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Lower bound approximation of nonlinear basket option with jump-diffusion

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Abstract. We extend the method presented by Xu and Zheng (Int. J. Theor. Appl. Finance 17 (2014) 21–36) for the general case. We develop a numerical-analytic formula for pricing nonlinear basket options with jump-diffusion model. We derive an easily computed method by using the asymptotic expansion to find the approximate value of the lower bound of nonlinear European basket call prices since a nonlinear basket option is generally not closed-form. We use Split Step Backward Euler and Compensated Split Step Backward Euler methods with Monte Carlo simulation to check the validity of the presented method.

Keywords: Basket option, nonlinear stochastic differential equations, Poisson process, Split Step Backward Euler method.
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1 Introduction

In general, there is no closed-form for stochastic differential equations with Jump (SDEwJ), and no explicit formula can be found to solve a differential equation, although attempts have been made to calculate this explicit form, which can be seen in [2]. As stated in [5–7], most of the time, there is no solution to a nonlinear differential equation, and it is evident, there is no closed-form solution to a nonlinear basket option. In the following, we introduce an analytical-approximate method for solving a non-linear equation basket and then examine the strengths and weaknesses of the method.

The price of a portfolio at maturity T, for each $S_i(T) \in R$, is

$$S(T) = \sum_{i=1}^{n} w_i S_i(T), \tag{1}$$

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where w_i are weights (generally positive). Also, the portfolio payoff at time T for the exercise price K is given by

$$E(S(T) - K)^+. \tag{2}$$

One of the important uses of formula (2) is the spread option, where the payoff involves the difference of two or more underlying assets [2, 3]. Moreover, if the average underlying price determines the payoff over some predicted period time, then the basket option is an Asian claim.

By using of a conditioning variable and conditional moment matching, Curran [4] proposed a method based on conditioning on the geometric mean. Assume that Λ is a random variable correlated with $S(T)$, where $S(T) \geq K$, and $\Lambda \geq d_\Lambda$, for some constant d_Λ. The option price is decomposed into two parts:

$$E[(S(T) - K)^+] = E[(S(T) - K)1_{\Lambda \geq d\Lambda}] + E[(S(T) - K)1_{\Lambda < d\Lambda}]. \tag{3}$$

The first part of the formula is calculated accurately by selecting the geometric average, but the second part cannot be calculated explicitly, and we use the conditional moment matching method to calculate the second part approximately. Of course, similar conditional arguments are used in [10], which derive the lower and upper bounds of an Asian option. Thompson [14] and Beisser [1] developed the idea of [10] into basket options and investigated the bound of the approximation in (3) that can be computed in closed-form in the lognormal framework.

Some authors using moment matching method attempted for approximating the basket. The main idea is to approximate $S(T)$ via $S^A(T)$, where $S^A(T)$ is a random variable with a suitable distribution. Ju [8] considered a Taylor expansion of the ratio of the arithmetic average and lognormal random variable. Zhou and Wang [18] by a log-extended-skew-normal distribution approximated the basket distribution. Similar to [10, 13], Xu and Zheng [15, 17, 18] showed a lower bound approximation. They also calculated the lower bound in a special jump-diffusion model with some constant volatility and several types of Poisson jumps. Also, they calculated an asymptotic expansion with a variance approximation to achieve the lower bound of basket option for the local volatility jump-diffusion models; see [16]. Studies have been done on the lower bound of basket option approximation methods have some weaknesses, such as follows:

1. Many methods cannot deal with the basket spread option case, because they require the positivity of the basket options weights.
2. Most studies have been done on the lognormal case rather than the general pricing methods.
3. In practice, solving problems in n-dimensional spaces is needed, however analytical formulas are available in the non-Gaussian case, which is of little help for applications involving a large number of assets.
4. Our method can compute very large classes of stochastic dynamics, such as mean reverting and nonlinear models. Therefore our procedure can particularly solve higher dimensional problems.

We must use a simple MATLAB code for the general method when the basket dimension is large. Hence in general, all existing methods face unaffordable computational costs. We test the lower bounds approximation on different models, including linear and nonlinear examples. Numerical examples were discussed and benchmarked by the Split Step Backward Euler (SSBE) and Compensated Split Step Backward Euler (CSSBE) methods [12].

The article is outlined as follows. Section 2 formulates the nonlinear basket option with a jump-diffusion model in asymptotic expansion. In Section 3, we introduce the SSBE and CSSBE methods from [7] to check the lower bound approximation. Finally, Section 4 presents numerical experiments.

2 Lower Bound Approximation (LBA)

2.1 Assumption and the Model

In this section, we present the nonlinear European basket options in a general jump-diffusion model. For $\epsilon \in [0,1]$, we define

$$dS_i^\epsilon(t) = \epsilon f_i(t, S_i^\epsilon(t^-))dt + \epsilon g_i(t, S_i^\epsilon(t^-))dW_i(t) + \epsilon h_i(t, S_i^\epsilon(t^-))dN(t), \tag{4}$$

with initial condition $S_i^\epsilon(0) = S_i(0)$ and $S_i^1(T) = S_i(T)$, where $i = 1, 2, \ldots, n$. We use the asymptotic expansion to parameterize asset price processes and apply the conditional expectation results of multiple Wiener–Ito integrals directly from [10] to approximate a lower bound for the nonlinear basket call option values in a jump-diffusion model (4). In what follows, we need some useful assumption to define the LBA.

