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Almost periodic positive solutions for a time-delayed SIR epidemic model with saturated treatment on time scales

Kapula Rajendra Prasad[†], **Mahammad Khuddush**[†*], **Kuparala Venkata Vidyasagar**[†‡]

[†] *College of Science and Technology, Department of Applied Mathematics, Andhra University, Visakhapatnam, India-530003*

[‡] *Department of Mathematics, Government Degree College for Women, Marripalem, Koyyuru Mandal, Visakhapatnam, India-531116*

Email(s): rajendra92@rediffmail.com, khuddush89@gmail.com, vidyavijaya08@gmail.com

Abstract. In this paper, we study a non-autonomous time-delayed SIR epidemic model which involves almost periodic incidence rate and saturated treatment function on time scales. By utilizing some dynamic inequalities on time scales, sufficient conditions are derived for the permanence of the SIR epidemic model and we also obtain the existence and uniform asymptotic stability of almost periodic positive solutions for the addressed SIR model by Lyapunov functional method. Finally numerical simulations are given to demonstrate our theoretical results.

Keywords: SIR model, time scale, almost periodic incidence rate, almost periodic positive solution, permanence, uniform asymptotic stability.
AMS Subject Classification 2010: 92D30, 34N05.

1 Introduction

Infectious diseases are caused by pathogenic microorganisms, such as bacteria, viruses, fungi and parasites. The diseases can spread directly or indirectly from one person to another or from birds or animals to humans and these diseases are a leading cause of death. Despite all the advancement in medicines, infectious disease outbreaks still constitute a significant threat to the public health and economy. Mathematical modeling has become a valuable tool to understand the dynamics of infectious disease and to support the development of control strategies and studied by many researchers [11, 16, 17] and references therein.

The differential, difference and dynamic equations on time scales are three equations play important role for modelling in the environment.Among them, the theory of dynamic equations

*Corresponding author.
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on time scales is the most recent and was introduced by Hilger in his PhD thesis in 1988 [7] with three main features: unification, extension and discretization. Since a time scale is any closed and nonempty subset of the real numbers set. So, by this theory, we can extend known results from continuous and discrete analysis to a more general setting. As a matter of fact, this theory allows us to consider time scales which possess hybrid behaviours (both continuous and discrete). These types of time scales play an important role for applications, since most of the phenomena in the environment are neither only discrete nor only continuous, but they possess both behaviours Hence, dynamic equations on a time scale have a potential for applications. In the population dynamics, the insect population can be better modelled using time scale calculus. The reason is that they evolve continuously while in season, die out in winter while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a non-overlapping population. Some of the good contributions in this field can be found in [1–3, 14, 15].

In 2016, Bohner and Streipert [4] considered the SIS model,

$$\begin{aligned} S^{\Delta}(t) &= I(t)\Big[-\beta S(\sigma(t)) + \gamma\Big], \quad S(t) > 0,\\ I^{\Delta}(t) &= I(t)\Big[\beta S(\sigma(t)) - \gamma\Big], \quad I(t) \ge 0, \end{aligned}$$

where $\beta > 0, \gamma > 0$ are the transmission and recovery rates of the disease and $\sigma(t)$ denotes the forward jump operator and discussed the stability of the steady states of the model. In [5], Bohner, Streipert and Torres derived exact solution of non-autonomous SIR epidemic model,

$$\begin{aligned} x^{\Delta}(t) &= -\frac{b(t)x(t)y(\sigma(t))}{x(t)+y(t)}\\ y^{\Delta}(t) &= \frac{b(t)x(t)y(\sigma(t))}{x(t)+y(t)} - c(t)y(\sigma(t))\\ z^{\Delta}(t) &= c(t)z(\sigma(t)), \quad x(t), y(t) > 0, \end{aligned}$$

and then analyzed the stability of the solutions to corresponding autonomous model. In the real world phenomena, since the almost periodic variation of the environment plays a crucial role in many biological and ecological dynamical systems and is more frequent and general than the periodic variation of the environment. The concept of almost periodic time scales was proposed by Li and Wang [10]. Based on this concept, some works have been done [12–15].

Recently, Bohner and Streipert [6] analysed the existence and globally asymptotic stability of a ω-periodic solution to the discrete SIS model,

$$\begin{aligned} \Delta S_t &= -\beta_t S_{t+1} I_t + \gamma_t I_t,\\ \Delta I_t &= \beta_t S_{t+1} I_t - \gamma_t I_t. \end{aligned}$$

Motivated by aforementioned works and mainly [9, 18], in this paper we study the following time-delayed SIR model on time scales.

2 Model description and preliminaries

In this section, we consider an SIR model with saturated and periodic incidence rate and saturated treatment function, whose corresponding continuous ($\mathbb{R}$) model has been studied in [9, 18].

The population is divided into three classes: the susceptible class S, the infectious class I, and the recovered class R. The transition dynamics associated with these subpopulations are illustrated in Fig. 1.

Based on the above discussion, we make the following assumptions:

(1) The infection is transmitted to humans by a vector, i.e., susceptible persons receive the infection from infectious vectors, and susceptible vectors receive the infection from infectious persons.

(2) When a susceptible vector is infected by a human, there is a fixed time τ during which the infectious agents develop in the vector, and it is after that time that the infected vector can infect the susceptible human population.

(3) The number of newly infected individuals per time unit is proportional to $S(t)\mathtt{u}(t)/\big(1+a(t)\mathtt{u}(t)\big)$, where $\mathtt{u}(t)$ the number of infectious vectors in the community at time t, and $(1+a(t)\mathtt{u}(t))^{-1}$ represents the saturation effect when the population of infectious vectors is large.

(4) The total vector population is very large and $\mathtt{u}(t)$ is proportional to $I(t-\tau)$.

Using above assumptions, we propose the delayed susceptible-infected-recovered (SIR) model with saturated treatment on time scales by

$$\left.\begin{aligned} S^{\Delta}(t) &= A(t) - \alpha(t)S(t) - \frac{\chi(t)S(t)I(t-\tau)}{1+a(t)I(t-\tau)},\\ I^{\Delta}(t) &= \frac{\chi(t)S(t)I(t-\tau)}{1+a(t)I(t-\tau)} - [\alpha(t)+\beta(t)+\gamma(t)]I(t) - \frac{b(t)I(t)}{1+c(t)I(t)},\\ R^{\Delta}(t) &= \gamma(t)I(t) + \frac{b(t)I(t)}{1+c(t)I(t)} - \alpha(t)R(t), \end{aligned}\right\} \tag{1}$$

where $t \in \mathbb{T}$(time scale). Motivated by biological realism, we take the contact rate as $\chi(t) = d+\delta\sin(\pi/6)t$, (for more details refer [9]) and all other parameters are positive. While contacting with infected individuals, the susceptible individuals become infected at a saturated incidence rate $\frac{\chi SI}{1+aI}$. Through treatment, the infected individuals recover at a saturated treatment function $\frac{bI}{1+cI}$. The interpretation and values of parameters are described in the Table 1.

