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Numerical solution of fractional partial differential equations by using radial basis functions combined with Legendre wavelets

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Abstract. This paper presents an approximate method to solve a class of fractional partial differential equations (FPDEs). First, we introduce radial basis functions (RBFs) combined with wavelets. Next, we obtain fractional integral operator (FIO) of wavelets-radial basis functions (W-RBFs) directly. In the next step, the W-RBFs and their FIO are used to transform the problem under consideration into a system of algebraic equations, which can be simply solved to achieve the solution of the problem. Finally, some numerical examples are presented to illustrate the efficiency and accuracy of the method.

Keywords: Fractional partial differential equations, radial basis functions, Legendre wavelets, numerical method, fractional integral operator.
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1 Introduction

Recently, there has been a great deal of interest in fractional calculus since there have been many applications in different fields of physics and engineering, for example viscoelastic flows, fluid-dynamic traffic model, porous

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media, biology, chemistry, acoustics and psychology [1]. Furthermore, fractional partial differential equations (FPDEs) have been the focus of numerous studies [26]. Some of FPDEs have been studied and solved, such as the fractional Fokker-Planck equation [39], the fractional telegraph equation [7], the fractional advection-diffusion equation [38], the fractional KdV equation [20], the fractional sine-Gordon equation, the fractional transport equation [16] and the space and time fractional diffusion-wave equation [24]. In general, most of fractional differential equations (FDEs) do not have exact solution. So, recent years, some scientists have devoted for solving of FDEs and dynamic systems containing fractional derivatives, and have proposed various numerical schemes such as homotopy analysis method [5], Adomian decomposition method [23], Chebyshev spectral approximation [15], Bernoulli wavelets method [27] and Legendre polynomials method [30].

In the last decade or so, radial basis functions (RBFs) have been extensively applied in various context and emerged as a potential alternative for approximation of partial differential equations (PDEs). The use of RBFs in the numerical solution of PDEs has achieved popularity [6] in science and engineering as it is meshless and can readily be extended to multi-dimensional problems. For example Kansa [17] modified Hardy's multiquadric scheme to solve PDEs. Wu [36] proved the convergence of RBFs Hermite-Birkhoff interpolation. Wendland [35] combined the theory of RBFs with the Galerkin schemes for numerical solution of PDEs. The authors of [12] studied the meshless collocation method using RBFs for numerical solution of systems of equations with linear differential or integral operators. Wendland [34] obtained error estimates for interpolation by a special class of compactly supported RBFs. The authors of [37] introduced a suitable variational formulation for the local error of scattered data interpolation by RBFs. Duan and Tan [10] combined the domain decomposition scheme with the meshless Galerkin scheme for numerical solution of PDEs using RBFs. The author of [3] applied the theory of RBFs together with Galerkin scheme to deal with PDEs with Dirichlet boundary conditions. Shokri and Dehghan [31] presented a numerical technique based on collocation and RBFs for solving the improved Boussinesq equation. The author of [8] applied collocation points and radial basis functions for solving nonlinear sine-Gordon equation.

In this manuscript, we introduce new RBFs combined with wavelets for solving time-space fractional partial differential equations. Then, we derive a new Riemann-Liouville fractional integral operator for wavelet-radial basis functions. Our scheme is based on reducing the main problem into the corresponding system of algebraic equations by expanding the solution as W-RBFs with unknown coefficients and using the fractional operator of integration, which can be simply solved to obtain the solution of the problem.

2 Radial basis functions

In this part, the RBFs schemes have been proposed for interpolation of scattered data. Table 1 lists some well-known RBFs. Let r be the Euclidean distance between $x^* \in R^d$ and any $x \in R^d$, i.e., $\|x - x^*\|_2$. A RBF on R^d is a function of the form

$$\phi^* = \phi(\|x - x^*\|_2),$$

which depends only on the distance between $x \in R^d$ and a fixed point $x^* \in R^d$ [13]. This property implies that the RBFs ϕ^* are radially symmetric about x^*. From Table 1, we see RBFs are infinitely differentiable, globally supported and depend on a free parameter ε.

Let $x_1, x_2, x_3, \ldots, x_N \in \Omega \subset R^d$ be a set of scattered data. The idea behind the use of RBFs is interpolation with a linear combination of RBFs of the same types as:

$$G(x) = \sum_{i=1}^{N} \lambda_i \phi_i(x),$$

where $\phi_i = \phi(\|x - x_i\|)$ and λ_i are unknown scalars for $i = 1, 2, \ldots, N$. Assume that we want to interpolate the given values $g_i = g(x_i), i = 1, 2, \ldots, N$. The unknown scalars λ_i are chosen, so that $G(x_j) = g_j,\ j = 1, 2, \ldots, N$ which results in the following linear system of equations:

$$A\lambda = g,$$

where for $i, j \in \{1, 2, 3, \ldots, N\}, A_{ij} = \phi_i(x_j), \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N]$ and $g = [g_1, g_2, \ldots, g_N]$. Authors of [19,25] demonstrated that the interpolation matrix is invertible for distinct interpolation points. The optimal choice of shape parameter is an open question and it is usually elected by brute force.

Assume that

$$\begin{aligned} &x_i, \quad i = 1, \ldots, 2^{k-1}M (\hat{m} = 2^{k-1}M); \\ &t_j, \quad j = 1, \ldots, 2^{k'-1}M' (\tilde{m} = 2^{k'-1}M'), \end{aligned}$$

are the zeros of Legendre wavelets [40]. Therefore we have wavelet- radial basis functions (W-RBFs). Also, we let

$$\begin{aligned} \Psi(x,t) = {} & [\psi_{1,1}(x,t), \ldots, \psi_{1,2^{k'-1}M'}(x,t), \psi_{2,1}(x,t), \ldots, \psi_{2,2^{k'-1}M'}(x,t), \\ & \ldots, \psi_{2^{k-1}M,1}(x,t), \ldots, \psi_{2^{k-1}M,2^{k'-1}M'}(x,t)]^T, \end{aligned} \tag{1}$$

Table 1: Some well-known functions that generate RBFs.

