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Positive solutions for generalized Caputo fractional differential equations with integral boundary conditions

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Abstract. This article is devoted to the study of a new class of nonlinear fractional-order differential equations with integral boundary conditions involving a generalized version of the Caputo type fractional derivative with respect to another function h. In such a path, we transform the proposed problem into an equivalent integral equation. Then we build the upper and lower control functions of the nonlinear term without any monotone requirement except the continuity. By utilizing the method of upper and lower solutions, the fixed point theorems of Schauder and Banach, we obtain the existence and uniqueness of positive solutions for the problem at hand. Finally, we present some examples to illuminate our results.

Keywords: Caputo fractional differential equation, integral boundary condition, existence of positive solution, control functions, fixed point theorem.
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1 Introduction

In recent decades, researchers have realized that fractional calculus has many applications in different fields of engineering and applied science.

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Several authors have used the fractional calculus as an empirical method of describing the properties of natural phenomena such as physics, chemistry, biology, electrochemistry, bioengineering, propagation, finance, etc, see [8, 14–16, 18] and many others. The interesting matter about this topic is that unlike ordinary derivatives because it deals with non-integer order. For example, the fractional derivative of arbitrary order depends not only on the graph of the function very close to the point but also on some history.

In recent years, there has been considerable growth in ODEs and PDEs involving the fractional derivatives operators, we refer here to the most well-known types such as Caputo, Liouville, Hilfer, Hadamard, Katugampola. Consequently, this had led to the study of many various types of fractional differential equations (FDEs) defined by several fractional operators. Mathematicians who are interested in this topic have introduced many generalizations of fractional derivatives such as Hilfer-Hadamard, Hilfer-Katugampola, ψ-Caputo, and ψ-Hilfer, see [4, 10, 11, 17, 20, 21]. The recent studies of the existence and uniqueness of solution for generalized fractional differential equations can be found in [1, 2, 5, 19, 23]. In particular, many interesting and recent results on the existence of positive solutions for different categories of fractional differential equations have been discussed, can be found in [3, 6, 7, 9, 13, 22, 24], and references therein. For example, Li and Wang in [13] discussed the existence and uniqueness of positive solution for the following nonlinear FDE

$$\begin{cases} \mathcal{D}_{0^+}^{r} u(t) = f(t, u(t)), \quad t \in [0,1], \\ u(0) = 0, \end{cases} \tag{1}$$

where $0 < r < 1$, $\mathcal{D}_{0^+}^{r}$ is the standard Riemann-Liouville of order r, and $f : [0,1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a given continuous function. Abdo, et al., in [3] investigated the existence and uniqueness of a positive solution for the following nonlinear FDE

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{r} u(t) = f(t, u(t)), \quad t \in [0,1], \quad 0 < r < 1, \\ u(0) = \lambda \displaystyle\int_0^1 u(s)ds + d, \quad \lambda \geq 0, \quad d \in \mathbb{R}^+, \end{cases} \tag{2}$$

where ${}^{c}\mathcal{D}_{0^+}^{r}$ is the fractional derivative of order r in the sense of Caputo, and f satisfies some appropriate assumptions. Ardjouni and Djoudi in [7] studied the existence and uniqueness of a positive solution for a nonlinear FDE of the type

$$\begin{cases} \mathcal{D}_{1^+}^{r} u(t) = f(t, u(t)), \quad t \in [1,e], \\ u(1) = \lambda \displaystyle\int_1^e u(s)ds + d, \quad \lambda \geq 0, d \in \mathbb{R}^+, \end{cases} \tag{3}$$

where $0 < r < 1$, $\mathcal{D}^r_{1^+}$ is the Caputo-Hadamard fractional derivative of order r, and f satisfies some suitable hypotheses.

Unfortunately, the existence of positive solutions of FDEs with generalized Caputo derivatives is still not studied until now. Motivated by the aforementioned works, this paper mainly investigates the existence and uniqueness of positive solutions for a more general problem involving generalized Caputo fractional derivative of the type

$$\begin{cases} {}^c\mathcal{D}^{r;h}_{0^+}u(t) = f(t, u(t)), \quad t \in [0, 1], \\ u(0) = \lambda \displaystyle\int_0^1 g(s)u(s)ds + d, \end{cases} \tag{4}$$

where $0 < r \leq 1$, ${}^c\mathcal{D}^{r;h}_{0^+}$ is the generalized Caputo fractional derivative of order r introduced by R. Almeida [5], $\lambda \geq 0$, $d \in \mathbb{R}^+$, $h : [0, 1] \longrightarrow \mathbb{R}^+$ is a strictly increasing function such that $h \in C^1[0, 1]$ and $h'(t) \neq 0$, for all $t \in [0, 1]$, $f : [0, 1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous, and $g \in L^1([0, 1], \mathbb{R}^+)$.

The rest of this paper is organized as follows: In Section 2, we present some notations, basic definitions, preliminary facts that will be used to prove our main results. In Section 3, we establish some sufficient conditions for the existence and uniqueness of positive solutions to the proposed problem (4), then we prove those results by applying the upper and lower solutions method and the standard fixed point theorems of Schauder and Banach. Finally, illustrative examples of verifying our results are given in Section 4.

2 Preliminaries

In this section, we collect some basic definitions and preliminary facts that are useful to prove of our main results in this research article. Let $[a, b] \subset \mathbb{R}^+ = [0, +\infty)$. $C[a, b]$ and $C^n[a, b]$ be the spaces of continuous functions and $n-$times continuously differentiable functions on $[a, b]$, respectively. Obviously, $(C[a, b], \|.\|)$ is a Banach space endowed with norm $\|u\|_\infty = \max\{|u(t)| : t \in [a, b]\}$ for $u \in C[a, b]$. We denote by $L^1[a, b]$ the space of all Lebesgue integrable functions on $[a, b]$ with the norm $\|u\|_{L^1} = \int_a^b |u(s)|\, ds < \infty$.

Definition 1. [12] Let $r > 0$ is a real number, $h \in C^1[a, b]$ an increasing function such that $h'(t) \neq 0$, for all $t \in [a, b]$. For an integrable function $u : [a, b] \longrightarrow \mathbb{R}$, the generalized Riemann-Liouville fractional integral of order r with respect to h is defined by

$$\mathcal{I}_{a^+}^{r;h}u(t) = \frac{1}{\Gamma(r)}\int_a^t (h(t)-h(s))^{r-1}h'(s)u(s)ds,$$

provided the right side integral is pointwise defined on (a,b), where $\Gamma(\cdot)$ is Gamma function.