Remark 1. *In order to unify the existence of almost periodic solutions for SIR model with saturated and periodic incidence rate and saturated treatment function modelled by ordinary differential equations and their discrete analogues in the form of difference equations, combination of both continuous and discrete and to extend these results to more general time scales, we required much developed theory on time scales. Therefore, the qualitative study of* (1) *on time scales is challenging one.*

Let $\mathcal{C} = \mathcal{C}([-\tau,0]_{\mathbb{T}},\mathbb{R}^3)$ denote the Banach space and assume that the initial conditions of (1) satisfy

$$S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad R(\theta) = \varphi_3(\theta),$$

Table 1: Descriptions and values of parameters in model (1).

Parameter	Parameter description
A	The recruitment rate of the population
a, b, c	The auxiliary parameters
α	The natural mortality rate
d	The baseline contact rate
δ	The magnitude of forcing
γ	The natural recovery rate of the infective
β	The disease-related death rate

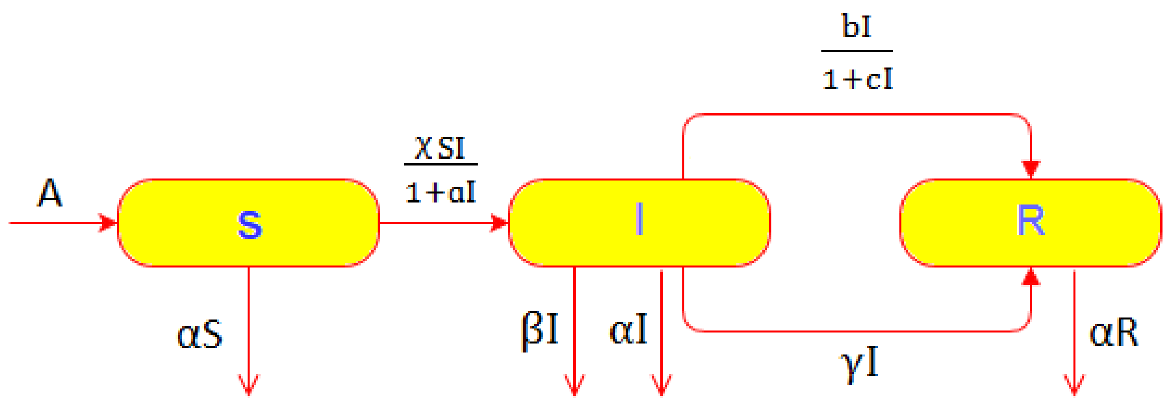

Figure 1: The transmission diagram.

$$\varphi_i(\theta) \geq 0,\ \theta \in [-\tau, 0],\ \varphi_i(0) > 0, i = 1, 2, 3,$$

where $(\varphi_1, \varphi_2, \varphi_3) \in \mathcal{C}$. For a function $f(t)$ defined on $\mathbb{T}$, we denote

$$f^L = \inf\{f(t) : t \in \mathbb{T}\},\ f^U = \sup\{f(t) : t \in \mathbb{T}\}.$$

Throughout the paper we suppose the following hold:

(H_1) $A, a, b, c, \alpha, \beta, \gamma : \mathbb{T} \to [0, \infty]$ are bounded almost periodic functions and satisfy $0 < A^L \leq A(t) \leq A^U$, $0 < a^L \leq a(t) \leq a^U$, $0 < b^L \leq b(t) \leq b^U$, $0 < c^L \leq c(t) \leq c^U$, $0 < \alpha^L \leq \alpha(t) \leq \alpha^U$, $0 < \beta^L \leq \beta(t) \leq \beta^U$, $0 < \gamma^L \leq \gamma(t) \leq \gamma^U$.

Next, we provide some definitions and lemmas which will be useful for later discussions.

Definition 1. *[2] A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined by*

$$\sigma(t) = \inf\{\xi \in \mathbb{T} : \xi > t\},\ \rho(t) = \sup\{\xi \in \mathbb{T} : \xi < t\},\quad \textit{and}\quad \mu(t) = \rho(t) - t,$$

respectively.

- *The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.*
- *A function $g : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.*

Definition 2. *[2] A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Also, we denote the set*

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0, \forall t \in \mathbb{T}\}.$$

Lemma 1. *[8] If $a > 0, b > 0$ and $-b \in \mathcal{R}^+$. Then*

$$\mathtt{u}^\Delta(t) \leq (\geq) a - b\,\mathtt{u}(t),\ \ \mathtt{u}(t) > 0,\ \ t \in [t_0, \infty)_{\mathbb{T}},$$

implies

$$\mathtt{u}(t) \leq (\geq) \frac{a}{b}\Big[1 + \Big(\frac{b\,\mathtt{u}(t_0)}{a} - 1\Big) e_{(-b)}(t, t_0)\Big],\ \ t \in [t_0, \infty)_{\mathbb{T}}.$$

Definition 3. *[10] A time scale $\mathbb{T}$ is called an almost periodic time scale if*

$$\prod = \{\xi \in \mathbb{R} : t + \xi \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}.$$

Definition 4. *[10] Let $\mathbb{T}$ be an almost periodic time scale. Then a function $\mathtt{u} \in \mathcal{C}(\mathbb{T}, \mathbb{R}^n)$ is called an almost periodic function if the ε-translation set of w i.e.,*

$$\mathcal{E}\{\varepsilon, \mathtt{u}\} = \Big\{\xi \in \prod : |\mathtt{u}(t + \xi) - \mathtt{u}(t)| < \varepsilon, \forall t \in \mathbb{T}\Big\},$$

is a relatively dense set in $\mathbb{T}$ for any positive real number ε.

Definition 5. *[10] Let $\mathbb{D}$ be an open set of $\mathbb{R}^n$ and $\mathbb{T}$ be a positive almost periodic time scale. Then a function $\phi \in \mathcal{C}(\mathbb{T} \times \mathbb{D}, \mathbb{R}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $w \in \mathbb{D}$ if the ε-translation set of ϕ*

$$\mathcal{E}\{\varepsilon, \phi, \mathbb{S}\} = \Big\{\xi \in \prod : |\phi(t + \xi, \mathtt{u}) - \phi(t, \mathtt{u})| < \varepsilon, \forall (t, \mathtt{u}) \in \mathbb{T} \times \mathbb{S}\Big\},$$

is a relatively dense set in $\mathbb{T}$ for any positive real number ε, and for each compact subset $\mathbb{S}$ of $\mathbb{D}$. that is, for any given $\varepsilon > 0$ and each compact subset $\mathbb{S}$ of $\mathbb{D}$, there exists a constant $l(\varepsilon, \mathbb{S}) > 0$ such that each interval of length $l(\varepsilon, \mathbb{S})$ contains a $\xi(\varepsilon, \mathbb{S}) \in E\{\varepsilon, \phi, \mathbb{S}\}$ such that

$$|\phi(t + \xi, \mathtt{u}) - \phi(t, \mathtt{u})| < \varepsilon,\ \ \forall (t, \mathtt{u}) \in \mathbb{T} \times \mathbb{S}.$$

Next, consider the system

$$\varpi^\Delta(t) = \mathtt{g}(t, \varpi),\ t \in \mathbb{T}^+, \tag{2}$$

where $\mathtt{g} : \mathbb{T} \times \mathcal{S}_{\mathtt{M}} \to \mathbb{R}$, $\mathcal{S}_{\mathtt{M}} = \{\varpi \in \mathbb{R}^n : \|\varpi\| < \mathtt{M}\}$, $\|\varpi\| = \sup_{t \in \mathbb{T}} |\varpi(t)|$, $\mathtt{g}(t, \varpi)$ is almost periodic in t uniformly for $\varpi \in \mathcal{S}_{\mathtt{M}}$ and is continuous in ϖ. To find the solution of the (2), we consider the product system of (2) as follows:

$$\varpi^\Delta(t) = \mathtt{g}(t, \varpi),\quad \vartheta^\Delta(t) = \mathtt{g}(t, \vartheta),$$

and we have the following lemma.