Assumption 2.1. We need the following notations to define the LBA method:

1. Jump intensity $\lambda > 0$, and $t \geqslant 0$.

2. $S_i^{(k)}(t) := \dfrac{\partial^k S_i^\epsilon(t)}{\partial \epsilon^k}|_{\epsilon=0}, \quad i = 1, 2, \ldots, n.$

3. $f_i^{(k)}(t) := \left.\dfrac{\partial^k f_i(t, S^\epsilon(t))}{\partial (S_i^\epsilon)^k}\right|_{\epsilon=0}, \quad g_i^{(k)}(t) := \left.\dfrac{\partial^k g_i(t, S^\epsilon(t))}{\partial (S_i^\epsilon)^k}\right|_{\epsilon=0},$
   $h_i^{(k)}(t) := \left.\dfrac{\partial^k h_i(t, S^\epsilon(t))}{\partial (S_i^\epsilon)^k}\right|_{\epsilon=0}$ for $k = 1, 2, \ldots$.

4. $S_i^{(0)}(t) = S_i^0(t) = S_i(0)$.

5. $f_i^{(0)}(t) = f_i(0, S_i(0))$, $g_i^{(0)}(t) = g_i(0, S_i(0))$, $h_i^{(0)}(t) = h_i(0, S_i(0))$ for $i = 1, 2, \ldots, n$.

6. $\overline{f}_i^{(k)}(t) = \displaystyle\sum_{j=1}^n w_j f_i^{(k)}(t) f_j^{(0)}(t)\rho_{ij}, \overline{g}_i^{(k)}(t) = \sum_{j=1}^n w_j g_i^{(k)}(t) g_j^{(0)}(t)\rho_{ij},$

   and $\overline{h}_i^{(k)}(t) = \displaystyle\sum_{j=1}^n w_j h_i^{(k)}(t) h_j^{(0)}(t)\rho_{ij}$ for $k = 0, 1, 2, \ldots$.

7. Maclauren expansion:
$S_i^\epsilon(T) \approx \sum_{j=0}^{n} \frac{S_i^{(j)}(T)}{j!}\epsilon^j$, $f_i^\epsilon(T) \approx \sum_{j=0}^{n} \frac{f_i^{(j)}(T)}{j!}\epsilon^j$,
$g_i^\epsilon(T) \approx \sum_{j=0}^{n} \frac{g_i^{(j)}(T)}{j!}\epsilon^j$, $h_i^\epsilon(T) \approx \sum_{j=0}^{n} \frac{h_i^{(j)}(T)}{j!}\epsilon^j$.

8. $\hat{f}_i^{(k)}(t) := \left.\frac{\partial^k f_i(t, S_i^\epsilon(t))}{\partial \epsilon^k}\right|_{\epsilon=0}$, $\hat{g}_i^{(k)}(t) := \left.\frac{\partial^k g_i(t, S_i^\epsilon(t))}{\partial \epsilon^k}\right|_{\epsilon=0}$,
$\hat{h}_i^{(k)}(t) := \left.\frac{\partial^k h_i(t, S_i^\epsilon(t))}{\partial \epsilon^k}\right|_{\epsilon=0}$ for $k = 1, 2, \ldots$.

9. $\tilde{g}_i^{(k)}(t) = \sum_{j=1}^{n} w_j g_i^{(k)}(t) g_i^{(0)}(t)\rho_{ij}$ for $k = 0, 1$: see [17].

The asymptotic expansion of $S_i^\epsilon(T)$ around $\epsilon = 0$ is

$$S_i^\epsilon(T) = S_i^{(0)}(T) + S_i^{(1)}(T)\epsilon + \frac{S_i^{(2)}(T)}{2!}\epsilon^2 + \cdots .$$

Eq. (4) implies

$$\begin{aligned} dS_i^{(0)}(t) + dS_i^{(1)}(t)\epsilon + \frac{dS_i^{(2)}(t)}{2!}\epsilon^2 + \ldots = {} & \epsilon(\hat{f}_i^{(0)}(t) + \hat{f}_i^{(1)}(t)\epsilon + \frac{\hat{f}_i^{(2)}(t)}{2!}\epsilon^2 + \cdots)dt \\ & +\epsilon(\hat{g}_i^{(0)}(t) + \hat{g}_i^{(1)}(t)\epsilon + \frac{\hat{g}_i^{(2)}(t)}{2!}\epsilon^2 + \cdots)dW_i(t) \\ & +\epsilon(\hat{h}_i^{(0)}(t) + \hat{h}_i^{(1)}(t)\epsilon + \frac{\hat{h}_i^{(2)}(t)}{2!}\epsilon^2 + \cdots)dN(t). \end{aligned} \tag{5}$$

Equalizing the powers of ϵ from both sides of (5), we obtain

$$\begin{aligned} dS_i^{(0)}(t) &= 0, \\ dS_i^{(1)}(t) &= \hat{f}_i^{(0)}(t)dt + \hat{g}_i^{(0)}(t)dW_i(t) + \hat{h}_i^{(0)}(t)dN(t), \\ \frac{dS_i^{(2)}(t)}{2} &= \hat{f}_i^{(1)}(t)dt + \hat{g}_i^{(1)}(t)dW_i(t) + \hat{h}_i^{(1)}(t)dN(t), \\ &\vdots \\ \frac{dS_i^{(n+1)}(t)}{n+1} &= \hat{f}_i^{(n)}(t)dt + \hat{g}_i^{(n)}(t)dW_i(t) + \hat{h}_i^{(n)}(t)dN(t). \end{aligned} \tag{6}$$