Definition 2. [12] Let $r > 0$ is a real, and $h \in C^1[a,b]$ an increasing function such that $h'(t) \neq 0$, for all $t \in [a,b]$. Then for a continuous function $u : [a,b] \longrightarrow \mathbb{R}$, the generalized Riemann-Liouville fractional derivative of order r with respect to h is given by

$$\mathcal{D}_{a^+}^{r;h}u(t) = \mathcal{D}^{n;h}\ \mathcal{I}_{a^+}^{n-r;h}u(t),$$

where $\mathcal{D}^{n;h} = \left[\frac{1}{h'(t)}\frac{d}{dt}\right]^n$, $n = [r]+1$ and $[r]$ denotes the integer part of r.

Definition 3. [4] Let $r > 0$. Given $u \in C^{n-1}[a,b]$ and $h \in C^n[a,b]$ be an increasing function such that $h'(t) \neq 0$, for all $t \in [a,b]$. Then, the generalized Caputo fractional derivative of order r with respect to h is defined as follows

$$^c\mathcal{D}_{a^+}^{r;h}u(t) = \mathcal{D}_{a^+}^{r;h}\left[u(t) - \sum_{k=0}^{n-1}\frac{u_h^{[k]}(a)}{k!}(h(t)-h(a))^k\right], \tag{5}$$

where $u_h^{[k]}(t) = \left[\frac{1}{h'(t)}\frac{d}{dt}\right]^k u(t)$ and $n = [r]+1$ for $r \notin \mathbb{N}$, $n = r$ for $r \in \mathbb{N}$. In particular, when $0 < r < 1$, the relation (5) takes the following form:

$$^c\mathcal{D}_{a^+}^{r;h}u(t) = \mathcal{D}_{a^+}^{r;h}\left[u(t) - u_h^{[0]}(a)\right], \quad u_h^{[0]}(a) = u(a).$$

Further, if $u \in C^n[a,b]$, then for $r \notin \mathbb{N}$ we have

$$^c\mathcal{D}_{a^+}^{r;h}u(t) = \mathcal{I}_{a^+}^{n-r;h}\ \mathcal{D}^{n;h}u(t),$$

and for $r = n \in \mathbb{N}$, one has

$$^c\mathcal{D}_{a^+}^{r;h}u(t) = u_h^{[n]}(t).$$

In particular, if $u \in C^1[a,b]$ $(0 < r < 1)$, we get

$$\begin{aligned} ^c\mathcal{D}_{a^+}^{r;h}u(t) &= \mathcal{I}_{a^+}^{1-r;h}\ \mathcal{D}^{1;h}u(t) \\ &= \frac{1}{\Gamma(1-r)}\int_a^t (h(t)-h(s))^{r-1}h'(s)u_h^{[1]}(s)ds. \end{aligned}$$

Lemma 1. *[5] Let $u : [a, b] \longrightarrow \mathbb{R}$. The following properties hold:*

1. *If $u \in C[a, b]$, then*
$$^{c}\mathcal{D}^{r;h}_{a^+}\mathcal{I}^{r;h}_{a^+}u(t) = u(t).$$

2. *If $u \in C^{n-1}[a, b]$, and $u^{[n]}_h$ exists, that is, on any bounded interval of $[a, b]$. Then*
$$\mathcal{I}^{r;h}_{a^+}\ {}^{c}\mathcal{D}^{r;h}_{a^+}u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{[k]}_h(a)}{k!}(h(t) - h(a))^k. \tag{6}$$

In particular, if $0 < r < 1$, then $\mathcal{I}^{r;h}_{a^+}\ {}^{c}\mathcal{D}^{r;h}_{a^+}u(t) = u(t) - u(a)$.

Lemma 2. *[2] Let $r > 0$, $u \in C[a, b]$ and $h \in C^1[a, b]$. Then for all $t \in [a, b]$,*

(i) *$\mathcal{I}^{r;h}_{a^+}(\cdot)$ is bounded from $C[a, b]$ to $C[a, b]$;*

(ii) *$\mathcal{I}^{r;h}_{a^+}u(a) = \lim\limits_{t\to a^+} I^{r;h}_{a^+}u(t) = 0$.*

Lemma 3. *[4, 12] Let $r, \kappa > 0$. For a given function $u : [a, b] \longrightarrow \mathbb{R}$ we have*

1. *$\mathcal{I}^{r;h}_{a^+}\ \mathcal{I}^{\kappa;h}_{a^+}u(t) = \mathcal{I}^{r+\kappa;h}_{a^+}u(t)$;*
2. *$\mathcal{I}^{r;h}_{a^+}\left[h(t) - h(a)\right]^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(r+\kappa)}\left[h(t) - h(a)\right]^{r+\kappa-1}$;*
3. *${}^{c}\mathcal{D}^{r;h}_{a^+}\left[h(t) - h(a)\right]^{k} = 0$, $\forall k \in \{0, 1, \ldots, n-1\}$, $n \in \mathbb{N}$.*
4. *${}^{c}\mathcal{D}^{r;h}_{a^+}h(t) = 0$ for any constant function $h(t)$.*

Definition 4. [25] Let $(X, \|.\|)$ be Banach space and $\Phi : X \longrightarrow X$. The operator Φ is a contraction operator if there is an $\rho \in (0, 1)$ such that $x, y \in X$ imply
$$\|\Phi x - \Phi y\| \leq \rho \|x - y\|. \tag{7}$$

Now, we state the fixed point theorems which enable us to prove the existence and uniqueness of positive solutions for the problem (4).

Theorem 1. *[25] (Banach fixed point theorem). Let S be a nonempty, closed and convex subset of a Banach space X and $\Phi : S \longrightarrow S$ be a contraction operator. Then there is a unique $u \in S$ with $\Phi u = u$.*

Theorem 2. *[25] (Schauder fixed point theorem). Let S be a nonempty bounded, closed and convex subset of a Banach space X and $\Phi : S \longrightarrow S$ be a completely continuous operator. Then Φ has a fixed point in S.*

In the present work, we deal with the positive solution of the problem (4) on $[0,1]$.

Definition 5. A function $u \in C[0,1] \cap C^1[0,1]$ is said to be a solution of (4) if u satisfies the equation ${}^c\mathcal{D}_{0^+}^{r;h} u(t) = f(t,u(t)),\ \ t \in [0,1]$ with the integral boundary conditions $u(0) = \lambda \int_0^1 g(s)u(s)ds + d$.

Definition 6. A function $u \in C[0,1] \cap C^1[0,1]$ is called a positive solution of (4) if $u(t) \geq 0$ for all $t \in [0,1]$ and u satisfies the problem (4).

3 Main results

In this section, we setup some adequate conditions for the existence and uniqueness of solution to the proposed problem (4).

