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New approach to existence of solution for weighted Cauchy-type problem

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Abstract. In this paper, we consider a singular differential equation involving Hilfer-Katugampola fractional derivative with the weighted initial condition. The Picard iterative technique has been successfully applied to obtain the existence of a unique solution. First, we derive an equivalent integral equation, then construct the successive approximations and use the ratio test to discuss its convergence. We demonstrate our results through a suitable illustrative example.

Keywords: Fractional integrals and derivatives, Picard iterative technique, singular fractional differential equation, Cauchy-type problem.
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1 Introduction

In the last few decades, the wings of fractional calculus (FC) have been opened as an emerging trend of applied mathematics research with deep applications in almost all branches of science and engineering. At this stage, it covers the complex problems of the real world in physics, chemical processes and materials, signal and image processing, dynamical systems and engineering. Many researchers devoted to theory and applications of FC [13, 19, 21] and reported through survey articles [1, 2], research papers [4–9, 11, 14, 17, 22, 25] and books [20, 23, 24] to name few.

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During the theoretical development of FC, plenty of fractional differential and corresponding integral operators had come into existence and used by many researchers in their works. The Riemann-Liouville, Hadamard, Caputo, Hilfer, Katugampola are the frontiers and their theory become more popular. The investigation of qualitative properties of fractional differential equations (FDEs) is always at the centre of development of FC. The existence of the unique solution of various FDEs involving aforesaid operators can be found in [1,3,6,7,9–11,14,17,25].

In 2014, Yang and Liu [25] studied the existence and uniqueness of singular initial value problems (IVP) involving Riemann-Liouville and Caputo fractional derivatives using Picard iterative processes. Then authors, in [8,10], discussed the criteria for local existence and uniqueness of solution to initial value problem involving Hilfer and Hilfer-Hadamard derivative, respectively, where the equivalence between IVP and the Volterra integral equation was proved in convenient weighted spaces.

Recently, in [22], Oliveira et. al proposed generalization of Hilfer and Hilfer-Hadamard fractional derivatives, popularly called Hilfer-Katugampola derivative. We observe that the existence and uniqueness of solution for IVP involving Hilfer-Katugampola derivative were discussed, but iterative scheme for approximating solutions was not reported in the literature.

Motivated by the works cited above, we consider the following weighted Cauchy-type problem

$$\begin{cases} \left({}^{\rho}D^{\alpha,\beta}_{a+}x\right)(t) = f(t,x), & t\in\Omega,\ \rho>0,\ 0<\alpha<1,\ 0\le\beta\le1, \\ \lim\limits_{t\to a^+}\Big(\dfrac{t^\rho-a^\rho}{\rho}\Big)^{1-\gamma}x(t) = x_a, & x_a\in E\subset\mathbb{R},\ \gamma=\alpha+\beta(1-\alpha), \end{cases} \tag{1}$$

where $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is the given function, ${}^{\rho}D^{\alpha,\beta}_{a+}$ is the Hilfer-Katugampola fractional derivative of order α and type β with $\Omega=[a,b], 0<a<b\le+\infty$. The nonlinear function f may be singular at $t=a$ satisfies assumptions given in Section 2 and initial condition of problem under consideration is more suitable in engineering applications. We prove the existence and uniqueness of a solution to Cauchy-type problem (1) using its equivalent integral representation, properties of beta function and the ratio test (as a convergence criterion). Further, we construct the computable iterative scheme to approximate the solution. To our knowledge, solution obtained for the proposed problem has not been reported in the literature.

In the next section, we list all the definitions and lemmas useful in subsequent sections. In Section 3, we obtain the equivalent integral equation and the existence of a unique solution by using Picard iterative technique. We then give an illustrative example in Section 4 to support our findings. Concluding remarks are given in the last section.

2 Preliminaries

In this section, we enlist the following definitions and properties from basic fractional calculus [19, 22].

As usual C denotes the Banach space of all continuous functions $x : \Omega \to E$, with the superemum (uniform) norm

$$\|x\|_\infty = \sup_{t\in\Omega} \|x(t)\|_E$$

and $AC(\Omega)$ be the space of absolutely continuous functions from Ω into E. Denote $AC^1(\Omega)-$ the space defined by

$$AC^1(\Omega) = \left\{ x : \Omega \to E | \frac{d}{dt} x(t) \in AC(\Omega) \right\}.$$

Throughout the paper, let $\delta_\rho^n = (t^{\rho-1} \frac{d}{dt})^n$, $n = [\alpha] + 1$, and mention $[\alpha]$ as integer part of α. Define the space

$$AC_{\delta_\rho}^n = \left\{ x : \Omega \to E | \delta_\rho^{n-1} x(t) \in AC(\Omega) \right\}, \quad n \in \mathbb{N}.$$

Note that $C_{0,\rho}(\Omega) = C(\Omega)$ is the space of continuous functions.