Lemma 2. *Let $\mathcal{V}(t,\varpi,\vartheta)$ be Lyapunov function defined on $\mathbb{T}^+\times\mathcal{S}_{\mathtt{M}}\times\mathcal{S}_{\mathtt{M}}$ and satisfies the following conditions*

(i) $\mathtt{A}\big(\|\varpi-\vartheta\|\big)\le\mathcal{V}(t,\varpi,\vartheta)\le\mathtt{B}\big(\|\varpi-\vartheta\|\big)$, *where* $\mathtt{A},\mathtt{B}\in\mathcal{P}$,

$$\mathcal{P}=\big\{\mathtt{G}\in\mathcal{C}(\mathbb{R}^+,\mathbb{R}^+):\mathtt{G}(0)=0 \textit{ and } \mathtt{G} \textit{ is increasing}\big\};$$

(ii) $|\mathcal{V}(t,\varpi,\vartheta)-\mathcal{V}(t,\varpi_1,\vartheta_1)|\le\mathcal{L}\big(\|\varpi-\varpi_1\|+\|\vartheta-\vartheta_1\|\big)$, *where* $\mathcal{L}>0$ *is a constant;*

(iii) $\mathcal{D}^+\mathcal{V}^\Delta(t,\varpi,\vartheta)\le-\lambda\mathcal{V}(t,\varpi,\vartheta)$, *where* $\lambda>0,-\lambda\in\mathcal{R}^+$.

Further, if there exists a solution $\varpi(t)\in\mathcal{S}$ of system (2) for $t\in\mathbb{T}^+$, where $\mathbb{S}\subset\mathcal{S}_{\mathtt{M}}$ is a compact set, then there exist a unique almost periodic solution $\mathtt{p}(t)\in\mathcal{S}$ of system (2), which is uniformly asymptotically stable. Also, if $\mathtt{g}(t,\varpi)$ is periodic in t uniformly for $\varpi\in\mathcal{S}_{\mathtt{M}}$, then $\mathtt{p}(t)$ is also periodic.

Proof. Let $\{\ell_n\}$ be a sequence in $\prod$ such that $\ell_n\to+\infty$ as $n\to+\infty$. Suppose that $\psi\in\mathcal{S}$ is a solution of (2) for $t\in\mathtt{T}^+$, then $\psi(t+\ell_n)\in\mathcal{S}$ is a solution of the equation $\varpi^\Delta(t)=\mathtt{g}(t,\varpi)$. Let $\mathtt{U}$ be a compact subset of $\mathbb{T}$. Then, for any $\epsilon>0$, there exists large enough integer $\mathtt{N}(\epsilon)$ such that

$$e_{(-\lambda)}(\ell_k,0)<\frac{\mathtt{A}(\epsilon)}{2\mathtt{B}(2\mathtt{M})},\ \big\|\mathtt{g}(t+\ell_k,\varpi)-\mathtt{g}(t+\ell_m,\varpi)\big\|<\frac{\lambda\mathtt{A}(\epsilon)}{2\mathcal{L}},$$

whenever $m\ge k\ge\mathtt{N}(\epsilon)$. Then from (ii) and (iii), we have

$$\begin{aligned}\mathcal{D}^+\mathcal{V}^\Delta\big(t,\psi(t),\psi(t+\ell_m-\ell_k)\big)\ &\le\ -\lambda\mathcal{V}\big(t,\psi(t),\psi(t+\ell_m-\ell_k)\big)\\&\qquad+\mathcal{L}\big\|\mathtt{g}\big(t+\ell_m-\ell_k,\psi(t+\ell_m-\ell_k)\big)-\mathtt{g}\big(t,\psi(t+\ell_m-\ell_k)\big)\big\|\\&\le\ -\lambda\mathcal{V}\big(t,\psi(t),\psi(t+\ell_m-\ell_k)\big)+\frac{\lambda\mathtt{A}(\epsilon)}{2}.\end{aligned}$$

Next for $m\ge k\ge\mathtt{N}(\epsilon)$, $t\in\mathtt{U}$ and from Lemma 1, we have

$$\begin{aligned}\mathcal{V}\big(t+\ell_k,\psi(t+\ell_k),\psi(t+\ell_m)\big)\ &\le\ e_{(-\lambda)}(t+\ell_k,0)\mathcal{V}\big(0,\psi(0),\psi(\ell_m-\ell_k))\big)\\&\qquad+\frac{\mathtt{A}(\epsilon)}{2}\big(1-e_{(-\lambda)}(t+\ell_k,0)\big)\\&\le\ e_{(-\lambda)}(t+\ell_k,0)\mathcal{V}\big(0,\psi(0),\psi(\ell_m-\ell_k))\big)+\frac{\mathtt{A}(\epsilon)}{2}\\&<\ \frac{\mathtt{A}(\epsilon)}{2\mathtt{B}(2\mathtt{M})}\mathtt{B}(2\mathtt{M})+\frac{\mathtt{A}(\epsilon)}{2}=\mathtt{A}(\epsilon).\end{aligned}$$

By (i), for $m\ge k\ge\mathtt{N}(\epsilon)$ and $t\in\mathtt{U}$, we get $\|\psi(t+\ell_m)-\psi(t+\ell_k)\|<\epsilon$, which shows that $\psi(t)$ is asymptotically almost periodic. Then, $\psi(t)$ can be written as $\psi(t)=\mathtt{p}(t)+\mathtt{r}(t)$, where $\mathtt{p}(t)$ is almost periodic and $\mathtt{r}(t)\to0$ as $t\to0$. Thus, $\mathtt{p}(t)\in\mathcal{S}$ is an almost periodic solution of (2). Further, it can be proved easily that $\mathtt{p}(t)$ is uniformly asymptotically stable and every solution in $\mathcal{S}_{\mathtt{M}}$ tends to $\mathtt{p}(t)$, which means $\mathtt{p}(t)$ is unique. Moreover, if $\mathtt{g}(t,\varpi)$ is ω-periodic in t uniformly for $\varpi\in\mathcal{S}_{\mathtt{M}}$, $\mathtt{p}(t+\omega)\in\mathcal{S}$ is also a solution. By the uniqueness, we have $\mathtt{p}(t+\omega)=\mathtt{p}(t)$. This completes the proof. □

3 Permanence of solutions

In this section, we derive sufficient conditions for system (1) to be permanent.

