^5 - 0.02340449824\, t^4 \\ &+ 0.1213062685\, t^3 - 0.4105174814\, t^2 + 0.8643431644\, t + 0.01173216948.\end{aligned}$$

The above approximations are visualized in Fig. 2. In addition to $\lambda = 100$, the approximated solutions correspond to $\lambda = 0.25, 2, 10$ are also plotted in Fig. 2. It can be seen that that there are a close relationship between the approximated solutions obtained by the Chebyshev collocation and LDG methods. Moreover, a comparison between our plots and those given by the previously well-established methods [6, 7, 20] shows that there is a strong level of agreement in our solutions.

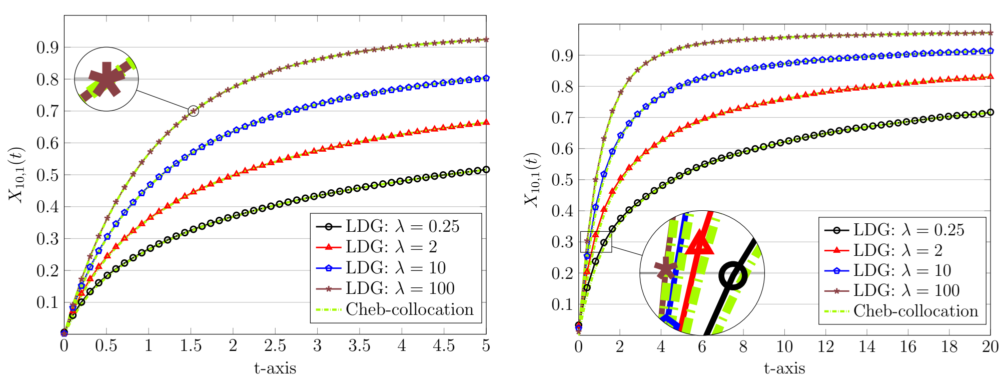

Figure 2: Comparison of approximated solutions using Chebyshev functions and LDG method for $T = 5$ (left) and $T = 20$ (right) for the Basset equation with ($N = 10$, $\alpha = 1$), ($q = 10$, $\mathcal{N} = 1$) and various $\lambda = 0.25, 2, 10, 100$.

Next, we see the impact of using different N on the computations. Comparison of residual error functions for the Basset equation using $\lambda = 10$, $\alpha = 1$, and various $N = 8, 10, \ldots, 16$ on the interval $[0, 20]$ are presented in Fig. 3. Another issue to be answered is about using different values of $\alpha \geqslant \beta$ as the local order of basis functions on our calculations in the Chebyshev collocation approach. For instance, using $\alpha = 1/2$ equals to β and $N = 10$, $\lambda = 100$, the following approximation is obtained on $0 \leqslant t \leqslant 5$

$$\begin{aligned}X_{10,\frac{1}{2}}(t) =&0.5581620014\, t^{1/2} - 0.5214023613\, t^{3/2} + 0.1455775015\, t^{5/2} \\ &- 0.2809450947\, t^{7/2} - 0.02153373848\, t^{9/2} + 0.00168599665\, t^5 + 0.1113886801\, t^4 \\ &+ 0.281731904\, t^3 - 0.3381478491\, t^2 + 0.895822155\, t + 4.8141 \times 10^{-109},\end{aligned}$$