Name of RBF	Definition
Multiquadric (MQ)	$\phi(r) = \sqrt{\varepsilon^2 + r^2}$
Inverse quadratic (IQ)	$\phi(r) = \frac{1}{(\varepsilon^2+r^2)}$
Inverse multiquadric (IMQ)	$\phi(r) = \frac{1}{\sqrt{\varepsilon^2+r^2}}$
Gaussian (GA)	$\phi(r) = e^{-\varepsilon^2 r^2}$
Thin plate splines (TPS)	$\phi(r) = r^2 \log r$

where

$$\psi_{i,j}(x,t) = e^{-\varepsilon^2((x-x_i)^2+(t-t_j)^2)},$$

for $i = 1, 2, \ldots, 2^{k-1}M$; $j = 1, 2, \ldots, 2^{k'-1}M'$.

3 Riemann-Liouville fractional integral operator for W-RBFs

The Riemann-Liouville fractional integral operator ${}_0^R\mathcal{I}_x^\alpha$ for $\Psi(x,t)$ in Eq. (1) is given by

$$ {}_0^R\mathcal{I}_x^\alpha \Psi(x,t) = \mathcal{R}(\alpha, x, t), \tag{2}$$

where

$$\begin{aligned} \mathcal{R}(\alpha, x, t) = {} & [{}_0^R\mathcal{I}_x^\alpha \psi_{1,1}(x,t), \ldots, {}_0^R\mathcal{I}_x^\alpha \psi_{1,2^{k'-1}M'}(x,t), {}_0^R\mathcal{I}_x^\alpha \psi_{2,1}(x,t), \ldots, \\ & {}_0^R\mathcal{I}_x^\alpha \psi_{2,2^{k'-1}M'}(x,t), \ldots, {}_0^R\mathcal{I}_x^\alpha \psi_{2^{k-1}M,1}(x,t), \\ & \ldots, {}_0^R\mathcal{I}_x^\alpha \psi_{2^{k-1}M,2^{k'-1}M'}(x,t)]^T. \end{aligned} \tag{3}$$

To obtain ${}_0^R\mathcal{I}_x^\alpha \psi_{i,j}(x,t)$, we use the definition of Riemann-Liouville fractional integral [29]. So, we obtain

$$ {}_0^R\mathcal{I}_x^\alpha \psi_{i,j}(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \psi_{i,j}(s,t) ds. \tag{4}$$

In above relation, we transfer the interval $[0, x]$ into $[-1, 1]$ as

$$\begin{aligned} {}_0^R\mathcal{I}_x^\alpha \psi_{i,j}(x,t) &= \frac{1}{\Gamma(\alpha)} \int_{-1}^1 (x - \frac{x}{2} - \frac{x}{2}\tau)^{\alpha-1} \psi_{i,j}(\frac{x}{2} + \frac{x}{2}\tau, t) \frac{x}{2} d\tau \\ &= \frac{x}{2\Gamma(\alpha)} \int_{-1}^1 (\frac{x}{2} - \frac{x}{2}\tau)^{\alpha-1} \psi_{i,j}(\frac{x}{2} + \frac{x}{2}\tau, t) d\tau. \end{aligned} \tag{5}$$

Now, we use the Gauss-Legendre numerical integration [33] to approximate the integral in Eq. (5) as

$$\begin{aligned} {}_0^R\mathcal{I}_x^{\alpha}\psi_{i,j}(x,t) &\simeq \frac{x}{2\Gamma(\alpha)}\sum_{s=1}^{\hat{n}}\omega_s(\frac{x}{2}-\frac{x}{2}\tau_s)^{\alpha-1}e^{-\varepsilon^2((\frac{x}{2}+\frac{x}{2}\tau_s-x_i)^2+(t-t_j)^2)} \\ &= \left(\frac{x}{2\Gamma(\alpha)}\sum_{s=1}^{\hat{n}}\omega_s(\frac{x}{2}-\frac{x}{2}\tau_s)^{\alpha-1}e^{-\varepsilon^2(\frac{x}{2}+\frac{x}{2}\tau_s-x_i)^2}\right)e^{-\varepsilon^2(t-t_j)^2}, \end{aligned} \tag{6}$$

where τ_s and ω_s are nods and weights of Gauss-Legendre given in [33].

Similarly, the integral with respect to t of vectors $\Psi(x,t)$ in (1) may be given as:

$$ {}_0^R\mathcal{I}_t^{\alpha}\Psi(x,t) = \mathcal{R}'(\alpha,x,t), \tag{7}$$

where

$$ {}_0^R\mathcal{I}_t^{\alpha}\psi_{i,j}(x,t) \simeq e^{-\varepsilon^2(x-x_i)^2}\left(\frac{t}{2\Gamma(\alpha)}\sum_{s=1}^{\hat{n}}\omega_s(\frac{t}{2}-\frac{t}{2}\tau_s)^{\alpha-1}e^{-\varepsilon^2(\frac{t}{2}+\frac{t}{2}\tau_s-t_j)^2}\right). \tag{8}$$

Also, we let

$$ {}_0^R\mathcal{I}_x^{\alpha_1}\,{}_0^R\mathcal{I}_t^{\alpha_2}\Psi(x,t) = \mathcal{R}''(\alpha_1,\alpha_2,x,t). \tag{9}$$

By using Eqs. (6) and (8), we get

$$\begin{aligned} {}_0^R\mathcal{I}_x^{\alpha_1}\,{}_0^R\mathcal{I}_t^{\alpha_2}\psi_{i,j}(x,t) &\simeq {}_0^R\mathcal{I}_x^{\alpha_1}\Bigg(e^{-\varepsilon^2(x-x_i)^2}\Bigg(\frac{t}{2\Gamma(\alpha_2)}\sum_{s=1}^{\hat{n}}\omega_s(\frac{t}{2}-\frac{t}{2}\tau_s)^{\alpha_2-1} \\ &\quad e^{-\varepsilon^2(\frac{t}{2}+\frac{t}{2}\tau_s-t_j)^2}\Bigg)\Bigg) = \Bigg(\frac{x}{2\Gamma(\alpha_1)}\sum_{s=1}^{\hat{n}}\omega_s \\ &\quad (\frac{x}{2}-\frac{x}{2}\tau_s)^{\alpha_1-1}e^{-\varepsilon^2(\frac{x}{2}+\frac{x}{2}\tau_s-x_i)^2}\Bigg)\Bigg(\frac{t}{2\Gamma(\alpha_2)}\sum_{s'=1}^{\hat{n}'}\omega_{s'} \\ &\quad (\frac{t}{2}-\frac{t}{2}\tau_{s'})^{\alpha_2-1}e^{-\varepsilon^2(\frac{t}{2}+\frac{t}{2}\tau_{s'}-t_j)^2}\Bigg). \end{aligned} \tag{10}$$