Definition 6. *System (1) is said to be permanent if there are positive constants k, K such that*

$$k \le \liminf_{t\to\infty} S(t) \le \limsup_{t\to\infty} S(t) \le K,\ \ k \le \liminf_{t\to\infty} I(t) \ \le \limsup_{t\to\infty} I(t) \le K,$$
$$k \le \liminf_{t\to\infty} R(t) \le \limsup_{t\to\infty} R(t) \le K,$$

for any solution $(S(t), I(t), R(t))$ of system (1).

Lemma 3. *Assume that $(S(t), I(t), R(t))$ be a positive solution of system (1). If $-\alpha^L$, $-(\alpha^L + \beta^L + \gamma^L) \in \mathcal{R}^+$, there exist $T_3 > 0$ and $K > 0$ such that $S(t) \le K$, $I(t) \le K$, $R(t) \le K$ for $t \in [T_3, \infty)_{\mathbb{T}}$.*

Proof. Assume that $(S(t), I(t), R(t))$ be any positive solution of system (1). It follows from the first equation of system (1) that

$$S^{\Delta}(t) \le A(t) - \alpha(t)S(t) \le A^U - \alpha^L S(t).$$

Therefore, by Lemma 1, for arbitrary small $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$S(t) \le \frac{A^U}{\alpha^L} + \varepsilon := K_1, \quad t \in [T_1, \infty)_{\mathbb{T}}. \tag{3}$$

Next, from the second equation of system (1) and (3), for $t \in [T_1, \infty)$,

$$\begin{aligned} I^{\Delta}(t) &\le \frac{\chi(t)S(t)I(t-\tau)}{1 + a(t)I(t-\tau)} - [\alpha(t) + \beta(t) + \gamma(t)]I(t) \\ &\le \frac{\chi(t)S(t)}{a(t)} - [\alpha(t) + \beta(t) + \gamma(t)]I(t) \le \frac{\chi^U K_1}{a^L} - [\alpha^L + \beta^L + \gamma^L]I(t). \end{aligned}$$

By Lemma 1, for arbitrary small $\varepsilon > 0$, there exists a $T_2 > T_1$ such that

$$I(t) \le \frac{\chi^U K_1}{a^L(\alpha^L + \beta^L + \gamma^L)} + \varepsilon := K_2, \quad t \in [T_2, \infty)_{\mathbb{T}}. \tag{4}$$

Finally, from the last equation of system (1) and (4), for $t \in [T_2, \infty]$,

$$\begin{aligned} R^{\Delta}(t) &= \gamma(t)I(t) + \frac{b(t)I(t)}{1 + c(t)I(t)} - \alpha(t)R(t) \\ &\le \gamma(t)I(t) + \frac{b(t)}{c(t)} - \alpha(t)R(t) \le \left[\gamma^U K_2 + \frac{b^U}{c^L}\right] - \alpha^L R(t) \end{aligned}$$

By Lemma 1, for arbitrary small $\varepsilon > 0$, there exists a $T_3 > T_2$ such that

$$R(t) \le \frac{c^L\gamma^U K_2 + b^U}{c^L \alpha^L} + \varepsilon := K_3, \quad t \in [T_3, \infty).$$

Let $K > \max\{K_1, K_2, K_3\}$, then

$$S(t) \le K,\ I(t) \le K,\ R(t) \le K \quad for \quad t \in [T_3, \infty)_{\mathbb{T}}.$$

This completes the proof. □

Lemma 4. *Assume that $(S(t), I(t), R(t))$ be a positive solution of system (1). If*

$$A^L c^L > b^U, \ \frac{A^L c^L - b^U}{c^L(\alpha^U + \beta^U + \gamma^U)} > K,$$

and $-\alpha^U$, $-(\alpha^U + \beta^U + \gamma^U) \in \mathcal{R}^+$, then there exist $T_6 > 0$ and $k > 0$ such that

$$S(t) \geq k, \ I(t) \geq k, \ R(t) \geq k \quad for \quad t \in [T_6, \infty)_{\mathbb{T}}.$$

Proof. Assume that $(S(t), I(t), R(t))$ be any positive solution of system (1). It follows from the first equation of system (1) and Lemma 3 that, for $t \in [T_3, \infty)$,

$$S^\Delta(t) \geq A^L - \alpha^U S(t) - \frac{\chi^U K S(t)}{1 + a^L K} \geq A^L - \Big[\frac{\alpha^U + (\alpha^U a^L + \chi^U)K}{1 + a^L K}\Big] S(t).$$

Therefore, by Lemma 1, for arbitrary small $\varepsilon > 0$, there exists a $T_4 > 0$ such that

$$S(t) \geq \frac{A^L(1 + a^L K)}{\alpha^U + (\alpha^U a^L + \chi^U)K} + \varepsilon := k_1, \quad t \in [T_4, \infty)_{\mathbb{T}}. \tag{5}$$

Next, define $P(t) = S(t) + I(t), \ t \in [T_4, \infty)_{\mathbb{T}}$, and calculating the delta derivative of $P(t)$ along the solutions of (1), we have

$$\begin{aligned} P^\Delta(t) &= A(t) - \alpha(t)S(t) - [\alpha(t) + \beta(t) + \gamma(t)]I(t) - \frac{b(t)I(t)}{1 + c(t)I(t)} \\ &\geq A(t) - [\alpha(t) + \beta(t) + \gamma(t)]S(t) - [\alpha(t) + \beta(t) + \gamma(t)]I(t) - \frac{b(t)I(t)}{1 + c(t)I(t)} \\ &\geq A(t) - [\alpha(t) + \beta(t) + \gamma(t)](I(t) + S(t)) - \frac{b(t)}{c(t)} \\ &\geq \Big[\frac{A^L c^L - b^U}{c^L}\Big] - [\alpha^U + \beta^U + \gamma^U]P(t). \end{aligned} \tag{6}$$

By Lemma 1, for arbitrary small $\varepsilon > 0$, there exists a $T_5 > T_4$, it follows from (6) that, for $t \in [T_5, \infty)$,

$$P(t) \geq \frac{A^L c^L - b^U}{c^L(\alpha^U + \beta^U + \gamma^U)} + \varepsilon.$$

From the definition of $P(t)$ and Lemma 3, it follows that

$$I(t) \geq \frac{A^L c^L - b^U}{c^L(\alpha^U + \beta^U + \gamma^U)} - K + \varepsilon := k_2. \tag{7}$$

By the third equation of the system (1), Lemma 3 and (7), for $t \in [T_5, \infty)$, we have

$$R^\Delta(t) \geq \Big[\gamma^L k_2 + \frac{b^L k_2}{1 + c^U K}\Big] - \alpha^U R(t). \tag{8}$$

By Lemma 1, for arbitrary small $\varepsilon > 0$, there exists a $T_6 > T_5$, it follows from (8) that, for $t \in [T_6, \infty)$,

$$R(t) \geq \frac{[\gamma^L(1 + c^U K) + b^L]k_2}{\alpha^U(1 + c^U K)} + \varepsilon := k_3.$$

Let $0 < k < \min\{k_1, k_2, k_3\}$, then $S(t) \geq k, \ I(t) \geq k, \ R(t) \geq k \ for \ t \in [T_6, \infty)_{\mathbb{T}}$. □

Theorem 1. *Assume that the conditions of Lemma 3 and Lemma 4 hold, then system (1) is permanent.*

Proof. Together with Lemma 3 and 4, we can obtain desired result. □

Define

$$\Omega =\Big\{ \big(S(t), I(t), R(t)\big) : \big(S(t), I(t), R(t)\big) \text{be a solution of } (1) \text{ and } 0 < s_* \le S(t) \le s^*,\ 0 < i_* \le I(t) \le i^*,\ 0 < r_* \le R(t) \le r^* \Big\}.$$

It is clear that Ω is invariant set of system (1).