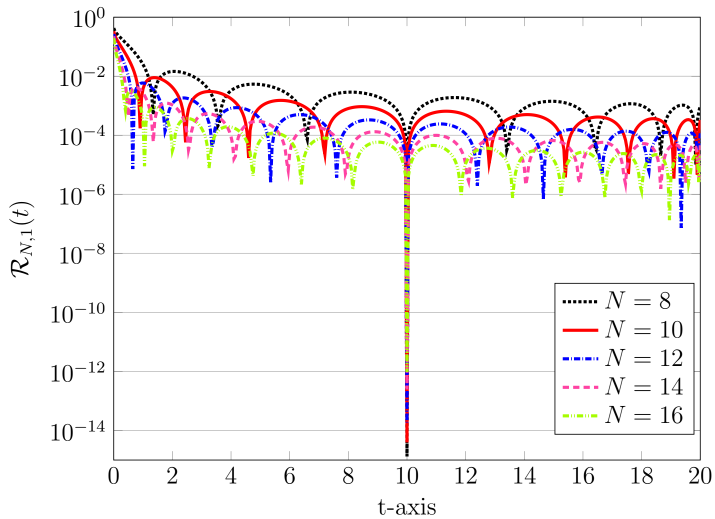

Figure 3: Comparison of residual error functions for the Basset equation for $\lambda = 10$, $\alpha = 1$, and various $N = 8, 10, \ldots, 16$.

A comparison with $X_{10,1}(t)$ shows that there is a considerable gap between these two solutions. Therefore, we consider only $\alpha = 1$ below. For $T = 20$, we further report the numerical results obtained by the Chebyshev-collocation and LDG schemes for two values of parameter $\lambda = 0.25, 10$ at some points $[0, 20]$. These results correspond to $N = 10$ and $q = 10, \mathcal{N} = 1$ are presented in Table 1.

Table 1: Numerical results obtained by Chebyshev-collocation and LDG methods for $T = 20$ for the Basset equation using $N = 10$, $q = 10, \mathcal{N} = 1$, and $\lambda = 0.25, 10$.

	Chebyshev		LDG	
t	$\lambda = 0.25$	$\lambda = 10$	$\lambda = 0.25$	$\lambda = 10$
1	0.24745687871926	0.44329246438781	0.26719235046467	0.464432885264128
2	0.36229987512529	0.62991614800135	0.37259196136653	0.637924971951327
4	0.47523639579227	0.76751184193410	0.47857875382588	0.769502482221561
6	0.54276011876724	0.82455908540295	0.54677542582372	0.826707334038536
8	0.58759812654452	0.85360218977282	0.58971027646290	0.853553666397951
10	0.62101313082419	0.87192719005300	0.62170013995796	0.872398628347251
12	0.64772063970328	0.88547512392764	0.64976044518700	0.886329630908577
14	0.66882831077307	0.89479053340209	0.67013493142816	0.894616219680941
16	0.68672759429012	0.90247325228560	0.68671347484272	0.902784006122455
18	0.70194334105779	0.90869786306967	0.70382316005383	0.908903744240047
20	0.71508339875847	0.91381257282151	0.71692377538879	0.914102415031257

We are going to describe the technique of residual correction to improve the current Chebyshev polynomial solution $X_{N,\alpha}(t)$ for $N = 10, \alpha = 1$ and $\lambda = 10$ on the interval $[0, T]$, where $T = 20$. In this case, we get

$$\begin{aligned} X_{10,1}(t) =& -4.4234 \times 10^{-11}\, t^{10} + 5.2175 \times 10^{-9}\, t^9 - 2.6840 \times 10^{-7}\, t^8 \\ &+ 7.9065 \times 10^{-6}\, t^7 - 1.4730 \times 10^{-4}\, t^6 + 1.8101 \times 10^{-3}\, t^5 - 0.01487437148\, t^4 \\ &+ 0.08137765888\, t^3 - 0.2913971569\, t^2 + 0.6665159043\, t. \end{aligned}$$