4 Description of the scheme

This part is devoted to the study of time-space FPDEs as:

$$\frac{\partial^{\alpha}\zeta(x,t)}{\partial x^{\alpha}} = \mathcal{F}(x,t,\zeta(x,t),\frac{\partial^{\beta}\zeta(x,t)}{\partial x^{\beta}},\frac{\partial^{\nu}\zeta(x,t)}{\partial t^{\nu}}), \qquad 0<\beta,\nu\le 1, 1<\alpha\le 2, \tag{11}$$

with the initial and boundary conditions as

$$\zeta(x,0) = f_0(x), \qquad 0 \le x \le \ell, \tag{12}$$
$$\zeta(0,t) = g_0(t), \qquad \zeta(\ell,t) = g_1(t), \quad 0 \le t \le \hat{\ell}, \tag{13}$$

where the function $\mathcal{F}$ is continuously differentiable with respect to all its arguments, f_0 and g_i are given functions in $L^2[0,\ell]$ and $L^2[0,\hat{\ell}]$, respectively. For solving this problem, we expand

$$\frac{\partial^{\alpha+\nu}\zeta(x,t)}{\partial x^\alpha \partial t^\nu} \simeq C^T\Psi(x,t), \tag{14}$$

where C is an unknown vector. By fractional integration of order ν of Eq. (14) with respect to t, we get

$$\begin{aligned} \frac{\partial^\alpha\zeta(x,t)}{\partial x^\alpha} &\simeq \frac{\partial^\alpha\tilde{\zeta}(x,t)}{\partial x^\alpha} = C^T\mathcal{R}'(\nu,x,t) + \frac{\partial^\alpha\zeta(x,t)}{\partial x^\alpha}\Big|_{t=0} \\ &= C^T\mathcal{R}'(\nu,x,t) + \frac{\partial^\alpha f_0(x)}{\partial x^\alpha}. \end{aligned} \tag{15}$$

By fractional integration of order α of Eq. (14) with respect to x, achieves

$$\frac{\partial^\nu\zeta(x,t)}{\partial t^\nu} \simeq C^T\mathcal{R}(\alpha,x,t) + \frac{\partial^\nu\zeta(x,t)}{\partial t^\nu}\Big|_{x=0} + x\frac{\partial}{\partial x}(\frac{\partial^\nu\zeta(x,t)}{\partial t^\nu})\Big|_{x=0}. \tag{16}$$

Putting $x=\ell$ in Eq. (16) and considering Eq. (13), we have

$$\frac{\partial}{\partial x}(\frac{\partial^\nu\zeta(x,t)}{\partial t^\nu})\Big|_{x=0} = \frac{1}{\ell}(\frac{\partial^\nu g_1(t)}{\partial t^\nu} - C^T\mathcal{R}(\alpha,\ell,t) - \frac{\partial^\nu g_0(t)}{\partial t^\nu}). \tag{17}$$

By using Eq. (17), we can rewrite Eq. (16) as

$$\begin{aligned} \frac{\partial^\nu\zeta(x,t)}{\partial t^\nu} \simeq \frac{\partial^\nu\tilde{\zeta}(x,t)}{\partial t^\nu} &= C^T\mathcal{R}(\alpha,x,t) - \frac{x}{\ell}C^T\mathcal{R}(\alpha,\ell,t) \\ &+(1-\frac{x}{\ell})\frac{\partial^\nu g_0(t)}{\partial t^\nu} + \frac{x}{\ell}\frac{\partial^\nu g_1(t)}{\partial t^\nu}. \end{aligned} \tag{18}$$

By fractional integration of order α of Eq. (15) with respect to x and considering Eq. (13), we get

$$\zeta(x,t) \simeq C^T\mathcal{R}''(\alpha,\nu,x,t) + f_0(x) - f_0(0) - xf_0'(0) + g_0(t) + x\frac{\partial\zeta(x,t)}{\partial x}\Big|_{x=0}. \tag{19}$$

Putting $x = \ell$ in Eq. (19) and considering the initial and boundary conditions, we have

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = \frac{1}{\ell}(g_1(t) - C^T\mathcal{R}''(\alpha,\nu,\ell,t) - f_0(\ell) + f_0(0) + \ell f_0'(0) - g_0(t)). \tag{20}$$

Now, we can rewrite Eq. (19) as

$$\zeta(x,t) \simeq \tilde{\zeta}(x,t) = C^T\mathcal{R}''(\alpha,\nu,x,t) - \frac{x}{\ell}C^T\mathcal{R}''(\alpha,\nu,\ell,t) + \mathcal{H}(x,t), \tag{21}$$

where

$$\begin{aligned}\mathcal{H}(x,t) \;&=\; g_0(t) + f_0(x) - f_0(0) - xf_0'(0) + \frac{x}{\ell}(g_1(t) - g_0(t)) \\ &\quad + \frac{x}{\ell}(-f_0(\ell) + f_0(0) + \ell f_0'(0)).\end{aligned}$$

By fractional differentiation of order β with respect to x from Eq. (21), we obtain

$$\begin{aligned}\frac{\partial^\beta \zeta(x,t)}{\partial x^\beta} \;\simeq\; \frac{\partial^\beta \tilde{\zeta}(x,t)}{\partial x^\beta} &= C^T\mathcal{R}''(\alpha-\beta,\nu,x,t) \\ &- \frac{x^{1-\beta}}{\Gamma(2-\beta)}C^T\mathcal{R}''(\alpha,\nu,\ell,t) + \frac{\partial^\beta \mathcal{H}(x,t)}{\partial x^\beta}.\end{aligned} \tag{22}$$