Lemma 5. *If hypothesis of Lemmas 3 and 4 holds. Then $\Omega \neq \emptyset$.*

Proof. It can be easily proved. So, we omit it here. □

4 Uniform asymptotic stability

In this section, we establish sufficient conditions for the existence and uniform asymptotic stability of the unique positive almost periodic solution to system (1).

Theorem 2. *If (H_1) and the following holds:*

(H_2) *$\eta > 0$ and $-\eta \in \mathcal{R}^+$, where $\eta = \min\{\eta_1, \eta_2, \alpha^L\}$ where*

$$\eta_1 = \alpha^L + \frac{a^L\chi^L i_*^2 + \chi^L i_*}{(1 + a^U i^*)^2} - \frac{a^U\chi^U i^{*2} + \chi^U i^*}{(1 + a^L i_*)^2},$$

$$\eta_2 = (\alpha^L + \beta^L + \gamma^L) + \frac{b^L}{(1 + c^U i^*)^2} + \frac{\chi^L s_*}{(1 + a^U i^*)^2} - \frac{b^U}{(1 + c^L i_*)^2} - \frac{\chi^U s^*}{(1 + a^L i_*)^2} - \gamma^U$$

then the dynamic system (1) has a unique almost periodic solution $\big(S(t), I(t), R(t)\big) \in \Omega$ and is uniformly asymptotically stable.

Proof. According to Theorem 1 and Lemma 5, every solution $\big(S(t), I(t), R(t)\big)$ of system (1) satisfies $s_* \le S(t) \le s^*$, $i_* \le I(t) \le i^*$, $r_* \le R(t) \le r^*$. Hence, $|S(t)| \le A_i$, $|I(t)| \le B_i$, $|R(t)| \le C_i$ where $A_i = \max\{|s_*|, |s^*|\}$, $B_i = \max\{|i_*|, |i^*|\}$ and and $C_i = \max\{|r_*|, |r^*|\}$.

$$\text{Denote } \ \|\big(S(t), I(t), R(t)\big)\| = \sup_{t\in\mathbb{T}^+} |S(t)| + \sup_{t\in\mathbb{T}^+} |I(t)| + \sup_{t\in\mathbb{T}^+} |R(t)|.$$

Suppose that $X = \big(S(t), I(t), R(t)\big)$, $\hat{X} = \big(\hat{S}(t), \hat{I}(t), \hat{R}(t)\big)$ are any two positive solutions of system (1), then

$$\|X\| \le A + B + C \quad \text{and} \quad \|\hat{X}\| \le A + B + C.$$

In view of system (1), we have

$$\left.\begin{aligned} \hat{S}^\Delta(t) &= A(t) - \alpha(t)\hat{S}(t) - \frac{\chi(t)\hat{S}(t)\hat{I}(t-\tau)}{1 + a(t)\hat{I}(t-\tau)}, \\ \hat{I}^\Delta(t) &= \frac{\chi(t)\hat{S}(t)\hat{I}(t-\tau)}{1 + a(t)\hat{I}(t-\tau)} - [\alpha(t) + \beta(t) + \gamma(t)]\hat{I}(t) - \frac{b(t)\hat{I}(t)}{1 + c(t)\hat{I}(t)}, \\ \hat{R}^\Delta(t) &= \gamma(t)\hat{I}(t) + \frac{b(t)\hat{I}(t)}{1 + c(t)\hat{I}(t)} - \alpha(t)\hat{R}(t). \end{aligned}\right\} \tag{9}$$

Define the Lyapunov function $\mathcal{V}(t, X, \hat{X})$ on $\mathbb{T}^+ \times \Omega \times \Omega$ as

$$\mathcal{V}(t, X, \hat{X}) = |S(t) - \hat{S}(t)| + |I(t) - \hat{I}(t)| + |R(t) - \hat{R}(t)|.$$

Define the norm

$$\|X(t) - \hat{X}(t)\| = \sup_{t\in\mathbb{T}^+} |S(t) - \hat{S}(t)| + \sup_{t\in\mathbb{T}^+} |I(t) - \hat{I}(t)| + \sup_{t\in\mathbb{T}^+} |R(t) - \hat{R}(t)|.$$

It is easy to see that there exist two constants $l > 0$, $m > 0$ such that

$$l\|X(t) - \hat{X}(t)\| \le V(t, X, \hat{X}) \le m\|X(t) - \hat{X}(t)\|.$$

Let $\mathtt{A}, \mathtt{B} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$, $\mathtt{A}(x) = lx$, $\mathtt{B}(x) = mx$, then the assumption (i) of Lemma 2 is satisfied. On the other hand, we have

$$\begin{aligned}
\big|\mathcal{V}\big(t, X(t), \hat{X}(t)\big) - \mathcal{V}\big(t, X^*(t), \hat{X}^*(t)\big)\big| &= \big||S(t) - \hat{S}(t)| + |I(t) - \hat{I}(t)| + |R(t) - \hat{R}(t)| \\
&\quad - |S^*(t) - \hat{S}^*(t)| - |I^*(t) - \hat{I}^*(t)| - |R^*(t) - \hat{R}^*(t)|\big| \\
&\le \big|S(t) - S^*(t)\big| + \big|I(t) - I^*(t)\big| + \big|R(t) - R^*(t)\big| \\
&\quad + \big|\hat{S}(t) - \hat{S}^*(t)\big| + \big|\hat{I}(t) - \hat{I}^*(t)\big| + + \big|\hat{R}(t) - \hat{R}^*(t)\big| \\
&= L\big[\|X - X^*(t)\| + \|\hat{X}(t) - \hat{X}^*(t)\|\big],
\end{aligned}$$

where $\mathcal{L} = 1$, so condition (ii) of Lemma 2 is satisfied. Now consider a function $\mathcal{W}(t) = \mathcal{W}_1(t) + \mathcal{W}_2(t) + \mathcal{W}_3(t)$, where

$$\mathcal{W}_1(t) = |S(t) - \hat{S}(t)|, \ \mathcal{W}_2(t) = |I(t) - \hat{I}(t)|,$$

and

$$\mathcal{W}_3(t) = |R(t) - \hat{R}(t)| + \left[\frac{\chi^U s^*}{(1 + a^L i_*)^2} - \frac{\chi^L s_*}{(1 + a^U i^*)^2}\right] \int_{t-\tau}^{t} |I(t) - \hat{I}(t)|\Delta t.$$