Before stating and proving our main results, we need the following lemmas:

Lemma 4. *Let $0 < r \leq 1,\ J = [0,1],\ \mu = \lambda \int_0^1 g(s)ds$ and $\sigma \in C(J, \mathbb{R}^+)$. Then $u \in C(J, \mathbb{R}^+)$ is a solution of the linear problem corresponding to (4) if and only if u satisfies the following fractional integral equation*

$$\begin{aligned} u(t) &= \frac{d}{1-\mu} + \frac{\lambda}{1-\mu} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} \sigma(\tau) d\tau \\ &\quad + \frac{1}{\Gamma(r)} \int_0^t (h(t)-h(s))^{r-1} h'(s)\sigma(s)ds, \quad t \in J, \end{aligned} \tag{8}$$

where $1-\mu > 0$ and

$$Q(\tau) := \int_\tau^1 g(s)h'(\tau)(h(s)-h(\tau))^{r-1}ds.$$

Proof. By virtue of the definition of $\mathcal{I}_{0^+}^{r;h}$ and using Lemma 1, with the integral boundary condition, one can obtain

$$u(t) = \lambda \int_0^1 g(s)u(s)ds + d + \mathcal{I}_{0^+}^{r;h}\sigma(t). \tag{9}$$

Set

$$A_u := \lambda \int_0^1 g(s)u(s)ds, \tag{10}$$

Using the equations (9), (10), Fubini's theorem, and definition of $Q(\tau)$, we get

$$
\begin{aligned}
A_u &= \lambda \int_0^1 g(s)u(s)ds \\
&= \lambda \int_0^1 g(s) \left[A_u + d + \mathcal{I}_{0^+}^{r;h}\sigma(s) \right] ds \\
&= \mu A_u + \mu d + \lambda \int_0^1 g(s) \left(\frac{1}{\Gamma(r)} \int_0^s h'(\tau)(h(s)-h(\tau))^{r-1}\sigma(\tau)d\tau \right) ds \\
&= \mu A_u + \mu d + \lambda \int_0^1 \int_\tau^1 \frac{g(s)h'(\tau)(h(s)-h(\tau))^{r-1}}{\Gamma(r)}\sigma(\tau)dsd\tau \\
&= \mu A_u + \mu d + \lambda \int_0^1 \frac{Q(\tau)}{\Gamma(r)}\sigma(\tau)d\tau, \qquad (11)
\end{aligned}
$$

which implies

$$A_u = \frac{\mu d}{1-\mu} + \frac{\lambda}{1-\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(r)}\sigma(\tau)d\tau. \qquad (12)$$

Substituting the value of A_u into (9), we obtain (8).

Conversely, assume that $u(t)$ satisfies equation (8). Then we apply the operator ${}^c\mathcal{D}_{0^+}^{r;h}$ to both sides of equation (8), it follows from fact that ${}^c\mathcal{D}_{0^+}^{r;h}$ is the left inverse of $\mathcal{I}_{0^+}^{r;h}$ (see Lemma 1) and Lemma 3, that

$${}^c\mathcal{D}_{0^+}^{r;h}u(t) = \sigma(t).$$

Moreover, taking $t \to 0$ in equation (8), it follows from Lemma 2, and definition of A_u that the condition of (4) is satisfied. This completes the proof. □

Lemma 5. *Let $h \in C^1(J, \mathbb{R}^+)$. If $Q(\tau) = \int_\tau^1 g(s)h'(\tau)(h(s)-h(\tau))^{r-1}ds$ for $\tau \in J$ such that $g \in L^1(J, \mathbb{R}^+)$ and $\sup_{s\in J} g(s) \le 1$, then*

$$\frac{Q(\tau)}{\Gamma(r)} < e, \;\; \textit{for } \tau \in J \textit{ and } 0 < r \le 1.$$

Proof. Since h is an increasing function and $h \in C^1(J, \mathbb{R}^+)$, $\sup_{\tau\in[0,s]} |h'(\tau)| \le |h'(s)|$ for $0 \le \tau \le s \le 1$. It follows from direct computation that

$$
\begin{aligned}
\frac{Q(\tau)}{\Gamma(r)} &= \frac{\int_\tau^1 g(s)h'(\tau)(h(s)-h(\tau))^{r-1}ds}{\int_0^\infty s^{r-1}\exp(-s)ds} < \frac{\int_\tau^1 h'(s)(h(s)-h(\tau))^{r-1}ds}{\int_0^\infty s^{r-1}\exp(-s)ds} \\
&= \frac{\int_0^{h(1)-h(\tau)} s^{r-1}ds}{\int_0^\infty s^{r-1}\exp(-s)ds} \le \frac{e\int_0^{h(1)-h(\tau)} s^{r-1}\exp(-s)ds}{\int_0^\infty s^{r-1}\exp(-s)ds} < e.
\end{aligned}
$$

□

Now we are ready to move forward in proving our main results that depend on the upper and lower solutions method and fixed point techniques of Schauder and Banach.

3.1 Existence of positive solution

Let $X = C(J, \mathbb{R}^+)$ be the Banach space of all real-valued continuous funcions defined on the compact interval J endowed with the maximum norm. Define the cone

$$\mathcal{E} = \{u \in X : u(t) \geq 0, \ \forall t \in J\}.$$

We express (8) as

$$u(t) = (\Phi u)(t),$$

where the operator $\Phi : \mathcal{E} \longrightarrow \mathcal{E}$ defined by

$$\begin{aligned}(\Phi u)(t) = {} & \frac{d}{1-\mu} + \frac{\lambda}{1-\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(r)} f(\tau, u(\tau))d\tau \\ & + \frac{1}{\Gamma(r)}\int_0^t (h(t)-h(s))^{r-1} h'(s) f(s, u(s))ds. \end{aligned} \tag{13}$$

In the sequel, we need the following lemma:

Lemma 6. *Assum that $f : J \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. Then the operator $\Phi : \mathcal{E} \longrightarrow \mathcal{E}$ is compact.*

Proof. By taking into account that f and h are continuous nonnegative functions, we get that $\Phi : \mathcal{E} \longrightarrow \mathcal{E}$ is continuous. The function $f : J \times \mathbb{B}_\eta \longrightarrow \mathbb{R}^+$ is bounded, then there exists $\rho > 0$ such that

$$0 \leq f(t, u) \leq \rho,$$

where $\mathbb{B}_\eta = \{u \in \mathcal{E} : \|u\| \leq \eta\}$. Hence, we obtain

$$\begin{aligned}|(\Phi u)(t)| &= \left|\frac{d}{1-\mu} + \frac{\lambda}{1-\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(r)} f(\tau, u(\tau))d\tau + \mathcal{I}_{0^+}^{r;h} f(t, u(t))\right| \\ &\leq \frac{d}{1-\mu} + \frac{\lambda}{1-\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(r)} |f(\tau, u(\tau))| d\tau \\ &\quad + \frac{1}{\Gamma(r)}\int_0^t (h(t)-h(s))^{r-1} h'(s) |f(s, u(s))| ds \\ &\leq \frac{d}{1-\mu} + \frac{\lambda\rho e}{1-\mu} + \frac{(h(t)-h(0))^r}{\Gamma(r+1)}\rho, \end{aligned}$$

which implies

$$\|\Phi u\| \leq \frac{d}{1-\mu} + \frac{\lambda \rho e}{1-\mu} + \frac{(h(1)-h(0))^r}{\Gamma(r+1)}\rho.$$

Hence $\Phi(\mathbb{B}_\eta)$ is uniformaly bounded.