Substituting Eqs. (15), (18), (21) and (22) in Eq. (11) and collocating this equation at the $\hat{m}$ and $\tilde{m}$ zeros of the shifted Legendre polynomials $P_{\hat{m}}(x)$ and $P_{\tilde{m}}(t)$, respectively, arrives at

$$\begin{aligned}\frac{\partial^\alpha \tilde{\zeta}(x_i,t_j)}{\partial x^\alpha} - \mathcal{F}(x_i,t_j,\tilde{\zeta}(x_i,t_j), \frac{\partial^\beta \tilde{\zeta}(x_i,t_j)}{\partial x^\beta}, \frac{\partial^\nu \tilde{\zeta}(x_i,t_j)}{\partial t^\nu}) = 0,& \\ i = 1,2,\dots,\hat{m}, \quad j = 1,2,\dots,\tilde{m}.&\end{aligned} \tag{23}$$

Eq. (23) gives $\hat{m}\times\tilde{m}$ equations, which can be solved for $c_{i,j}$, $i = 1,2,\dots,\hat{m}$, $j = 1,2,\dots,\tilde{m}$.

5 Error bound

In this section, we estimate the bound of the applied equation for the proposed scheme method in the Sobolev space. The Sobolev norm in the domain $\Delta = (a,b)^d$ in R^d with $d = 2,3$ for $\mu \geq 1$ is defined as [4, 28]

$$\|\zeta\|_{H^\mu(\Delta)} = \left(\sum_{k=0}^{\mu}\sum_{i=1}^{d} \|D_i^{(k)}\zeta\|^2_{L^2(\Delta)}\right)^{\frac{1}{2}}, \tag{24}$$

where $D_i^{(k)}\xi$ denotes the kth derivative of ζ respect to variable of ith. The symbol $|\zeta|_{H^{\mu;M}(\Delta)}$ was defined by [4,28]

$$|\zeta|_{H^{\mu;M}(\Delta)} = \Bigg(\sum_{k=min(\mu,M+1)}^{\mu} \sum_{i=1}^{d} \|D_i^{(k)}\zeta\|^2_{L^2(\Delta)} \Bigg)^{\frac{1}{2}}.$$

For simplicity of work, we let $\ell = \ell' = 1$ and $\hat{m} = \tilde{m}$. Of course for $\ell, \ell' \neq 1$ and $\hat{m} \neq \tilde{m}$ the procedure is similar.

Theorem 1. *Consider $\zeta \in H^\mu(\Delta)$ with $\mu \geq 1$ and $\Delta = [0,1] \times [0,1]$. If $\tilde{\zeta} = \sum_{i=0}^{\hat{m}} \sum_{j=0}^{\tilde{m}} c_{ij}\psi_{i,j}$, is the best approximation of ζ then*

$$\|\zeta - \tilde{\zeta}\|_{L^2(\Delta)} \leq c\hat{m}^{1-\mu} |\zeta|_{H^{\mu;\hat{m}}(\Delta)}. \tag{25}$$

Also, for $1 \leq l \leq \mu$,

$$\|\zeta - \tilde{\zeta}\|_{H^l(\Delta)} \leq c\hat{m}^{\varrho(l)-\mu} |\zeta|_{H^{\mu;\hat{m}}(\Delta)}, \tag{26}$$

where c depends on μ and

$$\varrho(l) = \begin{cases} 0, & l = 0, \\ 2l - \frac{1}{2}, & l > 0. \end{cases}$$

Proof. Let $P_N = P_N(\Delta)$ be the space of all algebraic polynomials of degree up to N in each variable x_i for $i = 1, 2, \ldots, d$. If ζ' is the best approximation of ζ upon P_N, so for all $\zeta \in H^\mu(\Delta), \mu \geq 1$, we have [4]

$$\|\zeta - \zeta'\|_{L^2(\Delta)} \leq cN^{1-\mu} |\zeta|_{H^{\mu;N}(\Delta)}, \tag{27}$$

and for $1 \leq l \leq \mu$,

$$\|\zeta - \zeta'\|_{H^l(\Delta)} \leq cN^{\varrho(l)-\mu} |\zeta|_{H^{\mu;N}(\Delta)}. \tag{28}$$

Since the best approximation is unique [18] we have

$$\|\zeta - \tilde{\zeta}\|_{L^2(\Delta)} = \|\zeta - \zeta'\|_{L^2(\Delta)} \leq c\hat{m}^{1-\mu} |\zeta|_{H^{\mu;\hat{m}}(\Delta)},$$

and for $1 \leq l \leq \mu$,

$$\|\zeta - \tilde{\zeta}\|_{H^l(\Delta)} = \|\zeta - \zeta'\|_{H^l(\Delta)} \leq c\hat{m}^{\varrho(l)-\mu} |\zeta|_{H^{\mu;\hat{m}}(\Delta)},$$

where

$$\varrho(l) = \begin{cases} 0, & l = 0, \\ 2l - \frac{1}{2}, & l > 0. \end{cases},$$

which completes the proof. $\square$

Theorem 2. *Let $\zeta \in H^{\mu}(\Delta)$ with $\mu \geq 1$ and $n-1 < q \leq n$, then*

$$\| {}_0^C D_x^q \zeta(x,t) - {}_0^C D_x^q \tilde{\zeta}(x,t) \|_{L^2(\Delta)} \leq \frac{c\hat{m}^{\varrho(l)-\mu}}{\Gamma(n-q+1)} |\zeta|_{H^{\mu;\hat{m}}(\Delta)}, \tag{29}$$

where $1 \leq l \leq \mu$.