For $t \in \mathbb{T}^+$, calculating the delta derivative $D^+\mathcal{W}_1(t)^\Delta$ of $W_1(t)$ along system (9), we get

$$\begin{aligned}
D^+\mathcal{W}_1^\Delta(t) &\le sign\big(S(\sigma(t)) - \hat{S}(\sigma(t))\big)\big[S(t) - \hat{S}(t)\big]^\Delta \\
&\le sign\big(S(\sigma(t)) - \hat{S}(\sigma(t))\big)\Bigg[-\alpha(t)\big(S(t) - \hat{S}(t)\big) - \frac{\chi(t)S(t)I(t-\tau)}{1 + a(t)I(t-\tau)} \\
&\qquad + \frac{\chi(t)\hat{S}(t)\hat{I}(t-\tau)}{1 + a(t)\hat{I}(t-\tau)}\Bigg] \\
&\le sign\big(S(\sigma(t)) - \hat{S}(\sigma(t))\big)\Bigg[-\alpha(t)\big(S(t) - \hat{S}(t)\big) \\
&\qquad - \frac{(a(t)\chi(t)I(t-\tau)\hat{I}(t-\tau) + \chi(t)\hat{I}(t-\tau))}{(1 + a(t)I(t-\tau))(1 + a(t)\hat{I}(t-\tau))}\big(S(t) - \hat{S}(t)\big) \\
&\qquad - \frac{\chi(t)S(t)}{(1 + a(t)I(t-\tau))(1 + a(t)\hat{I}(t-\tau))}\big(I(t-\tau) - \hat{I}(t-\tau)\big)\Bigg] \\
&\le -\left[\alpha^L + \frac{a^L\chi^L i_*^2 + \chi^L i_*}{(1 + a^U i^*)^2}\right]\big|S(t) - \hat{S}(t)\big| - \frac{\chi^L s_*}{(1 + a^U i^*)^2}\big|I(t-\tau) - \hat{I}(t-\tau)\big|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
D^+\mathcal{W}_2^\Delta(t) &\le sign\big(I(\sigma(t))-\hat{I}(\sigma(t))\big)\big[I(t)-\hat{I}(t)\big]^\Delta\\
&\le sign\big(I(\sigma(t))-\hat{I}(\sigma(t))\big)\bigg[\frac{\chi(t)S(t)I(t-\tau)}{1+a(t)I(t-\tau)}-\frac{\chi(t)\hat{S}(t)\hat{I}(t-\tau)}{1+a(t)\hat{I}(t-\tau)}\\
&\quad-\big(\alpha(t)+\beta(t)+\gamma(t)\big)\big(I(t)-\hat{I}(t)\big)-\frac{b(t)I(t)}{1+c(t)I(t)}+\frac{b(t)\hat{I}(t)}{1+c(t)\hat{I}(t)}\bigg]\\
&\le sign\big(I(\sigma(t))-\hat{I}(\sigma(t))\big)\bigg[\frac{(a(t)\chi(t)I(t-\tau)\hat{I}(t-\tau)+\chi(t)\hat{I}(t-\tau))}{(1+a(t)I(t-\tau))(1+a(t)\hat{I}(t-\tau))}\big(S(t)-\hat{S}(t)\big)\\
&\quad+\frac{\chi(t)S(t)\big(I(t-\tau)-\hat{I}(t-\tau)\big)}{(1+a(t)I(t-\tau))(1+a(t)\hat{I}(t-\tau))}-\big(\alpha(t)+\beta(t)+\gamma(t)\big)\big(I(t)-\hat{I}(t)\big)\\
&\qquad-\frac{b(t)}{(1+c(t)I(t))(1+c(t)\hat{I}(t))}\big(I(t)-\hat{I}(t)\big)\bigg]\\
&\le\bigg[\frac{a^U\chi^Ui^{*2}+\chi^Ui^*}{(1+a^Li_*)^2}\bigg]\big|S(t)-\hat{S}(t)\big|+\frac{\chi^Us^*}{(1+a^Li_*)^2}\big|I(t-\tau)-\hat{I}(t-\tau)\big|\\
&\quad-\bigg[(\alpha^L+\beta^L+\gamma^L)+\frac{b^L}{(1+c^Ui^*)^2}\bigg]\big|I(t)-\hat{I}(t)\big|,
\end{aligned}$$

and

$$\begin{aligned}
D^+\mathcal{W}_3^\Delta(t) &\le sign\big(R(\sigma(t))-\hat{R}(\sigma(t))\big)\big[R(t)-\hat{R}(t)\big]^\Delta\\
&\quad+\bigg[\frac{\chi^Us^*}{(1+a^Li_*)^2}-\frac{\chi^Ls_*}{(1+a^Ui^*)^2}\bigg]\Big[|I(t)-\hat{I}(t)|-|I(t-\tau)-\hat{I}(t-\tau)|\Big]\\
&\le sign\big(R(\sigma(t))-\hat{R}(\sigma(t))\big)\bigg[\gamma(t)\big(I(t)-\hat{I}(t)\big)\\
&\qquad+\frac{b(t)I(t)}{1+c(t)I(t)}-\frac{b(t)\hat{I}(t)}{1+c(t)\hat{I}(t)}-\alpha(t)\big(R(t)-\hat{R}(t)\big)\bigg]\\
&\le\bigg[\gamma^U+\frac{b^U}{(1+c^Li_*)^2}+\frac{\chi^Us^*}{(1+a^Li_*)^2}-\frac{\chi^Ls_*}{(1+a^Ui^*)^2}\bigg]\big|I(t)-\hat{I}(t)\big|\\
&\quad-\bigg[\frac{\chi^Us^*}{(1+a^Li_*)^2}-\frac{\chi^Ls_*}{(1+a^Ui^*)^2}\bigg]\big|I(t-\tau)-\hat{I}(t-\tau)\big|-\alpha^L\big|R(t)-\hat{R}(t)\big|.
\end{aligned}$$

Since $\mathcal{V}(t)\le\mathcal{W}(t)$ for $t\in\mathbb{T}^+$ and by assumption (H_2), it follows that

$$\begin{aligned}
D^+\big(\mathcal{V}(t)\big)^\Delta &\le D^+\big(\mathcal{W}(t)\big)^\Delta=D^+\big(\mathcal{V}_1(t)+\mathcal{V}_2(t)+\mathcal{V}_3(t)\big)^\Delta\\
&\le-\bigg[\alpha^L+\frac{a^L\chi^Li_*^2+\chi^Li_*}{(1+a^Ui^*)^2}-\frac{a^U\chi^Ui^{*2}+\chi^Ui^*}{(1+a^Li_*)^2}\bigg]\big|S(t)-\hat{S}(t)\big|\\
&\quad-\bigg[(\alpha^L+\beta^L+\gamma^L)+\frac{b^L}{(1+c^Ui^*)^2}+\frac{\chi^Ls_*}{(1+a^Ui^*)^2}\\
&\qquad-\frac{b^U}{(1+c^Li_*)^2}-\frac{\chi^Us^*}{(1+a^Li_*)^2}-\gamma^U\bigg]\big|I(t)-\hat{I}(t)\big|-\alpha^L\big|R(t)-\hat{R}(t)\big|
\end{aligned}$$

$$\leq -\eta_1 \big|S(t) - \hat{S}(t)\big| - \eta_2 \big|I(t) - \hat{I}(t)\big| - \alpha^L \big|R(t) - \hat{R}(t)\big|$$
$$\leq -\eta V(t).$$