From Parts 2, 3 and 8 of Assumption 2.1, it follows that

$$\hat{f}_i^{(1)}(t) = \frac{\partial f_i(t, S_i^\epsilon(t))}{\partial \epsilon} = \frac{\partial f_i(t, S_i^\epsilon(t))}{\partial S_i^\epsilon(t)} \frac{\partial S_i^\epsilon(t)}{\partial \epsilon} = f_i^{(1)}(t)S_i^{(1)}(t).$$

By a similar manner for $\hat{f}_i^{(2)}(t)$, we have

$$\begin{aligned}\hat{f}_i^{(2)}(t) &= \frac{\partial^2 f_i(t, S_i^\epsilon(t))}{\partial^2 \epsilon} = \frac{\partial}{\partial \epsilon}\left(\frac{\partial f_i(t, S_i^\epsilon(t))}{\partial \epsilon}\right) = \frac{\partial}{\partial S_i^\epsilon(t)}\left(\frac{\partial f_i(t, S_i^\epsilon(t))}{\partial S_i^\epsilon(t)}\frac{\partial S_i^\epsilon(t)}{\partial \epsilon}\right)\frac{\partial S_i^\epsilon(t)}{\partial \epsilon}\\ &= \left(\frac{\partial^2 f_i(t, S_i^\epsilon(t))}{\partial^2 S_i^\epsilon(t)}\frac{\partial S_i^\epsilon(t)}{\partial \epsilon} + \frac{\partial^2 S_i^\epsilon(t)}{\partial S_i^\epsilon(t)\partial \epsilon}\frac{\partial f_i(t, S_i^\epsilon(t))}{\partial S_i^\epsilon(t)}\right)\frac{\partial S_i^\epsilon(t)}{\partial \epsilon}.\end{aligned}$$

Since

$$\frac{\partial^2 S_i^\epsilon(t)}{\partial S_i^\epsilon(t)\partial \epsilon} = 0,$$

we can write

$$\hat{f}_i^{(2)}(t) = \frac{\partial^2 f_i(t, S_i^\epsilon(t))}{\partial^2 S_i^\epsilon(t)}\left(\frac{\partial S_i^\epsilon(t)}{\partial \epsilon}\right)^2 = f_i^{(2)}(t)\left(S_i^{(1)}(t)\right)^2.$$

In the gereral form, for nth derivative, we obtain

$$\begin{aligned}\hat{f}_i^{(n)}(t) &= \frac{\partial^n f_i(t, S_i^\epsilon(t))}{\partial^n \epsilon} = \frac{\partial}{\partial \epsilon}\left(\frac{\partial^{n-1} f_i(t, S_i^\epsilon(t))}{\partial^{n-1} \epsilon}\right) = \frac{\partial}{\partial S_i^\epsilon(t)}\left(\frac{\partial^{n-1} f_i(t, S_i^\epsilon(t))}{\partial^{n-1} S_i^\epsilon(t)}\frac{\partial^{n-1} S_i^\epsilon(t)}{\partial^{n-1} \epsilon}\right)\frac{\partial S_i^\epsilon(t)}{\partial \epsilon}\\ &= \left(\frac{\partial^n f_i(t, S_i^\epsilon(t))}{\partial^n S_i^\epsilon(t)}\frac{\partial^{n-1} S_i^\epsilon(t)}{\partial^{n-1} \epsilon} + \frac{\partial^n S_i^\epsilon(t)}{\partial S_i^\epsilon(t)\partial^{n-1} \epsilon}\frac{\partial^{n-1} f_i(t, S_i^\epsilon(t))}{\partial^{n-1} S_i^\epsilon(t)}\right)\frac{\partial S_i^\epsilon(t)}{\partial \epsilon},\end{aligned}$$

Since

$$\frac{\partial^n S_i^\epsilon(t)}{\partial S_i^\epsilon(t)\partial^{n-1} \epsilon} = 0,$$

so

$$\hat{f}_i^{(n)}(t) = \frac{\partial^n f_i(t, S^\epsilon(t))}{\partial^n S_i^\epsilon(t)}\left(\frac{\partial^{n-1} S_i^\epsilon(t)}{\partial^{n-1} \epsilon}\right)\frac{\partial S_i^\epsilon(t)}{\partial \epsilon} = f_i^{(n)}(t)S_i^{(n-1)}(t)S_i^{(1)}(t).$$

Substituting the above equations in (6) implies that

$$\begin{aligned}dS_i^{(1)}(t) &= f_i^{(0)}(t)dt + g_i^{(0)}(t)dW_i(t) + h_i^{(0)}(t)dN(t),\\ \frac{dS_i^{(2)}}{2}(t) &= f_i^{(1)}(t)S_i^{(1)}(t)dt + g_i^{(1)}(t)S_i^{(1)}(t)dW_i(t) + h_i^{(1)}(t)S_i^{(1)}(t)dN(t),\\ &\vdots\\ \frac{dS_i^{(n+1)}(t)}{n+1} &= f_i^{(n)}(t)S_i^{(n-1)}(t)S_i^{(1)}t)dt + g_i^{(n)}(t)S_i^{(n-1)}(t)S_i^{(1)}t)dW_i(t)\\ &\quad + h_i^{(n)}S_i^{(n-1)}(t)S_i^{(1)}(t)dN(t).\end{aligned}$$