Proof. By using definition of fractional integral and [2]

$$\|f * g\|_p \leq \|f\|_p \|g\|_1,$$

for $n-1 < q \leq n$, we obtain

$$\begin{aligned}
&\| {}_0^C D_x^q \zeta(x,t) - {}_0^C D_x^q \tilde{\zeta}(x,t) \|^2_{L^2(\Delta)} \\
&\quad = \| {}_0^R \mathcal{I}_x^{n-q} ({}_0^C D_x^n \zeta(x,t) - {}_0^C D_x^n \tilde{\zeta}(x,t)) \|^2_{L^2(\Delta)} \\
&\quad = \| \frac{1}{x^{1+q-n}\Gamma(n-q)} * ({}_0^C D_x^n \zeta(x,t) - {}_0^C D_x^n \tilde{\zeta}(x,t)) \|^2_{L^2(\Delta)} \\
&\quad \leq \left(\frac{1}{(n-q)\Gamma(n-q)} \right)^2 \| {}_0^C D_x^n \zeta(x,t) - {}_0^C D_x^n \tilde{\zeta}(x,t) \|^2_{L^2(\Delta)} \\
&\quad \leq \left(\frac{1}{\Gamma(n-q+1)} \right)^2 \| \zeta(x,t) - \tilde{\zeta}(x,t) \|^2_{H^l(\Delta)},
\end{aligned}$$

by using above equation and Eq. (26) we get Eq. (29). $\square$

Corollary 1. *From Eq. (29) for $\zeta \in H^{\mu}(\Delta)$ with $\mu \geq 1$ and $1 < \alpha \leq 2, 0 < \gamma, \beta \leq 1$ we can write*

$$\| {}_0^C D_x^\alpha \zeta(x,t) - {}_0^C D_x^\alpha \tilde{\zeta}(x,t) \|_{L^2(\Delta)} \leq \frac{c\hat{m}^{\varrho(l)-\mu}}{\Gamma(3-\alpha)} |\zeta|_{H^{\mu;\hat{m}}(\Delta)}, \tag{30}$$

$$\| {}_0^C D_x^\beta \zeta(x,t) - {}_0^C D_x^\beta \tilde{\zeta}(x,t) \|_{L^2(\Delta)} \leq \frac{c\hat{m}^{\varrho(l)-\mu}}{\Gamma(2-\beta)} |\zeta|_{H^{\mu;\hat{m}}(\Delta)}, \tag{31}$$

$$\| {}_0^C D_t^\nu \zeta(x,t) - {}_0^C D_t^\nu \tilde{\zeta}(x,t) \|_{L^2(\Delta)} \leq \frac{c\hat{m}^{\varrho(l)-\mu}}{\Gamma(2-\nu)} |\zeta|_{H^{\mu;\hat{m}}(\Delta)}. \tag{32}$$

Proof. It is a fast consequence of Theorem 2. $\square$

Theorem 3. *Let $\zeta \in H^{\mu}(\Delta)$ with $\mu \geq 1$ and $\mathcal{F}$ in Eq. (11) is a Lipschitzian function, with the Lipschitz constant η. The error bound is given by*

$$\begin{aligned}
\|E_{\hat{m}}\|_{L^2(\Delta)} \leq\ & \frac{c\hat{m}^{\varrho(l)-\mu}}{\Gamma(3-\alpha)} |\zeta|_{H^{\mu;\hat{m}}(\Delta)} + \eta c\hat{m}^{1-\mu} |\zeta|_{H^{\mu;\hat{m}}(\Delta)} \\
& + \eta \frac{c\hat{m}^{\varrho(l)-\mu}}{\Gamma(2-\beta)} |\zeta|_{H^{\mu;\hat{m}}(\Delta)} + \eta \frac{c\hat{m}^{\varrho(l)-\mu}}{\Gamma(2-\nu)} |\zeta|_{H^{\mu;\hat{m}}(\Delta)},
\end{aligned} \tag{33}$$

where $E_{\hat{m}} = R_\zeta - R_{\tilde{\zeta}}$ *such that*

$$R_\zeta = \frac{\partial^\alpha \zeta(x,t)}{\partial x^\alpha} - \mathcal{F}(x,t,\zeta(x,t),\frac{\partial^\beta \zeta(x,t)}{\partial x^\beta},\frac{\partial^\nu \zeta(x,t)}{\partial t^\nu}),$$

and

$$R_{\tilde{\zeta}} = \frac{\partial^\alpha \tilde{\zeta}(x,t)}{\partial x^\alpha} - \mathcal{F}(x,t,\tilde{\zeta}(x,t),\frac{\partial^\beta \tilde{\zeta}(x,t)}{\partial x^\beta},\frac{\partial^\nu \tilde{\zeta}(x,t)}{\partial t^\nu}).$$

Proof. By using Eqs. (11), (25) and (30)-(32), we get

$$\begin{aligned}
\|E_{\hat{m}}\|_{L^2(\Delta)} &= \|\frac{\partial^\alpha \zeta(x,t)}{\partial x^\alpha} - \mathcal{F}(x,t,\zeta(x,t),\frac{\partial^\beta \zeta(x,t)}{\partial x^\beta},\frac{\partial^\nu \zeta(x,t)}{\partial t^\nu}) \\
&\quad -\frac{\partial^\alpha \tilde{\zeta}(x,t)}{\partial x^\alpha} + \mathcal{F}(x,t,\tilde{\zeta}(x,t),\frac{\partial^\beta \tilde{\zeta}(x,t)}{\partial x^\beta},\frac{\partial^\nu \tilde{\zeta}(x,t)}{\partial t^\nu})\|_{L^2(\Delta)} \\
&\leq \|\frac{\partial^\alpha \zeta(x,t)}{\partial x^\alpha} - \frac{\partial^\alpha \tilde{\zeta}(x,t)}{\partial x^\alpha}\|_{L^2(\Delta)} + \eta\|\zeta(x,t) - \tilde{\zeta}(x,t)\|_{L^2(\Delta)} \\
&\quad +\eta\|\frac{\partial^\beta \zeta(x,t)}{\partial x^\beta} - \frac{\partial^\beta \tilde{\zeta}(x,t)}{\partial x^\beta}\|_{L^2(\Delta)} \qquad (34) \\
&\quad +\eta\|\frac{\partial^\nu \zeta(x,t)}{\partial t^\nu} - \frac{\partial^\nu \tilde{\zeta}(x,t)}{\partial t^\nu}\|_{L^2(\Delta)} \\
&\leq \frac{c\hat{m}^{\varrho(l)-\mu}}{\Gamma(3-\alpha)}|\xi|_{H^{\mu;\hat{m}}(\Delta)} + \eta c\hat{m}^{1-\mu}|\xi|_{H^{\mu;\hat{m}}(\Delta)} \\
&\quad +\eta\frac{c\hat{m}^{\varrho(l)-\mu}}{\Gamma(2-\beta)}|\xi|_{H^{\mu;\hat{m}}(\Delta)} + \eta\frac{c\hat{m}^{\varrho(l)-\mu}}{\Gamma(2-\nu)}|\xi|_{H^{\mu;\hat{m}}(\Delta)}. \qquad (35)
\end{aligned}$$