By (H_2), we see that Condition (iii) of Lemma 2 is satisfied. Hence, according to Lemma 2, there exists a unique uniformly asymptotically stable almost periodic solution $\big(S(t), I(t), R(t)\big)$ of system (1) and $\big(S(t), I(t), R(t)\big) \in \Omega$. The proof is complete. □

5 Numerical Simulations

In this section we provide some numerical simulations to illustrate the results obtained in the previous sections.

Example 1. Consider the dynamic susceptible-infected-recovered (SIR) model with saturated treatment on time scale $\mathbb{T}^+$:

$$\left.\begin{aligned}
S^{\Delta}(t) &= A(t) - \alpha(t)S(t) - \frac{\chi(t)S(t)I(t-0.004)}{1+a(t)I(t-0.004)},\\
I^{\Delta}(t) &= \frac{\chi(t)S(t)I(t-0.004)}{1+a(t)I(t-0.004)} - [\alpha(t)+\beta(t)+\gamma(t)]I(t) - \frac{b(t)I(t)}{1+c(t)I(t)},\\
R^{\Delta}(t) &= \gamma(t)I(t) + \frac{b(t)I(t)}{1+c(t)I(t)} - \alpha(t)R(t),
\end{aligned}\right\} \tag{10}$$

where $A(t) = 0.5 + |\sin\sqrt{2}t|$, $\alpha = 5 + |\cos\sqrt{5}t|$, $\beta = 0.1$, $\gamma = 0.02 + |\sin \pi t|$, $a(t) = 0.5$, $b(t) = 0.1$, $c(t) = 0.05$, $\chi(t) = 2\times 10^{-3} + 2\times 10^{-4}\sin((\pi/6)t)$. By direct calculations, we obtain $s^* = 0.3$, $i^* = 0.0004327868853$, $r^* = 0.1715168599$, and $s_* = 0.3446911567$, $i_* = 0.6900990099$, $r_* = 0.01363200507$. Let $K = 0.4$. Then $K > \max\{s^*, i^*, r^*\} = 0.3$, $A^L c^L - b^U = 0.25 > 0$ and

$$\frac{A^L c^L - b^U}{c^L(\alpha^U + \beta^U + \gamma^U)} = 0.9900990099 > K.$$

Therefore, by Theorem 1, (10) is permanent.

Now by these values, we get $\eta_1 = 5.001855105, \eta_2 = 4.106533372$. S0, $\eta = \min\{\eta_1, \eta_2, \alpha^L\} = \eta_2 > 0$. By Theorem 2, (10) has a unique almost periodic solution $\big(S(t), I(t), R(t)\big) \in \Omega$ and is uniformly asymptotically stable. From Fig. 2-5, we can see that for system (10), there exists a positive almost periodic solution denoted by $\big(S^*(t), I^*(t), R^*(t)\big)$. Moreover, Fig. 6-8 shows that any positive solution $\big(S(t), I(t), R(t)\big)$ tends to the above almost periodic solution $\big(S^*(t), I^*(t), R^*(t)\big)$.

In addition, from Fig. 1-8 when the initial conditions are different, the disease will tend toward different periodic solutions. So, besides related control measures, we can change the initial condition to change the tendency of the disease.

6 Conclusion

In the real nature, due to the interference of various factors, such as seasonal effects of the weather, food supplies, and mating habits, the coefficients of most of the systems are approximate to certain periodic functions. However, with the uncertainty of the interferences, the

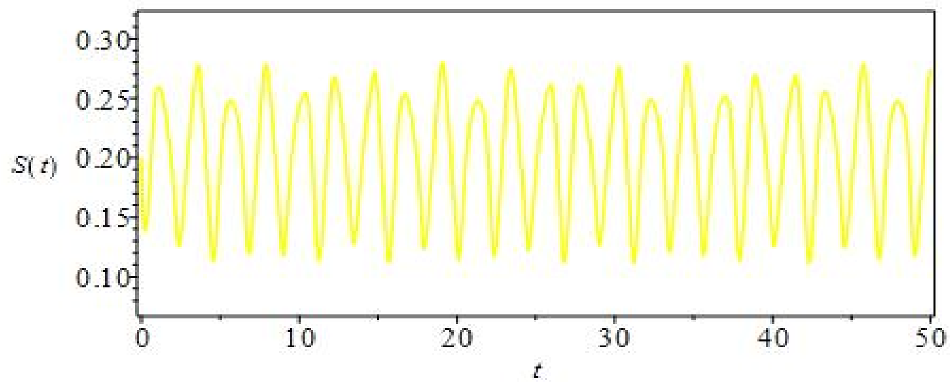

Figure 2: Positive almost periodic solution of system (10). Time series of $S(t)$ with initial value $S(0) = 0.12$ and t over $[0, 50]$.

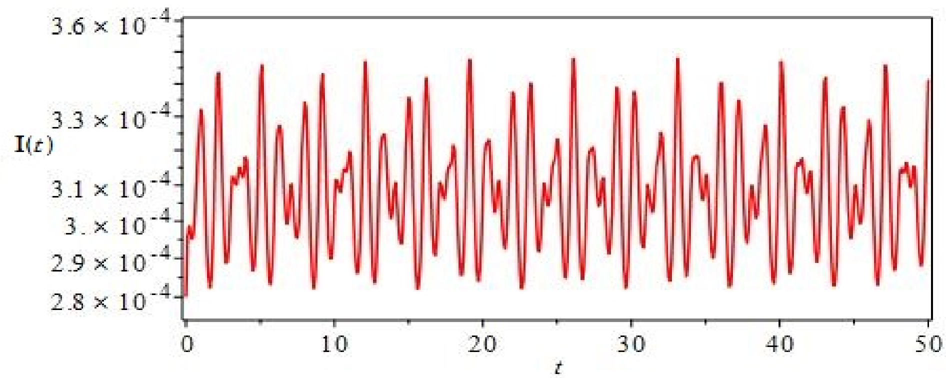

Figure 3: Positive almost periodic solution of system (10). Time series of $I(t)$ with initial value $I(0) = 0.00028$ and t over $[0, 50]$.