The corresponding residual error function $\mathcal{R}_{10,1}(t)$ takes the form

$$\begin{aligned} \mathcal{R}_{10,1}(t) =& -0.33348 + 0.083722\, t - 0.047264\, t^2 + 0.02188\, t^3 - 0.005824\, t^4 \\ &+ 9.2631 \times 10^{-4}\, t^5 - 9.1950 \times 10^{-5}\, t^6 + 5.7593 \times 10^{-6}\, t^7 - 2.2144 \times 10^{-7}\, t^8 \\ &+ 4.7751 \times 10^{-9}\, t^9 - 4.4234 \times 10^{-11}\, t^{10} + 0.49235\, t^{\frac{1}{2}} - 0.28701\, t^{\frac{3}{2}} \\ &+ 0.096182\, t^{\frac{5}{2}} - 0.020092\, t^{\frac{7}{2}} + 0.0027167\, t^{\frac{9}{2}} - 2.4117 \times 10^{-4}\, t^{\frac{11}{2}} \\ &+ 1.3941 \times 10^{-5}\, t^{\frac{13}{2}} - 5.048 \times 10^{-7}\, t^{\frac{15}{2}} + 1.039 \times 10^{-8}\, t^{\frac{17}{2}} - 9.2724 \times 10^{-11}\, t^{\frac{19}{2}}. \end{aligned}$$

The next task is to solve the error problem (24) for $\mathcal{E}_{N,\alpha}(t)$. Choosing $M = 15$ and using the Chebyshev-collocation procedure, the corresponding approximation solution is calculated as follows

$$\begin{aligned} \widehat{\mathcal{E}_{10,15,1}}(t) =& 1.0606 \times 10^{-15}\, t^{15} - 1.7843 \times 10^{-13}\, t^{14} + 1.3662 \times 10^{-11}\, t^{13} \\ &- 6.3028 \times 10^{-10}\, t^{12} + 1.9548 \times 10^{-8}\, t^{11} - 4.3064 \times 10^{-7}\, t^{10} + 6.9407 \times 10^{-6}\, t^9 \\ &- 8.3030 \times 10^{-5}\, t^8 + 7.3956 \times 10^{-4}\, t^7 - 4.8690 \times 10^{-3}\, t^6 + 0.023238\, t^5 \\ &- 0.077542\, t^4 + 0.16984\, t^3 - 0.21724\, t^2 + 0.12301\, t - 2.6612 \times 10^{-110}. \end{aligned}$$

Finally, after inserting the error function $\widehat{\mathcal{E}_{10,15,1}}(t)$ into (20), the corrected approximate solution $X_{10,15,1}(t)$ for $(N, M) = (10, 15)$ is obtained

$$\begin{aligned} X_{10,15,1}(t) =& 1.0606 \times 10^{-15}\, t^{15} - 1.7843 \times 10^{-13}\, t^{14} + 1.3662 \times 10^{-11}\, t^{13} \\ &- 6.3028 \times 10^{-10}\, t^{12} + 1.9548 \times 10^{-8}\, t^{11} - 4.3068 \times 10^{-7}\, t^{10} + 6.9459 \times 10^{-6}\, t^9 \\ &- 8.3298 \times 10^{-5}\, t^8 + 7.4747 \times 10^{-4}\, t^7 - 5.0163 \times 10^{-3}\, t^6 + 0.025048\, t^5 \\ &- 0.092416\, t^4 + 0.25121\, t^3 - 0.50863\, t^2 + 0.78952\, t - 2.6612 \times 10^{-110}. \end{aligned}$$

The above Chebyshev approximate solution $X_{10,1}(t)$ and its corrected approximation $X_{10,15,1}(t)$ are visualized in Fig. 4. It can be seen that these solutions are very close to each other.

Next, we verify the accuracy of the proposed Chebyshev-collocation and LDG methods compared to existing numerical methods in standard literature when applied to Basset equation. In Table 2, we report the numerical results correspond to $N = 10$ obtained by the Chebyshev-collocation procedure using $\alpha = 1$ for different $\lambda = 0.25, 2, 10, 100$. In the LDG scheme, we additionally utilize $p = 10$ and $\mathcal{N} = 1$. Comparison results of the ILT [23] utilizing 15-terms and the FFDM [23] for $\Delta x = 0.1, 0.01$ in some points for numerical solution of the Basset equation are also shown in Table 2.