Now, we prove that $\Phi(\mathbb{B}_\eta)$ is equicontinuous. Let $u \in \mathbb{B}_\eta$, and $t_1, t_2 \in J$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&\left|(\Phi u)(t_2) - (\Phi u)(t_1)\right| \\
&= \left| \frac{1}{\Gamma(r)} \int_0^{t_2} (h(t_2)-h(s))^{r-1} h'(s) f(s,u(s)) ds \right. \\
&\quad \left. -\frac{1}{\Gamma(r)} \int_0^{t_1} (h(t_1)-h(s))^{r-1} h'(s) f(s,u(s)) ds \right| \\
&\leq \frac{1}{\Gamma(r)} \int_0^{t_1} \left[(h(t_1)-h(s))^{r-1} - (h(t_2)-h(s))^{r-1}\right] h'(s) \left|f(s,u(s))\right| ds \\
&\quad +\frac{1}{\Gamma(r)} \int_{t_1}^{t_2} (h(t_2)-h(s))^{r-1} h'(s) \left|f(s,u(s))\right| ds \\
&\leq \frac{2\rho}{\Gamma(r+1)} (h(t_2)-h(t_1))^r.
\end{aligned}$$

It follows from continuity of h that $\left|(\Phi u)(t_2) - (\Phi u)(t_1)\right| \to 0$ as $t_2 \longrightarrow t_1$. Thus, the operator Φ is equicontinuous in $\mathbb{B}_\eta$. So, the compactness of Φ follows by Ascoli Arzela's theorem. □

Now for any $u \in [a,b] \subset \mathbb{R}^+$, we define respectively the upper and lower control functions as follows:

$$\overline{H}(t,u) = \sup_{a \leq v \leq u} f(t,v), \quad \text{and} \quad \underline{H}(t,u) = \inf_{u \leq v \leq b} f(t,v). \tag{14}$$

It is clear that these functions are non-decreasing on $[a,b]$ and $\underline{H}(t,u) \leq f(t,u) \leq \overline{H}(t,u)$.

Definition 7. *Let $\overline{u}, \underline{u} \in \mathcal{E}$, $a \leq \underline{u} \leq \overline{u} \leq b$ satisfy*

$$^c\mathcal{D}_{0^+}^{r;h} \overline{u}(t) \geq \overline{H}(t, \overline{u}(t)), \ \overline{u}(0) \geq \lambda \int_0^1 g(s) \overline{u}(s) ds + d,$$

or

$$\begin{aligned}
\overline{u}(t) \ \geq \ & \frac{d}{1-\mu} + \frac{\lambda}{(1-\mu)} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} \overline{H}(\tau, \overline{u}(\tau)) d\tau \\
& +\frac{1}{\Gamma(r)} \int_0^t (h(t)-h(s))^{r-1} h'(s) \overline{H}(s, \overline{u}(s)) ds, \quad t \in J,
\end{aligned}$$

and

$$ {}^{c}\mathcal{D}_{0^+}^{r;h}\underline{u}(t) \leq \underline{H}(t,\underline{u}(t)),\ \ \underline{u}(0) \leq \lambda \int_0^1 g(s)\underline{u}(s)ds + d, $$

or

$$ \begin{aligned} \underline{u}(t) \leq\ & \frac{d}{1-\mu} + \frac{\lambda}{(1-\mu)} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} \underline{H}(\tau,\underline{u}(\tau))d\tau \\ & + \frac{1}{\Gamma(r)} \int_0^t (h(t)-h(s))^{r-1} h'(s) \underline{H}(s,\underline{u}(s))ds, \quad t \in J. \end{aligned} $$

Then the functions $\overline{u}(t)$ and $\underline{u}(t)$ are called a pair of upper and lower solutions for problem (4).

Theorem 3. *Let $f : J \times \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous. Assume that $\overline{u}$ and $\underline{u}$ are respectively upper and lower solution of (4), then the problem (4) has at least positive solution in X. Moreover, $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$, $t \in J$.*

Proof. Let $K = \{u \in \mathcal{E}, \underline{u}(t) \leq u(t) \leq \overline{u}(t),\ t \in J\}$. As $K \subset \mathcal{E}$ and K is nonempty, bounded, closed and convex subset, Lemma 6 shows that $\Phi : K \longrightarrow K$ is compact. Next, we show that if $u \in K$, we have $\Phi u \in K$. For any $u \in K$, we have $\underline{u} \leq u \leq \overline{u}$. Hence

$$ \begin{aligned} (\Phi u)(t) =\ & \frac{d}{1-\mu} + \frac{\lambda}{1-\mu} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} f(\tau,u(\tau))d\tau \\ & + \frac{1}{\Gamma(r)} \int_0^t (h(t)-h(s))^{r-1} h'(s) f(s,u(s))ds \\ \leq\ & \frac{d}{1-\mu} + \frac{\lambda}{1-\mu} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} \overline{H}(\tau,\overline{u}(\tau))d\tau \\ & + \frac{1}{\Gamma(r)} \int_0^t (h(t)-h(s))^{r-1} h'(s) \overline{H}(s,\overline{u}(s))ds \\ \leq\ & \overline{u}(t), \end{aligned} \tag{15} $$

and

$$ \begin{aligned} (\Phi u)(t) =\ & \frac{d}{1-\mu} + \frac{\lambda}{1-\mu} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} f(\tau,u(\tau))d\tau \\ & + \frac{1}{\Gamma(r)} \int_0^t (h(t)-h(s))^{r-1} h'(s) f(s,u(s))ds \\ \geq\ & \frac{d}{1-\mu} + \frac{\lambda}{1-\mu} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} \underline{H}(\tau,\underline{u}(\tau))d\tau \\ & + \frac{1}{\Gamma(r)} \int_0^t (h(t)-h(s))^{r-1} h'(s) \underline{H}(s,\underline{u}(s))ds \\ \geq\ & \underline{u}(t). \end{aligned} \tag{16} $$