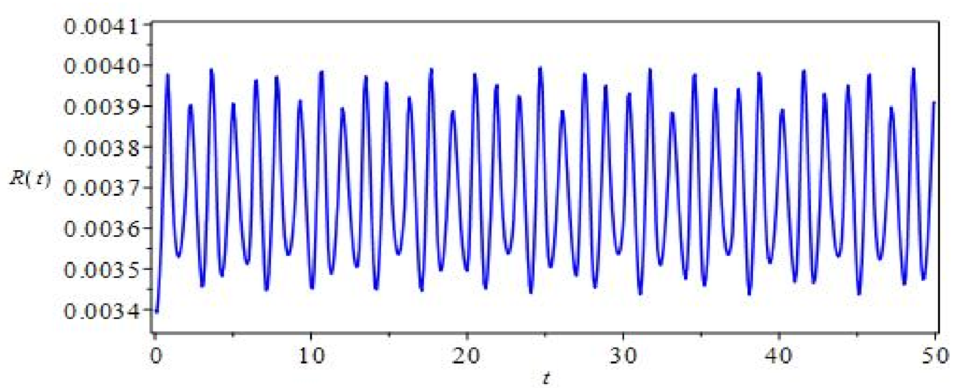

Figure 4: Positive almost periodic solution of system (10). Time series of $R(t)$ with initial value $R(0) = 0.0034$ and t over $[0, 50]$.

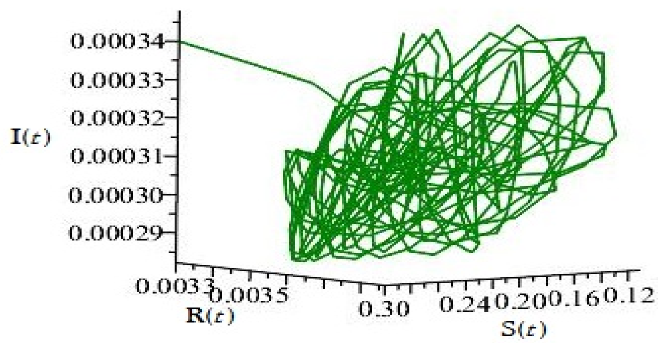

Figure 5: Positive almost periodic solution of system (10). Time series of $\big(S(t), I(t), R(t)\big)$ with initial value $\big(S(0), I(0), R(0)\big) = (0.3, 0.00034, 0.0033)$.

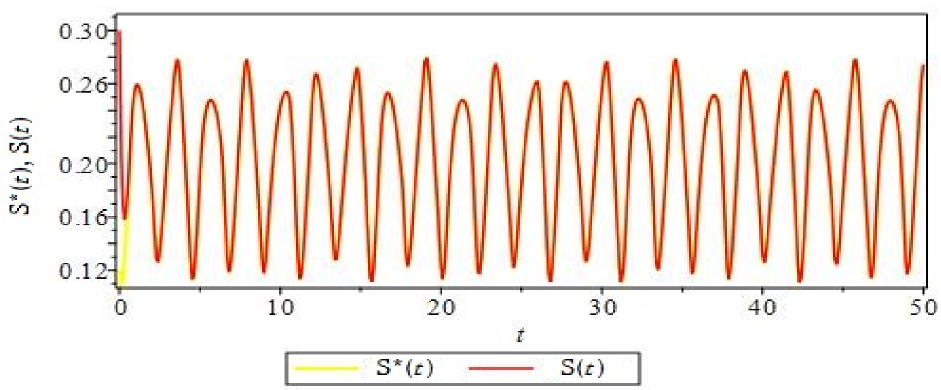

Figure 6: Uniformly asymptotic stability of system (10). Time series of $S^*(t)$ and $S(t)$ with initial values $S^*(0) = 0.12$, $S(0) = 0.3$, and t over $[0, 50]$.

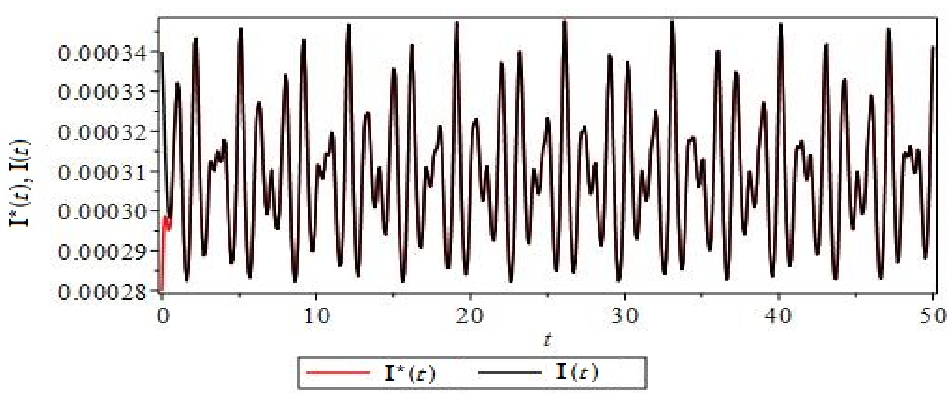

Figure 7: Uniformly asymptotic stability of system (10). Time series of $I^*(t)$ and $I(t)$ with initial values $I^*(0) = 0.00034$, $I(0) = 0.00028$, and t over $[0, 50]$.

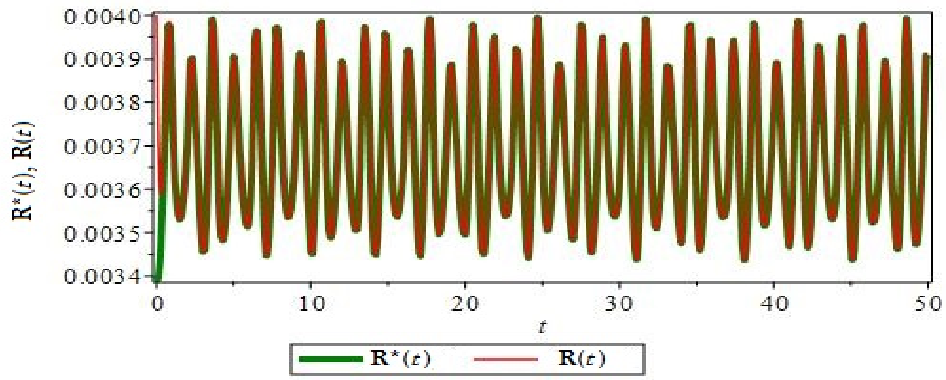

Figure 8: Uniformly asymptotic stability of system (10). Time series of $R^*(t)$ and $R(t)$ with initial values $R^*(0) = 0.004$, $R(0) = 0.0034$, and t over $[0, 50]$.

coefficients of the systems are not strictly periodic. Therefore, almost periodicity is a more common phenomenon than strict periodicity. Hence, we dealt with the almost periodic dynamics of a time-delayed SIR epidemic model with saturated treatment on time scales. By establishing some dynamic inequalities on time scales, a permanence result for the model is obtained. Furthermore, by means of the almost periodic functional theory on time scales and Lyapunov functional, some criteria is obtained for the existence, uniqueness and uniform asymptotic stability of almost periodic solutions of the model. Thus, the mathematical results in the paper are quite new, and it may have some application value and practical significance for the prediction and control strategy for corresponding ecoepidemic systems. Our future research will focus on the stability of the periodic solution and apply our mathematical methods to the research of special diseases.

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