Using integral and recursive formulas, for $n = 1, 2, 3, \ldots,$ one can obtain the following explicit formula of $S_i^{(n+1)}(t)$.

$$\begin{aligned}
S_i^{(1)}(t) - S_i^{(1)}(0) &= \int_0^t f_i^{(0)}(s)ds + \int_0^t g_i^{(0)}(s)dW_i(s) + \int_0^t h_i^{(0)}(s)dN(s),\\
S_i^{(2)}(t) - S_i^{(2)}(0) &= 2\Big(\int_0^t f_i^{(1)}(s)S_i^{(1)}(s)ds + \int_0^t g_i^{(1)}(s)S_i^{(1)}(s)dW_i(s)\\
&\quad + \int_0^t h_i^{(1)}(s)S_i^{(1)}(s)dN(s)\Big),\\
&\vdots\\
S_i^{(n+1)}(t) - S_i^{(n+1)}(0) &= (n+1)\Big(\int_0^t f_i^{(n)}(s)S_i^{(n-1)}(s)S_i^{(1)}(s)ds\\
&\quad + \int_0^t g_i^{(n)}(s)S_i^{(n-1)}(s)S_i^{(1)}(s)dW_i(s)\\
&\quad + \int_0^t h_i^{(n)}(s)S_i^{(n-1)}(s)S_i^{(1)}(s)dN(s)\Big),
\end{aligned} \tag{7}$$

for all $0 \le t \le T$. Indeed, it is very difficult to calculate formulas (7) for $n \ge 3$. Therefore, the approximation of the basket option of value $S(T)$, for maturity T and $\epsilon = 1$, is

$$S(T) \approx S^A(T) := S(0) + S^{(1)}(T) + \frac{S^{(2)}(T)}{2!} + \frac{S^{(3)}(T)}{3!}, \tag{8}$$

such that $S^{(j)}(T) := \sum_{i=1}^n w_i S_i^{(j)}(T)$ for $j = 1, 2, 3$. We can compute the conditional expectation $E[S^A(T)|\Lambda]$ for some conditioning variable Λ, and the LBA is

$$LB \approx LBA := E[(E[S^A(T)|\Lambda] - K)^+]. \tag{9}$$

In the next step, we must determine the conditional variable Λ for the LBA, where this method can be used for the general nonlinear basket call option with jump-diffusion models.

From [15], we choose $\Lambda = (N(T), \Delta(T))$, where

$$\Delta(T) = \sum_{i=1}^n w_i \int_0^T g_i^{(0)}(t)dW_i(t),$$

is a normal variable with mean 0 and variance

$$\nu^2 = \sum_{i=1}^n w_i \int_0^T \tilde{g}_i^{(0)}(t)dt,$$

and $N(T)$ is a Poisson variable with parameter λT. According to (8) and (9) with $k \in \mathbb{R}^+$ and $\nu \in \mathbb{R}^+$, we can write

$$\begin{aligned}
LBA = \sum_{k=0}^{\infty} p_k \int_{-\infty}^{\infty} \Big[\Big(& S(0) + E[S^{(1)}(T)|\Lambda = (\kappa, \nu x)]\\
&+ E[\frac{S^{(2)}(T)}{2!}|\Lambda = (\kappa, \nu x)] + E[\frac{S^{(3)}(T)}{3!}|\Lambda = (\kappa, \nu x)] - K\Big)^+\Big] d\Phi(x),
\end{aligned} \tag{10}$$

where

$$p_k = \exp(-\lambda T)\frac{(\lambda T)^k}{k!}, \text{ and } S^{(j)}(T) := \sum_{i=1}^{n} w_i S_i^{(j)}(T).$$

Moreover $\Phi(x)$ is the cumulative distribution function of a standard normal variable. We need to calculate the value $E[S_i^{(j)}(T)|\Lambda = (k, \nu x)]$ for $j = 1, 2, 3$ and $i = 1, 2, \ldots, n$.

From (10), let

$$E[S_i^{(1)}(T)|\Lambda = (k, \nu x)] = A_1 + A_2 + A_3,$$

where

$$A_1 = E\Big[\int_0^T f_i^{(0)}(s)ds|\Lambda = (k, \nu x)\Big] = \int_0^T f_i^{(0)}(s)ds,$$
$$A_2 = E\Big[\int_0^T g_i^{(0)}(s)dW_i(s)|\Lambda = (k, \nu x)\Big] = \frac{x}{\nu}\int_0^T \overline{g}_i^{(0)}(t)dt,$$
$$A_3 = E\Big[\int_0^T h_i^{(0)}(s)dN(s)|\Lambda = (k, \nu x)\Big] = \frac{k}{T}\int_0^T h_i^{(0)}(t)dt.$$

For the second part of (10), let

$$E[S_i^{(2)}(t)|\Lambda = (k, \nu x)] = B_1 + B_2 + B_3. \tag{11}$$

For more details, see Appendix A.