From (15) and (16), we conclude that $\underline{u}(t) \leq \Phi u(t) \leq \overline{u}(t)$, $t \in J$. Thus, $\Phi u \in K$ that is $\Phi : K \to K$. In view of the above steps and Theorem 2, there exists a fixed point u in K. Therefore, the problem (4) has at least one positive solution u in X and $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$, $t \in J$. $\square$

Corollary 1. *Assume that $f : J \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, and there exist $\lambda_2 \geq \lambda_1 > 0$ such that*

$$\lambda_1 \leq f(t,u) \leq \lambda_2 \quad \textit{for all } (t,u) \in J \times \mathbb{R}^+. \tag{17}$$

Then the problem (4) has at least a positive solution $u \in X$. Moreover

$$\lambda_1 \frac{(h(t)-h(0))^r}{\Gamma(r+1)} + d \leq u(t) \leq \lambda_2 \frac{(h(t)-h(0))^r}{\Gamma(r+1)} + d \tag{18}$$

and

$$u(t) \leq \frac{d}{1-\mu} + \frac{\lambda\lambda_2}{1-\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(r)} d\tau + \frac{\lambda_2(h(t)-h(0))^r}{\Gamma(r+1)}, \tag{19}$$

$$u(t) \geq \frac{d}{1-\mu} + \frac{\lambda\lambda_1}{1-\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(r)} d\tau + \frac{\lambda_1(h(t)-h(0))^r}{\Gamma(r+1)}. \tag{20}$$

Proof. Case (i) If $\lambda = 0$, then from (17) and definition of control functions, we have

$$\lambda_1 \leq \underline{H}(t,u) \leq \overline{H}(t,u) \leq \lambda_2 \tag{21}$$

Now, we consider the following FDE

$${}^c\mathcal{D}_{0^+}^{r;h}\overline{u}(t) = \lambda_2, \qquad \overline{u}(0) = d. \tag{22}$$

In view of Lemma 4, the equation (22) has a positive solution

$$\begin{aligned} \overline{u}(t) &= \frac{\lambda_2}{\Gamma(r)}\int_0^t (h(t)-h(s))^{r-1} h'(s) ds + d \\ &= \frac{\lambda_2(h(t)-h(0))^r}{\Gamma(r+1)} + d. \end{aligned} \tag{23}$$

Taking into account (21), we have

$$\overline{u}(t) \geq \frac{1}{\Gamma(r)}\int_0^t (h(t)-h(s))^{r-1} h'(s)\overline{H}(s,\overline{u}(s)) ds + d.$$

It is clear that $\overline{u}$ is the upper solution of (4). Also, we consider the following FDE

$${}^c\mathcal{D}_{0^+}^{r;h}\underline{u}(t) = \lambda_1, \quad \underline{u}(0) = d,$$

which has also a positive solution (due to Lemma 4)

$$\begin{aligned}\underline{u}(t) &= \frac{\lambda_1}{\Gamma(r)}\int_0^t (h(t)-h(s))^{r-1}h'(s)ds + d\\ &= \frac{\lambda_1(h(t)-h(0))^r}{\Gamma(r+1)} + d. \end{aligned}\tag{24}$$

By (21) and the same way that we used to reach the upper solution, we get

$$\underline{u}(t) \le \frac{1}{\Gamma(r)}\int_0^t (h(t)-h(s))^{r-1}h'(s)\underline{H}(s,\underline{u}(s))ds + d.$$

Therefore, $\underline{u}$ is the lower solution of (4). An application of Theorem 3 yields that the problem (4) has at least one positive solution $u(t) \in X$ which satisfies the inequality (18).

Case (ii) $\lambda > 0$. Let

$${}^c\mathcal{D}_{0^+}^{r;h}\overline{u}(t) = \lambda_2, \quad \overline{u}(0) = \lambda\int_0^1 g(s)\overline{u}(s)ds + d. \tag{25}$$

The equation (25) has a positive solution

$$\overline{u}(t) = \frac{d}{1-\mu} + \frac{\lambda\lambda_2}{(1-\mu)}\int_0^1 \frac{Q(\tau)}{\Gamma(r)}d\tau + \frac{\lambda_2(h(t)-h(0))^r}{\Gamma(r+1)}.$$

Indeed, the problem (25) is equivalent to the following fractional integral equation

$$\overline{u}(t) = \lambda\int_0^1 g(s)\overline{u}(s)ds + d + \mathcal{I}_{0^+}^{r;h}\lambda_2. \tag{26}$$

Set $A_{\overline{u}} := \lambda\int_0^1 g(s)\overline{u}(s)ds$. By the same arguments used in (11), we obtain

$$A_{\overline{u}} = \frac{\mu d}{1-\mu} + \frac{\lambda\lambda_2}{(1-\mu)}\int_0^1 \frac{Q(\tau)}{\Gamma(r)}d\tau. \tag{27}$$

where μ, and $Q(\tau)$ as in Lemma 4. Hence, the equation (26) becomes

$$\begin{aligned}\overline{u}(t) &= A_{\overline{u}} + d + \mathcal{I}_{0^+}^{r;h}\lambda_2\\ &= \frac{d}{1-\mu} + \frac{\lambda\lambda_2}{(1-\mu)}\int_0^1 \frac{Q(\tau)}{\Gamma(r)}d\tau\\ &\quad + \frac{\lambda_2}{\Gamma(r)}\int_0^t (h(t)-h(s))^{r-1}h'(s)ds\\ &= \frac{d}{1-\mu} + \frac{\lambda\lambda_2}{(1-\mu)}\int_0^1 \frac{Q(\tau)}{\Gamma(r)}d\tau + \frac{\lambda_2(h(t)-h(0))^r}{\Gamma(r+1)}. \end{aligned}\tag{28}$$

On the other hand, we can verify that $\overline{u}(0) = \lambda \int_0^1 g(s)\overline{u}(s)ds + d$. In fact, taking $t \to 0$ in (28) then using (27), it follows that

$$\begin{aligned} \overline{u}(0) &= \frac{d}{1-\mu} + \frac{\lambda\lambda_2}{(1-\mu)} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} d\tau \\ &= \frac{d}{1-\mu} - \frac{\mu d}{1-\mu} + \frac{\mu d}{1-\mu} + \frac{\lambda\lambda_2}{(1-\mu)} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} d\tau \\ &= d + A_{\overline{u}} \\ &= d + \lambda \int_0^1 g(s)\overline{u}(s)ds. \end{aligned}$$

Taking into account (21), we have

$$\begin{aligned} \overline{u}(t) \geq\ & \frac{d}{1-\mu} + \frac{\lambda}{(1-\mu)} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} \overline{H}(\tau, \overline{u}(\tau)) d\tau \\ & + \frac{1}{\Gamma(r)} \int_0^t (h(t)-h(s))^{r-1} h'(s) \overline{H}(s, \overline{u}(s)) ds. \end{aligned}$$