This complete the proof. □

6 Numerical results

In this section, five examples are given to show the efficiency and reliability of our method. The computations associated with the examples were performed using Mathematica 10.

Problem 1. Consider the following time-fractional diffusion equation [14]

$$\frac{\partial^\nu \zeta(x,t)}{\partial t^\nu} - \frac{\partial^2 \zeta(x,t)}{\partial x^2} = 2(\frac{1}{\Gamma(3-\nu)}t^{2-\nu} - 1),$$

with the initial and boundary conditions as

$$\zeta(x,0) = x^2, \quad \zeta(0,t) = t^2, \quad \zeta(1,t) = 1 + t^2.$$

The exact solution for this problem is $\zeta(x,t) = x^2 + t^2$. We solve this problem by applying the presented scheme in Section 4. By using $k = k' = 2$; $M = M' = 1$; $0 < \varepsilon \leq 1$; $0 < \nu \leq 1$, we obtain the exact solution. We compare the absolute error of the present scheme with biorthogonal flatlet multiwavelets method [14] for $k = k' = 2$; $M = M' = 1$; $\nu = 0.5$; $\varepsilon = 1$ and $t = 0.25$ in Table 2.

Table 2: Comparison of the absolute error for $k = k' = 2$; $\nu = 0.5$, $t = 0.25$ with Ref. [14] for Problem 1.

x	Ref. [14]			Our method
	$J=1, m=2$	$J=1, m=3$	$J=2, m=2$	$M = M' = 1$
0.2	3.3×10^{-2}	4.4×10^{-3}	8.8×10^{-2}	0
0.4	1.9×10^{-2}	5.1×10^{-2}	9.8×10^{-2}	0
0.6	1.6×10^{-2}	7.1×10^{-2}	3.4×10^{-1}	0
0.8	1.2×10^{-1}	2.8×10^{-2}	4.3×10^{-1}	0
CPU times	–	–	–	0.564

Problem 2. Consider the following time-fractional equation [22]

$$\frac{\partial^\nu \zeta(x,t)}{\partial t^\nu} + x\frac{\partial \zeta(x,t)}{\partial x} + \frac{\partial^2 \zeta(x,t)}{\partial x^2} = 2t + 2x^2 + 2,$$

with the initial and boundary conditions $\zeta(x,0) = x^2$, and

$$\begin{aligned}\zeta(0,t) &= 3t^2 - 6\frac{t^{3-\nu}}{\Gamma(4-\nu)} + 2\frac{t^{4-2\nu}}{\Gamma(5-2\nu)},\\ \zeta(1,t) &= 1 + 3t^2 - 6\frac{t^{3-\nu}}{\Gamma(4-\nu)} + 2\frac{t^{4-2\nu}}{\Gamma(5-2\nu)}.\end{aligned}$$

The exact solution for this problem is

$$\zeta(x,t) = x^2 + 3t^2 - 6\tfrac{t^{3-\nu}}{\Gamma(4-\nu)} + 2\tfrac{t^{4-2\nu}}{\Gamma(5-2\nu)}.$$

Table 3 displays the absolute error and CPU time (in seconds) of the presented scheme with different choices M, M' and $\nu = 1; k = k' = 2; \varepsilon = 1$. Also, Figure 1 shows the graph of the absolute error and approximate solution for $k = k' = 2$; $M = M' = 3$ and $\nu = 0.5$; $\varepsilon = 1$. We compare our numerical results with usual RBF method for $k = k' = 2$, $M = M' = 3$; $\varepsilon = 1$ and $\nu = 0.7, 1$ in Table 4.

Table 3: Comparison of the absolute error for $k = k' = 2$ and different values of M, M', ν for Problem 2.

(x,t)	$M = M' = 1$	$M = M' = 2$	$M = M' = 3$
$(0.1, 0.1)$	3.47×10^{-18}	0	0
$(0.3, 0.3)$	45.55×10^{-17}	0	0
$(0.5, 0.5)$	0	0	0
$(0.7, 0.7)$	0	1.11×10^{-16}	0
$(0.9, 0.9)$	2.22×10^{-16}	2.22×10^{-16}	0
CPU times	0.5	0.61	1.09

Table 4: Comparison of the absolute error for $k = k' = 2$, $M = M' = 3$ for Problem 2.

(x,t)	Usual RBF method		Our method	
	$\nu = 0.7$	$\nu = 1$	$\nu = 0.7$	$\nu = 1$
$(0.1, 0.1)$	2.05×10^{-3}	6.94×10^{-18}	0	0
$(0.3, 0.3)$	3.90×10^{-3}	2.78×10^{-17}	0	0
$(0.5, 0.5)$	1.66×10^{-3}	1.11×10^{-16}	0	0
$(0.7, 0.7)$	4.53×10^{-3}	1.11×10^{-16}	0	0
$(0.9, 0.9)$	6.69×10^{-4}	2.22×10^{-16}	2.22×10^{-16}	0

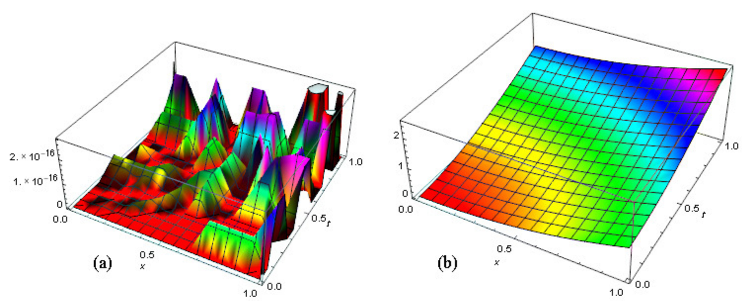

Figure 1: (a): absolute error and (b): approximate solutions of the presented method with $k = k' = 2$; $M = M' = 3$ and $\nu = 0.5; \varepsilon = 1$ for Problem 2.