Similarly, we can write for the third part of (10)

$$E(S_i^{(3)}(t)) = C_1 + C_2 + C_3, \tag{12}$$

where

$$C_1 = 6E\Big[\int_0^T f_i^{(2)}(s)(S_i^{(1)}(s))^2 ds|\Lambda = (k, \nu x)\Big], \tag{13}$$

$$C_2 = 6E\Big[\int_0^T g_i^{(2)}(s)(S_i^{(1)}(s))^2 dW_i(s)|\Lambda = (k, \nu x)\Big], \tag{14}$$

$$C_3 = 6E\Big[\int_0^T h_i^{(2)}(s)(S_i^{(1)}(s))^2 dN(s)|\Lambda = (k, \nu x)\Big]. \tag{15}$$

We know that

$$S_i^{(1)}(t) = \int_0^t f_i^{(0)}(t)dt + \int_0^t g_i^{(0)}(t)dW_i(t) + \int_0^t h_i^{(0)}(t)dN(t). \tag{16}$$

Substituting (16) in (13), (14) and (15), we can achieve $E(S_i^{(3)}(t))$ and in it is obtained in the general form with substituting the equations into equation (10) and sorting them in terms of powers x; see appendix B for each equations.

We can now write the following equation by using the formulas calculated above and inserting them into (10) and get the lower bound approximation formula as follows:

$$LBA = \sum_{k=0}^{\infty} p_k \int_{-\infty}^{\infty} \Bigg(a_3(k)x^3 + a_2(k)x^2 + a_1(k)x + a_0(k) \Bigg)^{+} d\Phi(x). \tag{17}$$

There is no need to do numerical integration for computing the lower bound approximation (17). The integrand is the positive part of a cubic equation, for every fixed k. Then we have only four cases: three roots, two roots, one root, and no root. Hence we can compute the LBA with a simple MATLAB code.

3 SSBE and CSSBE methods

In this section, we investigate the validity of the method presented in this article by using the methods presented in [7]. For this purpose, we define the following SDEwJs equation:

$$dS_i(t) = f_i(S_i(t^-))dt + g_i(S_i(t^-))dW_i(t) + h_i(S_i(t^-))dN(t), \quad i = 1, \ldots, n. \tag{18}$$

The SSBE method for (18) by $Y_0 = S_i(0^-)$ can be defined with the constant step size $h = \Delta t$, such that

$$P_{\Delta t} = \{t_k = k\Delta t,\ k = 0, 1, 2, \ldots\},$$

on $[0, \infty)$ and jump time, $J_t = \{t_i : i = 1, 2, 3, \ldots, l\} \subseteq [0, T]$. For $P = P_{\Delta t} \cup J_t$, we have

$$\begin{aligned} y_n^* &= y_n + f(y^*)\Delta t, \\ y_{n+1} &= y_n^* + g(y_n^*)\Delta w_n + h(y_n^*)\Delta N_n. \end{aligned} \tag{19}$$

From [7], for compensated form of SDEwJs equation, we have

$$dS_i(t) = f_{i\lambda}(S_i(t^-))dt + g_i(S_i(t^-))dW_i(t) + h_i(S_i(t^-))d\widehat{N}(t), \quad i = 1, \ldots, n, \tag{20}$$

where $\widehat{N}(t)$ is a compensated Poisson process as follows:

$$\widehat{N}(t) = N(t) - \lambda t,$$

where $\widehat{N}(t)$ is a martingale and

$$f_{i\lambda}(x) = f_i(x) + \lambda h_i(x).$$

The CSSBE method for (20) by $Y_0 = S_i(0^-)$ can be defined as follows:

$$\begin{aligned} y_n^* &= y_n + f_\lambda(y^*)\Delta t, \\ y_{n+1} &= y_n^* + g(y_n^*)\Delta w_n + h(y_n^*)\Delta \widehat{N}_n, \end{aligned} \tag{21}$$

where $\widehat{N}_n = \widehat{N}(t_{n+1}) - \widehat{N}(t_n)$. According to (19) and (21), the following formula can be deduced

$$y_n^* - f(y^*)\Delta t = y_n.$$

From [9], there exists $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = x - f(x)\Delta t$ and let

$$F(y_n^*) = y_n.$$

Since the inverse function cannot be found explicitly, the inverse function F can be identified by using root-finding algorithms, such as Newton's method or any function that satisfies the fixed point theorem and substituting it into (19) or (21) part 2, to solve the two numerical methods.

4 Numerical experiments

In this section, we present two examples to check the accuracy of the lower bound approximation method. In the first example, we express the linear basket option which has an exact solution and calculates the LBA and compares it with the SSBE, CSSBE methods and also with the exact solution of the equation. The second example is a nonlinear basket option that does not have an exact closed-form solution. We calculate the LBA value and check its accuracy with SSBE and CSSBE methods. In practice, calculations are complicated for their higher accuracy. So we do the calculation up to the third order. Using the Monte Carlo method and calculating 100 sample paths in MATLAB software and calculating the expectation value, we give the approximate solution as follows:

$$E(X_n) = \frac{1}{100} \sum_{i=1}^{100} X_n(w_i).$$

The following data are used in all numerical tests: Two number of assets in the basket, the portfolio weights of each asset $w_i = 0.5$ for $i = 1, 2$, the correlation coefficients of Brownian motions $\rho_{ij} = 0.5$ for $i, j = 1, 2$, the initial asset prices $X(0) = 1$ and $Y(0) = 1$ Dollars, the exercise price $K = 1$, and the jump intensity with two values $\lambda = 5, 10$. One hundred simulations for each test case and maturity at $T = 1$, $T = \frac{1}{2}$, $T = \frac{1}{4}$, and $T = \frac{1}{12}$ years. The Poisson summation is approximated by truncating the infinite series after the first 10 terms and finally the step size is 2^{-8}. Table 1, shows the performance of the LBA, which is the lower bound for the results proposed with SSBE, CSSBE. In Table 2, it shows the LBA values for the nonlinear basket option, because, we do not have an exact solution, so we compare the LBA results with the SSBE and CSSBE methods.