It is clear that $\overline{u}$ is the upper solution of (4). Now let

$$ {}^{c}\mathcal{D}_{0^+}^{r;h} \underline{u}(t) = \lambda_1, \quad \underline{u}(0) = \lambda \int_0^1 g(s)\underline{u}(s)ds + d. $$

which has also a positive solution

$$\begin{aligned} \underline{u}(t) &= \frac{d}{1-\mu} + \frac{\lambda\lambda_1}{(1-\mu)} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} d\tau \\ &\quad + \frac{\lambda_1}{\Gamma(r)} \int_0^t (h(t)-h(s))^{r-1} h'(s) ds \\ &= \frac{d}{1-\mu} + \frac{\lambda\lambda_1}{(1-\mu)} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} d\tau + \frac{\lambda_1 (h(t)-h(0))^r}{\Gamma(r+1)}. \end{aligned}$$

By (21) and the same way that we used to reach the upper solution, we obtain

$$\begin{aligned} \underline{u}(t) \leq\ & \frac{d}{1-\mu} + \frac{\lambda}{(1-\mu)} \int_0^1 \frac{Q(\tau)}{\Gamma(r)} \underline{H}(\tau, \underline{u}(\tau)) d\tau \\ & + \frac{1}{\Gamma(r)} \int_0^t (h(t)-h(s))^{r-1} h'(s) \underline{H}(s, \underline{u}(s)) ds. \end{aligned}$$

Therefore, $\underline{u}$ is the lower solution of (4). An application of Theorem 3 shows that the problem (4) has at least one positive solution $u(t) \in X$ which verifies the inequalities (19) and (20). $\square$

Theorem 4. *Let a is a positive constant. Assume that $f(t,u) : J \times \mathbb{R}^+ \to [a, +\infty)$ is continuous and $g : J \to \mathbb{R}^+$ with $g \in L^1(J, \mathbb{R}^+)$ and $\mu^* = \int_0^1 g(s)ds < +\infty$. If*

$$a < \lim_{u \to +\infty} \max_{0 \le t \le 1} \frac{f(t,u)}{u} < +\infty, \tag{29}$$

then the problem (4) has at least one positive solution $u(t) \in C([0,\sigma], \mathbb{R}^+) \cap C^1([0,\sigma], \mathbb{R}^+)$ where $0 < \sigma < 1$.

Proof. According to assumption (29), there there exist $M_f > 0$ and $c_f > 0$ such that for any $u(t) \in X$, we have

$$f(t, u(t)) \le M_f u(t) + c_f.$$

By the definition of control function, we have

$$\overline{H}(t, u(t)) \le M_f u(t) + c_f. \tag{30}$$

On the other hand, we consider the following FDE

$${}^c\mathcal{D}_{0^+}^{r;h} u(t) = M_f u(t) + c_f, \quad 0 < r < 1, \quad 0 \le t \le 1. \tag{31}$$

According to Lemma 4, the equation (31) is equivalent to the following fractional integral equation

$$u(t) = d + \lambda \int_0^1 g(s)u(s)ds + \frac{1}{\Gamma(r)} \int_0^t (h(t) - h(s))^{r-1} h'(s) \left[M_f u(s) + c_f \right] ds.$$

Let $\Phi^* : \mathcal{E} \longrightarrow \mathcal{E}$ be an operator defined by

$$\begin{aligned} \left(\Phi^* u\right)(t) \; &= d + \lambda \int_0^1 g(s)u(s)ds \\ &\quad + \frac{1}{\Gamma(r)} \int_0^t (h(t) - h(s))^{r-1} h'(s) \left[M_f u(s) + c_f \right] ds. \end{aligned} \tag{32}$$

Now, we show that $\Phi^* : \mathcal{E} \longrightarrow \mathcal{E}$ is compact.

First, the operator $\Phi^* : \mathcal{E} \longrightarrow \mathcal{E}$ is continuous from the continuity of $h(t)$ and hypothesis of $g(t)$ with the Lebesgue dominated convergence theorem.

Next, let $S \subset \mathcal{E}$ be bounded, that is, there exists a positive constant R such that $\|u\| \le R$ for any $u \in S$, and setting

$$L_f := \max_{(t,u) \in [0,1] \times R} f(t, u(t)) + 1.$$

Then, for any $u \in S$ and $t \in [0,\sigma]$, we have

$$\begin{aligned}
\left|(\Phi^* u)(t)\right| &\le d + \lambda \int_0^1 g(s)\,|u(s)|\,ds \\
&\quad + \frac{1}{\Gamma(r)} \int_0^t (h(t)-h(s))^{r-1} h'(s)\left[M_f\,|u(s)| + c_f\right] ds \\
&\le d + \lambda \mu^* R + \frac{(h(\sigma)-h(0))^r}{\Gamma(r+1)}\left[M_f R + c_f\right] := \ell.
\end{aligned}$$

Thus $\|\Phi^* u\| \le \ell$. Hence, $\Phi^*(S)$ is uniformly bounded. Finally, we prove that $\Phi^*(S)$ is equicontinuous. For each $t \in [0,\sigma]$ and using (32), we can estimate the operator derivative as

$$\begin{aligned}
\left|(\Phi^* u)'(t)\right| &\le \frac{1}{\Gamma(r-1)} \int_0^t h'(s)(h(t)-h(s))^{r-2}\left[M_f\,|u(s)| + c_f\right] ds \\
&\le \frac{(h(\sigma)-h(0))^{r-1}}{\Gamma(r)}\left[M_f\,\|u\| + c_f\right].
\end{aligned}$$

Hence, for each $t_1, t_2 \in [0,\sigma]$ with $0 < t_1 < t_2 < \sigma$ and for $u \in S$, we get

$$\begin{aligned}
\left|\Phi^* u(t_2) - \Phi^* u(t_1)\right| &= \int_{t_1}^{t_2} \left|(\Phi^* u)'(s)\right| ds \\
&\le \frac{(h(\sigma)-h(0))^{r-1}}{\Gamma(r)}\left[M_f R + c_f\right]|t_2 - t_1|.
\end{aligned}$$

So, we can deduce that the right hand side tends to zero as $t_2 \to t_1$. Thus, $|\Phi^* u(t_2) - \Phi^* u(t_1)| \to 0$ as $t_2 \to t_1$, that is, the family $\{\Phi^* u;\, u \in S\}$ is equicontinuous. The Arzela-Ascoli Lemma implies that Φ^* is compact.