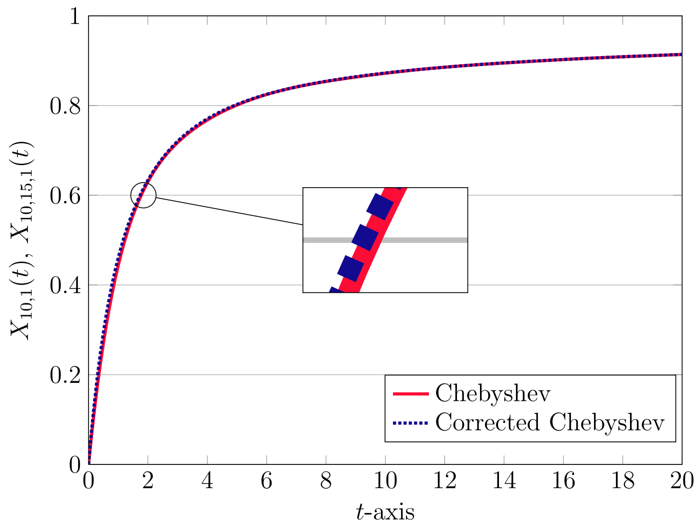

Figure 4: The approximated Chebyshev and its corrected solutions $X_{10,1}(t)$ and $X_{10,15,1}(t)$ of the Basset equation using $(N, M) = (10, 15), \alpha = 1$, and $\beta = 1/2$.

It should be noticed that, we have further utilized various basis functions such as Chelyshkov [12], novel Bessel [17], and Legendre [14] functions rather than Chebyshev polynomials in the proposed collocation scheme. However, our experiments show that the same results are obtained when different bases are used in the computations.

5 Conclusions

In this paper, two approximation procedures were proposed for the solution of the Basset equation as a fractional-order differential equation with initial condition arising in the modeling of the unsteady motion of a particle accelerating a viscous fluid under the action of gravity. The first method was based upon Chebyshev polynomials and the other one was an upwinded LDG scheme. In the first approach, utilizing the Chebyshev functions together with the collocation points, the differential equations are transformed into an algebraic system of linear equations. On the other hand, the main feature of the LDG is an element-by-element solution, hence, there is no need for a full global solution. Moreover, the stability of the scheme was proven in the L_∞ norm. Numerical examples were given to illustrate the efficiency and accuracy of the presented methods and a comparison between the presented methods and other existing schemes was carried out.

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		ILT [23]	FFDM [23]		Chebyshev	LDG
t	λ	15-terms	$\Delta x = 0.1$	$\Delta x = 0.01$	$N = 10$	$p = 10$
5.0	0.25	0.516464	0.514752	0.516290	0.515925931777874	0.516566679888025
5.0	2	0.663762	0.661594	0.663544	0.663421522128128	0.663793996932885
5.0	10	0.803013	0.800625	0.802775	0.802865104315721	0.803019822050154
5.0	100	0.924250	0.922054	0.924036	0.924220909644161	0.924251126125401
10	0.25	0.622844	0.621963	0.622748	0.622202765692998	0.623221859903706
10	2	0.760001	0.759121	0.759907	0.759634975384172	0.760145199698311
10	10	0.872679	0.872007	0.872609	0.872545403739373	0.872711962261722
10	100	0.957297	0.956991	0.957266	0.957279801007083	0.957299619149681
15	0.25	0.679197	0.678624	0.679127	0.678523913948532	0.679955563260405
15	2	0.803911	0.803412	0.803851	0.803541025959130	0.804247498391441
15	10	0.899151	0.898827	0.899112	0.899020407177704	0.899237205380435
15	100	0.966791	0.966671	0.966777	0.966778291194122	0.966796522625656
20	0.25	0.715764	0.715346	0.715705	0.715083398758470	0.716923775388791
20	2	0.830228	0.829891	0.830178	0.829859685360606	0.830806263916319
20	10	0.913938	0.913735	0.913908	0.913812572821510	0.914102415031257
20	100	0.971834	0.971763	0.971823	0.971834581922573	0.971837512848187

Table 2: Comparison of numerical solutions in Chebyshev-collocation method for $N = 10$, $\alpha = 1$, and different $\lambda = 0.25, 2, 10, 100$ evaluated at $t = 5, 10, 15, 20$.

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