Example 1. Suppose that the linear basket option of two assets X(t) and Y(t) also satisfies the following SDEwJ:

$$dX(t) = X(t^-)dt + X(t^-)dW_1(t) + X(t^-)dN(t),$$
$$dY(t) = Y(t^-)dt + Y(t^-)dW_2(t) - \frac{3}{2}Y(t^-)dN(t).$$

We run the MATLAB program 100 times and list the numerical experiments in Table 1.

Remark 1. As in Table 1, according to the data in Example 1, the LBA column values are lower than the CSSBE and SSBE column values, in fact, this is a lower column for the columns related to the methods presented in Section 3. The exact solution of Example 1 gives us the assurance that the LBA value for the linear problem is less than the exact solution.

Table 1: LBA for linear basket option.

Time(Year)	λ	LBA	CSSBE	SSBE	ES	Elapsed Time
(1)One year	5	-0.0477	2.2894	1.0338	2.24165	22.652555
	10	-3.2114e-04	10.01515	7.7975	8.80805	22.522510
(1/2) 6 Month	5	-0.0127	0.7415	0.99783	0.9304	22.583417
	10	-8.5648e-05	0.65515	1.0446	0.82155	22.594633
(1/4) 3 Month	5	-4.2319e-06	0.89815	1.2854	1.30265	22.445786
(1/12) One Month	5	0.1755	0.1795	0.2123	0.9945	22.435280
	10	0.0012	0.49415	0.8975	0.79485	22.334668

Example 2. Consider a nonlinear basket call option of two assets that satisfies the following SDEwJ:

$$dX(t) = 2X(t)dt + (X^2(t) + 1)dW(t) - X(t)dN(t),$$

$$dY(t) = \frac{Y(t)}{5}dt + (\frac{Y^2(t) + 1}{2})dW(t) - \frac{Y(t)}{3}dN(t).$$

Then we run the MATLAB program 100 times and list the numerical results in Table 2.

Table 2: LBA for nonlinear basket option.

Time(Year)	λ	w_i	LBA	CSSBE	SSBE	Elapsed Time(s)
(1) One year	5	1/2	0.0337	0.1735	0.0342	16.808443
	10	1/2	2.2713e-04	0.18285	0.0078	16.886436
(1/2) 6 Month	5	1/2	0.0524	0.060125	0.5068	16.775061
	10	1/2	3.5297e-04	0.0222	0.0490	16.818628
(1/4) 3 Month	5	1/2	0.0623	0.2312	0.0985	16.710625
(1/12) One Month	5	1/2	0.0656	0.2072	0.1127	16.764730
	10	1/2	4.4221e-04	0.07475	0.0835	16.755422

Remark 2. In Example 2, we have a nonlinear basket option, without the exact closed-form solution, but in Example 1, we examined the accuracy of the method on a linear example. The results obtained in column LBA are less than the values of the columns CSSBE and SSBE. It should be noted that the LBA is only an approximation to the exact lower bound and is, therefore, possible to have values greater than the exact solution of the basket option.

5 Conclusion

The main aim of this work was to use asymptotic expansions to achieve the LBA for nonlinear basket options with the Poisson jump-diffusion model. It is important to note that, since nonlinear problems do not have a closed-form solution, we used SSBE, CSSBE, and Monte Carlo methods to test the accuracy of the solution. In Tables 1 and Table 2, we examined the accuracy of the method presented in the article for a linear and nonlinear basket option with 100 random trajectories, and as you can see, the values in column LBA are less than CSSBE, SSBE and exact solution(ES), using by MATLAB code. The important subject is that the above calculation formulas are for quantities up to the third-order accuracy and computations for the higher-order are very complicated.

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A Further details to compute B_i's

The various parts of (11) can be explained as follows:

$$\begin{aligned}
B_1 &= 2E\Big[\int_0^T f_i^{(1)}(s)S_i^{(1)}(s)ds|\Lambda=(k,\nu x)\Big] = 2E\Big[\int_0^T f_i^{(1)}(s)\int_0^s f_i^{(0)}(v)dvds \\
&+ \int_0^T f_i^{(1)}(s)\int_0^s g_i^{(0)}(v)dW_i(v)ds + \int_0^T f_i^{(1)}(s)\int_0^s h_i^{(0)}(v)dN(v)ds\Big|\Lambda=(k,\nu x)\Big] \\
&= 2\int_0^T f_i^{(1)}(s)\int_0^s f_i^{(0)}(v)dvds + 2\frac{x}{\nu}\int_0^T f_i^{(1)}(t)\int_0^s \overline{g}_i^{(0)}(v)dvds + 2\frac{k}{T}\int_0^T f_i^{(1)}(t)\int_0^s h_i^{(0)}(v)dvds,
\end{aligned} \tag{22}$$