To apply Schauder's fixed point theorem, we need to verify that there exists a closed convex bounded subset in $B_\zeta \subset C([0,\sigma],\mathbb{R}^+)$ such that $\Phi^* B_\zeta \subset B_\zeta$. To this end, we define

$$B_\zeta = \left\{ u(t) \in \mathcal{E}, \left\| u - \frac{(h(t)-h(0))^r}{\Gamma(r+1)} c_f \right\| \le \zeta < +\infty \right\},$$

with $\zeta \ge \max\{2A, 2B\}$, where

$$A := d + \left[\lambda \mu^* + M_f \frac{(h(\sigma)-h(0))^r}{\Gamma(r+1)}\right] \frac{(h(1)-h(0))^r}{\Gamma(r+1)} c_f$$

and

$$B := \left[\lambda \mu^* + M_f \frac{(h(\sigma)-h(0))^r}{\Gamma(r+1)}\right] < 1. \tag{33}$$

Clearly, B_ζ is convex, bounded, and closed subset of the Banach space $C([0,\sigma],\mathbb{R}^+)$ where $0<\sigma<1$. Then for any $u\in B_\zeta$, we have

$$\|u\|\leq \frac{(h(t)-h(0))^r}{\Gamma(r+1)}c_f+\zeta\leq \frac{(h(\sigma)-h(0))^r}{\Gamma(r+1)}c_f+\zeta\leq \frac{(h(1)-h(0))^r}{\Gamma(r+1)}c_f+\zeta.$$

Thus

$$\begin{aligned}\left\|\Phi^* u-\frac{(h(t)-h(0))^r}{\Gamma(r+1)}c_f\right\| &\leq d+\lambda\mu^*\left[\frac{(h(1)-h(0))^r}{\Gamma(r)}c_f+\zeta\right]\\ &\quad +M_f\frac{(h(t)-h(0))^r}{\Gamma(r+1)}\left[\frac{(h(1)-h(0))^r}{\Gamma(r+1)}c_f+\zeta\right]\\ &\leq d+\left[\lambda\mu^*+M_f\frac{(h(\sigma)-h(0))^r}{\Gamma(r+1)}\right]\frac{(h(1)-h(0))^r}{\Gamma(r+1)}c_f\\ &\quad +\left[\lambda\mu^*+M_f\frac{(h(\sigma)-h(0))^r}{\Gamma(r+1)}\right]\zeta\\ &= A+B\zeta\leq\zeta.\end{aligned}$$

An application of Schauder's fixed point theorem shows that there exists at least a fixed point, and then (31) has at least one positive solution $u^*(t)$, where $0<t<\sigma$. Therefore, we have

$$u^*(t)=d+\lambda\int_0^1 g(s)u^*(s)ds+\frac{1}{\Gamma(r)}\int_0^t (h(t)-h(s))^{r-1}h'(s)\left[M_f u^*(s)+c_f\right]ds.$$

Combining condition (30), we get

$$u^*(t)\geq d+\lambda\int_0^1 g(s)u^*(s)ds+\frac{1}{\Gamma(r)}\int_0^t (h(t)-h(s))^{r-1}h'(s)\overline{H}(t,u^*(s))ds.$$

Obviously, $u^*(t)$ is the upper solution of the problem (4), and

$$\begin{aligned}v^*(s) &= d+\lambda\int_0^1 g(s)v^*(s)ds+\mathcal{I}_{0^+}^{r;h}(a)\\ &= d+\lambda\int_0^1 g(s)v^*(s)ds+\frac{a(h(1)-h(0))^r}{\Gamma(r+1)}>0\end{aligned}$$

is the lower solution of the problem (4). By Theorem 3, the problem (4) has at least one positive solution $u(t)\in C([0,\sigma],\mathbb{R}^+)$, where $0<\sigma<1$ and $v^*(t)\leq u(t)\leq u^*(t)$. □

3.2 Uniqueness of positive solution

In this subsection, we shall prove the uniqueness of positive solution by using the Banach fixed point theorem.

Theorem 5. *Let $f : J \times \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function that satisfy the following Lipschitz type condition*

$$|f(t,u) - f(t,v)| \leq M\,|u - v|\,, \tag{34}$$

where $t \in J$, $u, v \in \mathbb{R}^+$ and $M > 0$. Then the problem (4) has a unique positive solution $u \in C(J, \mathbb{R}^+)$, provided that

$$\Lambda := \left(\frac{\lambda e}{1-\mu} + \frac{(h(1) - h(0))^r}{\Gamma(r+1)}\right) M < 1, \tag{35}$$

where $\mu = \lambda \int_0^1 g(s)ds$ and $1 - \mu > 0$.

Proof. From Theorem 3, it follows that (4) has a least one positive solution in K. Hence, we need only to prove that the operator Φ defined in (13) is a contraction mapping on K. In fact, since for any $u, v \in K$, (34) and (35) are verified, then we have

$$\begin{aligned}
|\Phi u(t) - \Phi v(t)| &\leq \frac{\lambda}{1-\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(r)}\,|f(\tau,u(\tau)) - f(\tau,v(\tau))|\,d\tau \\
&\quad + \frac{1}{\Gamma(r)}\int_0^t (h(t) - h(s))^{r-1}h'(s)\,|f(s,u(s)) - f(\tau,v(\tau))|\,ds \\
&\leq \frac{\lambda}{1-\mu}eM\,\|u - v\| + \frac{(h(t) - h(0))^r}{\Gamma(r+1)}M\,\|u - v\| \\
&\leq \left(\frac{\lambda e}{1-\mu} + \frac{(h(t) - h(0))^r}{\Gamma(r+1)}\right) M\,\|u - v\| \\
&\leq \Lambda\,\|u - v\|\,,
\end{aligned}$$

which implies

$$\|\Phi u - \Phi v\| \leq \Lambda\,\|u - v\|\,.$$

As $\Lambda < 1$, the operator Φ is a contraction mapping and by Banach fixed point theorem, there exists a unique fixed point $u \in K$ such that $\Phi u(t) = u(t)$. Therefore u is the unique positive solution to the problem (4) on J. □

4 Examples

In this section, we present some examples to illustrate our results.