Problem 3. Consider the time-fractional Navier-Stokes equation [11, 21]

$$\frac{\partial^\nu \zeta(x,t)}{\partial t^\nu} = p + \frac{\partial^2 \zeta(x,t)}{\partial x^2} + \frac{1}{x}\frac{\partial \zeta(x,t)}{\partial x},$$

with the initial and boundary conditions

$$\zeta(x,0)=1-x^2,\qquad \zeta(0,t)=1+(p-4)\frac{t^{\nu}}{\Gamma(1+\nu)},\qquad \zeta(1,t)=(p-4)\frac{t^{\nu}}{\Gamma(1+\nu)}.$$

The exact solution for this problem is

$$\zeta(x,t)=1-x^2+(p-4)\frac{t^{\nu}}{\Gamma(1+\nu)}.$$

We solve this problem by using the presented scheme in Section 4. By taking $k=k'=2$; $M=M'=2$; $0<\varepsilon\le 1$; $0<\nu\le 1$, we obtain the same solution that obtained by the ADM [21] which is the exact solution. Figure 2 shows the absolute error and the numerical results for $k=k'=2; M=M'=1$ and $\nu=1;\varepsilon=1$.

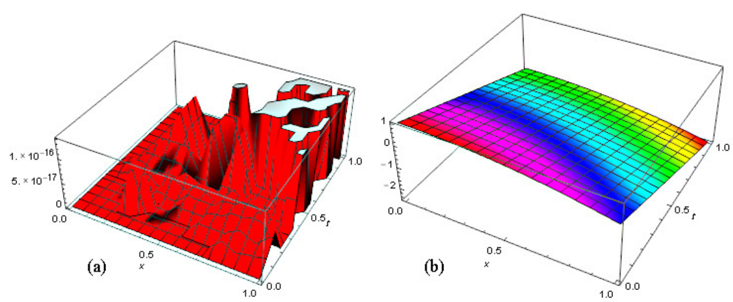

Figure 2: (a): absolute error and (b): approximate solutions of the present method with $k=k'=2$; $M=M'=1$ and $\nu=1$; $\varepsilon=1$ for Problem 3.

Problem 4. Consider the fractional Burgers' equation [9]

$$\frac{\partial^{\nu}\zeta(x,t)}{\partial t^{\nu}}+\zeta(x,t)\frac{\partial\zeta(x,t)}{\partial x}=\frac{\partial^2\zeta(x,t)}{\partial x^2},$$

with the initial and boundary conditions

$$\zeta(x,0)=2x,\qquad \zeta(0,t)=0,\qquad \zeta(1,t)=\frac{2}{1+2t}.$$

The exact solution for case $\nu=1$ is $\zeta(x,t)=2x/(1+2t)$. We solve this problem by applying the presented scheme in Section 4. By using $k=k'=2$; $M=M'=1$; $\varepsilon=1$; $\nu=1$, we obtain the exact solution. Also,

Figure 3 shows the absolute error and the numerical solutions by W-RBFs for $k = k' = 2$; $M = M' = 2$ and $\nu = 1$; $\varepsilon = 1$.

The exact solutions for values of $\nu \neq 1$ do not exist, therefore, we measured the reliability by defining the residual error, that as following:

$$R(x,t) = \left| \frac{\partial^\nu \tilde{\zeta}(x,t)}{\partial t^\nu} + \tilde{\zeta}(x,t)\frac{\partial \tilde{\zeta}(x,t)}{\partial x} - \frac{\partial^2 \tilde{\zeta}(x,t)}{\partial x^2} \right|,$$

where $\tilde{\zeta}(x,t)$ is the numerical solution of the problem. Table 5 displays numerical values of at some selected points $R(x,t)$ for $k = k' = 2$; $M = M' = 2$; $\varepsilon = 1$ with different values of ν.

Table 5: Numerical values of at some selected points $R(x,t)$ for $k = k' = 2$, $M = M' = 2$ with different values of ν for Problem 4.

(x,t)	$\nu = 0.7$	$\nu = 0.8$	$\nu = 0.9$
$(0.1, 0.1)$	1.50×10^{-2}	1.55×10^{-3}	6.94×10^{-4}
$(0.3, 0.3)$	1.07×10^{-2}	1.63×10^{-3}	6.94×10^{-4}
$(0.5, 0.5)$	2.12×10^{-3}	8.54×10^{-4}	6.94×10^{-4}
$(0.7, 0.7)$	1.49×10^{-2}	2.79×10^{-5}	6.94×10^{-4}
$(0.9, 0.9)$	2.60×10^{-2}	7.09×10^{-4}	6.94×10^{-4}

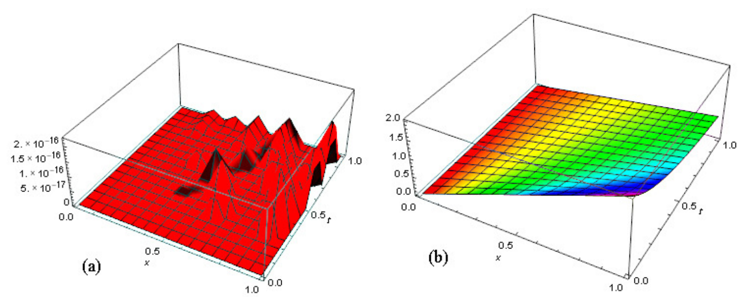

Figure 3: (a): absolute error and (b): approximate solutions of the present method with $k = k' = 2$; $M = M' = 2$ and $\nu = 1$; $\varepsilon = 1$ for Problem 4.