$$\begin{aligned}
B_2 &= 2E\Big[\int_0^T g_i^{(1)}(s)S_i^{(1)}(s)dW_i(s)|\Lambda=(k,\nu x)\Big] = 2E\Big[\int_0^T g_i^{(1)}(t)\int_0^t f_i^{(0)}(v)dvdW_i(s) \\
&\quad + \int_0^T g_i^{(1)}(s)\int_0^s g_i^{(0)}(v)dW_i(v)dW_i(s) + \int_0^T g_i^{(1)}(s)\int_0^s h_i^{(0)}(v)dN(v)dW_i(s)|\Lambda=(k,\nu x)\Big] \\
&= 2(\frac{x}{\nu})\int_0^T \overline{g}_i^{(1)}(t)\int_0^s f_i^{(0)}(v)dvds + 2(\frac{1}{\nu^2})\int_0^T \overline{g}_i^{(1)}(t)\int_0^s \overline{g}_i^{(0)}(v)dvds(x^2-1) \\
&\quad + 2(\frac{kx}{T\nu})\int_0^T \overline{g}_i^{(1)}(s)\int_0^s h_i^{(0)}(v)dvds,
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
B_3 &= 2E\Big[\int_0^T h_i^{(1)}(s)S_i^{(1)}(s)dN(s)|\Lambda=(k,\nu x)\Big] = 2E\Big[\int_0^T h_i^{(1)}(s)\int_0^s f_i^{(0)}(v)dvdN(s) \\
&\quad + \int_0^T h_i^{(1)}(s)\int_0^s g_i^{(0)}(v)dW_i(v)dN(v) + \int_0^T h_i^{(1)}(s)\int_0^T h_i^{(0)}(v)dN(v)dN(s)|\Lambda=(k,\nu x)\Big] \\
&= 2(\frac{k}{T})\int_0^T h_i^{(1)}(t)\int_0^s f_i^{(0)}(v)dvdt + 2(\frac{kx}{T\nu})\int_0^T h_i^{(1)}(s)\int_0^s \overline{g}_i^{(0)}(v)dvdt \\
&\quad + 2(\frac{k^2}{T^2})\int_0^T h_i^{(1)}(t)\int_0^s h_i^{(0)}(v)dvdt.
\end{aligned} \tag{24}$$

Now substitute the above equations into (11).

B Further detail for computing C_i

In a similar manner, the various parts of (13), (14), and (15) can be explained as follows. Form (13), we have

$$E(S_i^{(3)}(t)) = C_1 + C_2 + C_3, \tag{25}$$

and

$$\begin{aligned} C_1 = 6E\Big[\int_0^T f_i^{(2)}(s) S_i^{(2)}(s) ds | \Lambda = (k, \nu x)\Big] &= 6E\Big[\int_0^T f_i^{(2)}(s)\int_0^s f_i^{(1)}(v) S_i^{(1)}(v) dv ds \\ &+ \int_0^T f_i^{(2)}(s)\int_0^s g_i^{(1)}(v) S_i^{(1)}(v) dW_i(v) ds \\ &+ \int_0^T f_i^{(2)}(s)\int_0^s h_i^{(1)}(v) S_i^{(1)}(v) dN(v) ds | \Lambda = (k, \nu x)\Big] = C_{11} + C_{12} + C_{13}, \end{aligned} \tag{26}$$

such that

$$\begin{aligned} C_{11} = 6\Big(&\int_0^T f_i^{(2)}(s)\int_0^s f_i^{(1)}(v)\int_0^v f_i^{(0)}(u) du dv ds + \frac{x}{\nu}\int_0^T f_i^{(2)}(s)\int_0^s f_i^{(1)}(v)\int_0^v \overline{g}_i^{(0)}(u) du dv ds \\ &+ \frac{\kappa}{T}\int_0^T f_i^{(2)}(s)\int_0^s f_i^{(1)}(v)\int_0^v h_i^{(0)}(u) du dv ds\Big), \end{aligned} \tag{27}$$

$$\begin{aligned} C_{12} = 6\Big(&\frac{x}{\nu}\int_0^T f_i^{(2)}(s)\int_0^s \overline{g}_i^{(1)}(v)\int_0^v f_i^{(0)}(u) du dv ds + \frac{x^2}{\nu^2}\int_0^T f_i^{(2)}(s)\int_0^s \overline{g}_i^{(1)}(v)\int_0^v \overline{g}_i^{(0)}(u) du dv ds \\ &+ \frac{\kappa x}{\nu T}\int_0^T f_i^{(2)}(s)\int_0^s \overline{g}_i^{(1)}(v)\int_0^v h_i^{(0)}(u) du dv ds\Big), \end{aligned} \tag{28}$$

$$\begin{aligned} C_{13} = 6\Big(&\frac{\kappa}{T}\int_0^T f_i^{(2)}(s)\int_0^s h_i^{(1)}(v)\int_0^v f_i^{(0)}(u) du dv ds + \frac{\kappa x}{T\nu}\int_0^T f_i^{(2)}(s)\int_0^s h_i^{(1)}(v)\int_0^v \overline{g}_i^{(0)}(u) du dv ds \\ &+ \frac{\kappa^2}{T^2}\int_0^T f_i^{(2)}(s)\int_0^s h_i^{(1)}(v)\int_0^v h_i^{(0)}(u) du dv ds\Big). \end{aligned} \tag{29}$$