Example 1. Consider the following fractional problem

$$\begin{aligned} {}^{c}\mathcal{D}_{0^{+}}^{\frac{1}{2},2^{t}}u(t) &= \frac{1}{10}\left(\left|\sin u(t)\right|+1\right), \quad t\in[0,1], \\ u(0) &= \int_0^1 su(s)ds+1, \end{aligned} \tag{36}$$

where $r=\frac{1}{2}$, $,\lambda=d=1$. Set $f(t,u)=\frac{1}{10}\left(\left|\sin u\right|+1\right)$ and $g(t)=t$. Now, we show that (34) and (35) are satisfied. In fact, for any $u,v\in\mathbb{R}^{+}$ and $t\in[0,1]$ we have

$$\left|f(t,u)-f(t,v)\right|\leq\frac{1}{10}\left|\sin u-\sin v\right|\leq M\left|u-v\right|.$$

Set $h(t)=2^{t}$ for all $t\in[0,1]$. By a direct calculation, we can get

$$\begin{aligned} \Lambda &:= \left(\frac{\lambda e}{1-\mu}+\frac{(h(1)-h(0))^{r}}{\Gamma(r+1)}\right)M \\ &= \left(\frac{2e}{1}+\frac{1}{\Gamma(\frac{3}{2})}\right)\frac{1}{10}=\frac{1}{5}e+\frac{1}{5\sqrt{\pi}}<1. \end{aligned}$$

Also,

$$1-\mu=1-\int_0^1 sds=\frac{1}{2}>0.$$

Therefore, all conditions of Theorem 5 are satisfied. Hence the problem (36) has a unique positive solution on $[0,1]$.

Example 2. Consider the following fractional problem

$$\begin{cases} {}^{c}\mathcal{D}_{0^{+}}^{r;h}u(t)=f(t,u(t)), \quad t\in[0,1], \\ u(0)=\lambda\int_0^1 g(s)u(s)ds+d, \end{cases} \tag{37}$$

(i) Consider $r=\frac{1}{2}$, $\lambda=d=1$,

$$f(t,u)=1+\frac{4u}{\left[1+\cos(\frac{1}{u})\right]},$$

$g(t)=t$, and $h(t)=\frac{1}{40}\sqrt{t+1}$, for all $t\in[0,1]$. We show that (29) and (33) are satisfied. In fact, for any $u\in\mathbb{R}^{+}$ and $t\in[0,1]$ we have

$$1\leq\lim_{u\to+\infty}\max_{0\leq t\leq 1}\frac{f(t,u)}{u}=\lim_{u\to+\infty}\left(\frac{1}{u}+\frac{4}{(1+\cos(\frac{1}{u}))}\right)=2<+\infty,$$

and

$$f(t,u) \leq 1 + 2u = c_f + M_f u.$$

A simple computation showed $\mu^* = \frac{1}{2}$ and $B \approx 0.7 < 1$. With the use of Theorem 4, the problem (37) with hypotheses (i) has at least positive solution $u(t) \in C([0,\frac{1}{2}],\mathbb{R}^+)$ $(\sigma = \frac{1}{2})$.

(ii) Consider $r = \frac{1}{2}$, $d = 1$, $f(t,u) = \frac{4}{7}\left(t\,|\sin u| + 1\right)$, $g(t) = t$, and $h(t) = E_{\frac{1}{2}}(t)$, for all $t \in [0,1]$. Now, since f is continuous and

$$\frac{4}{7} \leq f(t,u) \leq \frac{8}{7}, \quad \text{for all } (t,u) \in [0,1] \times \mathbb{R}^+.$$

Case 1. if $\lambda = 0$. then the problem (37) with hypotheses (ii) has a positive solution which verifies $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$ where

$$\overline{u}(t) = \frac{\lambda_2 (h(t) - h(0))^r}{\Gamma(r+1)} + d = \frac{8}{7\Gamma(\frac{2}{3})}\sqrt{E_{\frac{1}{2}}(t) - 1} + 1,$$

and

$$\underline{u}(t) = \frac{\lambda_1 (h(t) - h(0))^r}{\Gamma(r+1)} + d = \frac{4}{7\Gamma(\frac{2}{3})}\sqrt{E_{\frac{1}{2}}(t) - 1} + 1,$$

are respectively the upper and lower solutions of the problem (37). Thus, Theorem 3 can be applied to the problem (37).

Case 2. If $0 < \lambda < 1$, then the problem (37) with hypotheses (ii) has a positive solution which verifies $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$ where

$$\begin{aligned}\overline{u}(t) &= \frac{d}{1-\mu} + \frac{\lambda\lambda_2}{1-\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(r)}d\tau + \frac{\lambda_2 (h(t) - h(0))^r}{\Gamma(r+1)} \\ &= \frac{1}{1-\frac{\lambda}{2}} + \frac{8\lambda}{1-\frac{\lambda}{2}}\frac{4}{9\sqrt{\pi}} + \frac{16\sqrt{t}}{7\sqrt{\pi}},\end{aligned}$$

and

$$\begin{aligned}\underline{u}(t) &= \frac{d}{1-\mu} + \frac{\lambda\lambda_1}{1-\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(r)}d\tau + \frac{\lambda_1 (h(t) - h(0))^r}{\Gamma(r+1)} \\ &= \frac{1}{1-\frac{\lambda}{2}} + \frac{4\lambda}{1-\frac{\lambda}{2}}\frac{4}{9\sqrt{\pi}} + \frac{8\sqrt{t}}{7\sqrt{\pi}},\end{aligned}$$

where $h(t) = t$, for all $t \in [0,1]$,

$$Q(\tau) = \int_\tau^1 \frac{s}{\sqrt{s-\tau}}ds = \frac{2}{3}\,(2\tau + 1)\sqrt{1-\tau},$$

and
$$\int_0^1 \frac{Q(\tau)}{\Gamma(r)} d\tau = \frac{28}{9\sqrt{\pi}}.$$
Thus, Theorem 3 can be applied to the problem (37).

Remark 1.

1. Our obtained results correspond to those for

   i) a nonlinear Riemann-Liouville-type fractional differential equation (1) if we set $h(t) = t$ and $\lambda = d = 0$, see [13];

   ii) a nonlinear Caputo-type fractional differential equation (2) if we set $h(t) = t$ and $g(s) \equiv 1$, see [3];

   iii) a nonlinear Caputo-Hadamard-type fractional differential equation (3) if we set $h(t) = log(t)$ and $g(s) \equiv 1$, see [7].

2. The corresponding problems involving Caputo-Katugampola-type [11] and Caputo-Hadamard-type [10] appear as a special case of our proposed problem for $h(t) \to t^\rho, \rho > 0$, and $h(t) \to log(t)$, respectively.

5 Conclusion

The existence and uniqueness of positive solutions for the generalized fractional differential equations (4) are investigated via the upper and lower solution method and the fixed point theorem in a cone. To derive the appropriate formula for the positive solution for the problem at hand is very important. The advantages of the problem considered and the importance of obtained results have been provided in the introduction section. It is the first work concerning results on positive solutions of generalized Caputo fractional differential equations. The main contribution of the work is to concentrate our attention on the existence and uniqueness of positive solutions of the more general proposed problem (4) which involving generalized version of the Caputo type fractional derivative. Moreover, we built the upper and lower control functions of the nonlinear term without any monotone requirement except the continuity.

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