Problem 5. Consider the time-fractional Burgers equation with proportional delay [32]

$$\frac{\partial^\nu \zeta(x,t)}{\partial t^\nu} = \frac{\partial^2 \zeta(x,t)}{\partial x^2} + \zeta(\frac{x}{2}, \frac{t}{2})\frac{\partial \zeta(x, \frac{t}{2})}{\partial x} + \frac{1}{2}\zeta(x,t),$$

with the initial and boundary conditions

$$\zeta(x,0) = x, \quad \zeta(0,t) = 0, \quad \zeta(1,t) = e^t.$$

The exact solution for case $\nu = 1$ is $\zeta(x,t) = xe^t$. To demonstrate the accuracy of the presented scheme, in Table 6, we compare the absolute errors with the numerical method proposed in [32] and our results with $k = k' = 2$; $M = M' = 1$ and $\varepsilon = 1$, $\nu = 1$. The solution behavior of $\zeta(x,t)$ at $x = 1$ for various choices of $\nu = 0.8, 0.9, 1$ is depicted in Figure 4. The graph of surface solution for $k = k' = 2$; $M = M' = 1$; $\varepsilon = 1$ and $\nu = 0.8, 0.9, 1$ is shown in Figure 5. These figures and table show the efficiency and accuracy of the W-RBFs method for solving FPDEs.

The exact solutions for values of $\nu \neq 1$ do not exist, therefore, we measured the reliability by defining the residual error, that as following:

$$R(x,t) = \left| \frac{\partial^\nu \tilde{\zeta}(x,t)}{\partial t^\nu} - \frac{\partial^2 \tilde{\zeta}(x,t)}{\partial x^2} - \tilde{\zeta}(\tfrac{x}{2}, \tfrac{t}{2}) \frac{\partial \tilde{\zeta}(x, \frac{t}{2})}{\partial x} - \tfrac{1}{2}\tilde{\zeta}(x,t) \right|,$$

where $\tilde{\zeta}(x,t)$ is the numerical solution of the problem. Table 7 displays numerical values of at some selected points $R(x,t)$ for $k = k' = 2$; $M = M' = 5$; $\varepsilon = 1$ with different values of ν.

Table 6: Comparison of absolute error for $k = k' = 2$; $M = M' = 1$ and $\varepsilon = 1$, $\nu = 1$ with Ref. [32] for Problem 5.

x	t	Ref. [32]	Presented method
0.25	0.25	2.12×10^{-6}	0
	0.50	7.09×10^{-5}	0
	0.75	5.63×10^{-4}	0
0.50	0.25	4.24×10^{-6}	0
	0.50	1.42×10^{-4}	0
	0.75	1.13×10^{-3}	0
0.75	0.25	6.37×10^{-6}	0
	0.50	2.13×10^{-4}	0
	0.75	1.69×10^{-3}	0

7 Conclusion

In this article, an efficient numerical scheme based on the W-RBFs together with their FIO was proposed to obtain numerical solutions of FPDEs. Also,

Table 7: Numerical values of at some selected points $R(x,t)$ for $M = M' = 1$, $k = k' = 2$ with various values of ν for Problem 5.

(x,t)	$\nu = 0.8$	$\nu = 0.9$	$\nu = 0.99$
$(0.1, 0.1)$	1.58×10^{-3}	8.89×10^{-4}	9.69×10^{-5}
$(0.3, 0.3)$	2.17×10^{-3}	1.23×10^{-3}	1.35×10^{-4}
$(0.5, 0.5)$	5.85×10^{-3}	3.39×10^{-3}	3.76×10^{-4}
$(0.7, 0.7)$	3.09×10^{-3}	1.95×10^{-3}	2.26×10^{-4}
$(0.9, 0.9)$	5.19×10^{-3}	3.13×10^{-3}	3.59×10^{-4}

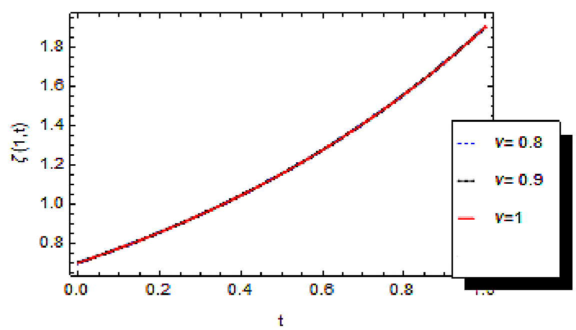

Figure 4: Approximate solution of the present method with $k = k' = 2$; $M = M' = 1$ and different values of ν for Problem 5.

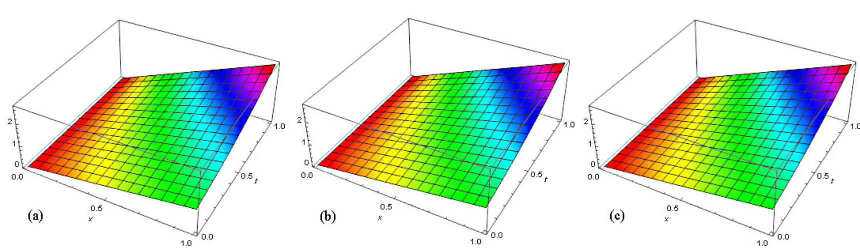

Figure 5: Approximate solution of the present method with $k = k' = 2$; $M = M' = 1$ and (a): $\nu = 0.8$, (b): $\nu = 0.9$, (c): $\nu = 1$ for Problem 5.

a new FIO in the Riemann-Liouville sense for W-RBFs was derived. The W-RBF and their FIO were applied to convert the problem under consideration into the corresponding system of algebraic equations, for achieving the solution of the problem. Our scheme is very convenient for solving the problem under study, since the initial and boundary conditions are taken into account automatically and only a small number of W-RBFs are needed to obtain a satisfactory result. Accuracy and priority of the scheme were checked on some examples. The obtained results of our scheme were in a good agreement with the exact solutions.

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