Also from (11), let

$$\begin{aligned} C_2 = 6E\Big[\int_0^T g_i^{(2)}(s) S_i^{(2)}(s) dW_i(s) | \Lambda = (k, \nu x)\Big] &= 6E\Big[\int_0^T g_i^{(2)}(s)\int_0^s f_i^{(1)}(s) S_i^{(1)}(v) dv dW_i(s) \\ &+ \int_0^T g_i^{(2)}(s)\int_0^s g_i^{(1)}(v) S_i^{(1)}(v) dW_i(v) dW_i(s) \\ &+ \int_0^T g_i^{(2)}(s)\int_0^s h_i^{(1)}(v) S_i^{(1)}(v) dN(v) dW_i(s) | \Lambda = (k, \nu x)\Big] = C_{21} + C_{22} + C_{23}, \end{aligned} \tag{30}$$

such that

$$\begin{aligned} C_{21} = 6\Big(&\frac{x}{\nu}\int_0^T \overline{g}_i^{(2)}(s)\int_0^s f_i^{(1)}(v)\int_0^v f_i^{(0)}(u) du dv ds + \frac{x^2}{\nu^2}\int_0^T \overline{g}_i^{(2)}(s)\int_0^s f_i^{(1)}(v)\int_0^v \overline{g}_i^{(0)}(u) du dv ds \\ &+ \frac{\kappa x}{T\nu}\int_0^T \overline{g}_i^{(2)}(s)\int_0^s f_i^{(1)}(v)\int_0^v h_i^{(0)}(u) du dv ds\Big), \end{aligned} \tag{31}$$

$$C_{22} = 6\Big(\frac{x^2}{\nu^2}\int_0^T \overline{g}_i^{(2)}(s)\int_0^s \overline{g}_i^{(1)}(v)\int_0^v f_i^{(0)}(u)dudvds + \frac{x^3}{\nu^3}\int_0^T \overline{g}_i^{(2)}(s)\int_0^s \overline{g}_i^{(1)}(v)\int_0^v \overline{g}_i^{(0)}(u)dudvds + \frac{\kappa x^2}{T\nu^2}\int_0^T \overline{g}_i^{(2)}(s)\int_0^s \overline{g}_i^{(1)}(v)\int_0^v h_i^{(0)}(u)dudvds\Big), \tag{32}$$

and

$$C_{23} = 6\Big(\frac{\kappa x}{T\nu}\int_0^T \overline{g}_i^{(2)}(s)\int_0^s h_i^{(1)}(v)\int_0^v f_i^{(0)}(u)dudvds + \frac{\kappa x^2}{T\nu^2}\int_0^T \overline{g}_i^{(2)}(s)\int_0^s h_i^{(1)}(v)\int_0^v \overline{g}_i^{(0)}(u)dudvds + \frac{\kappa^2 x}{T^2\nu}\int_0^T \overline{g}_i^{(2)}(s)\int_0^s h_i^{(1)}(v)\int_0^v h_i^{(0)}(u)dudvds\Big). \tag{33}$$

Finally from (15), we have

$$C_3 = 6E\Big[\int_0^T h_i^{(2)}(s)S_i^{(2)}(s)dN(s)|\Lambda=(k,\nu x)\Big] = 6E\Big[\int_0^t h_i^{(2)}(s)\int_0^s f_i^{(1)}(v)S_i^{(1)}(v)dvdN(s) + \int_0^T h_i^{(2)}(s)\int_0^s g_i^{(1)}(v)S_i^{(1)}(v)dW_i(v)dN(s) + \int_0^T h_i^{(2)}(s)\int_0^s h_i^{(1)}(v)S_i^{(1)}(v)dN(v)dN(s)|\Lambda=(k,\nu x)\Big] = C_{31}+C_{32}+C_{33}, \tag{34}$$

such that

$$C_{31} = 6\Big(\frac{\kappa}{T}\int_0^T h_i^{(2)}(s)\int_0^s f_i^{(1)}(v)\int_0^v f_i^{(0)}(u)dudvds + \frac{\kappa x}{T\nu}\int_0^T h_i^{(2)}(s)\int_0^s f_i^{(1)}(v)\int_0^v \overline{g}_i^{(0)}(u)dudvds + \frac{\kappa^2}{T^2}\int_0^T h_i^{(2)}(s)\int_0^s f_i^{(1)}(v)\int_0^v h_i^{(0)}(u)dudvds\Big), \tag{35}$$

$$C_{32} = 6\Big(\frac{\kappa x}{T\nu}\int_0^T h_i^{(2)}(s)\int_0^s \overline{g}_i^{(1)}(v)\int_0^v f_i^{(0)}(u)dudvds + \frac{\kappa x^2}{T\nu^2}\int_0^T h_i^{(2)}(s)\int_0^s \overline{g}_i^{(1)}(v)\int_0^v \overline{g}_i^{(0)}(u)dudvds + \frac{\kappa^2 x}{T^2\nu}\int_0^T h_i^{(2)}(s)\int_0^s \overline{g}_i^{(1)}(v)\int_0^v h_i^{(0)}(u)dudvds\Big), \tag{36}$$

and

$$C_{33} = 6\Big(\frac{\kappa^2}{T^2}\int_0^T h_i^{(2)}(s)\int_0^s h_i^{(1)}(v)\int_0^v f_i^{(0)}(u)dudvds + \frac{\kappa^2 x}{T^2\nu}\int_0^T h_i^{(2)}(s)\int_0^s h_i^{(1)}(v)\int_0^v \overline{g}_i^{(0)}(u)dudvds + \frac{\kappa^3}{T^3}\int_0^T h_i^{(2)}(s)\int_0^s h_i^{(1)}(v)\int_0^v h_i^{(0)}(u)dudvds\Big). \tag{37}$$

Now substitute the above equations into (12).