Here $L^p(a,b)$, $p \geq 1$, is the space of Lebesgue integrable functions on (a,b). Let the Euler's gamma and beta functions are defined respectively, by

$$\Gamma(x) = \int_0^{+\infty} s^{x-1} e^{-s} ds, \quad \mathbb{B}(x,y) = \int_0^1 (1-s)^{x-1} s^{y-1} ds, \quad x > 0,\ y > 0.$$

It is well known that $\mathbb{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, for $x > 0,\ y > 0$.

Definition 1. *[15]* Let $\alpha \in \mathbb{R}_+, c \in \mathbb{R}$ and $g \in X_c^p(a,b)$, where $\mathbb{R}_+ = [0, \infty)$ and $X_c^p(a,b)$ is the space of Lebesgue measurable functions. The left-sided Katugampola fractional integral of order α is defined by

$$({}^\rho I_{a+}^\alpha g)(t) = \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} \frac{g(s)}{\Gamma(\alpha)} ds, \qquad t > a,\ \rho > 0,$$

where $\Gamma(\cdot)$ is a Euler's gamma function.

Definition 2. *[16]* Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\rho > 0$. The left-sided Katugampola fractional derivative ${}^\rho D_{a+}^\alpha$ of order α is defined by

$$\begin{aligned} ({}^\rho D_{a+}^\alpha g)(t) &= \delta_\rho^n ({}^\rho I_{a+}^{n-\alpha} g)(t) \\ &= \left(t^{\rho-1} \frac{d}{dt} \right)^n \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho} \right)^{n-\alpha-1} \frac{g(s)}{\Gamma(n-\alpha)} ds. \end{aligned}$$

Definition 3. *[22] The Hilfer-Katugampola fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ with respect to t is defined by*

$$({}^{\rho}D^{\alpha,\beta}_{a+}g)(t) = ({}^{\rho}I^{\beta(1-\alpha)}_{a+}\delta_{\rho}{}^{\rho}I^{(1-\beta)(1-\alpha)}_{a+}g)(t), \quad \rho > 0, \tag{2}$$

for the function for which right hand side expression exists.

Remark 1. The left-sided Hilfer-Katugampola operator ${}^{\rho}D^{\alpha,\beta}_{a+}$ can be written as

$${}^{\rho}D^{\alpha,\beta}_{a+} = {}^{\rho}I^{\beta(1-\alpha)}_{a+}\delta_{\rho}{}^{\rho}I^{1-\gamma}_{a+} = {}^{\rho}I^{\beta(1-\alpha)}_{a+}{}^{\rho}D^{\gamma}_{a+}, \quad \gamma = \alpha + \beta - \alpha\beta.$$

Remark 2. [22] The fractional derivative ${}^{\rho}D^{\alpha,\beta}_{a+}$ is an interpolator of the following fractional derivatives: Hilfer ($\rho \to 1$) [13], Hilfer-Hadamard ($\rho \to 0^+$) [14], Katugampola ($\beta = 0$) [16], Caputo-Katugampola ($\beta = 1$) [3], Riemann-Liouville ($\beta = 0, \rho \to 1$) [19], Hadamard ($\beta = 0, \rho \to 0^+$) [18], Caputo ($\beta = 1, \rho \to 1$) [19], Caputo-Hadamard ($\beta = 1, \rho \to 0^+$) [12], Liouville ($\beta = 0, \rho \to 1, a = 0$) and Weyl ($\beta = 0, \rho \to 1, a = -\infty$) [19].

Lemma 1. *[15] If $\alpha, \beta > 0, 1 \le p \le \infty, 0 < a < b < \infty$ and $\rho, c \in \mathbb{R}$ for $\rho \ge c$. Then, for $g \in X^p_c(a,b)$ the following relation hold:*

$$({}^{\rho}I^{\alpha}_{a+}{}^{\rho}I^{\beta}_{a+}g)(t) = ({}^{\rho}I^{\alpha+\beta}_{a+}g)(t).$$

Lemma 2. *[22] Let $t > a$, ${}^{\rho}I^{\alpha}_{a+}$ and ${}^{\rho}D^{\alpha}_{a+}$ are as in Definition 1 and Definition 2, respectively. Then the following hold:*

$$(i) \quad \left({}^{\rho}I^{\alpha}_{a+}\Big(\frac{s^{\rho}-a^{\rho}}{\rho}\Big)^{\sigma}\right)(t) = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+\alpha+1)}\Big(\frac{t^{\rho}-a^{\rho}}{\rho}\Big)^{\sigma+\alpha}, \quad \alpha \ge 0, \sigma > 0,$$

$$(ii) \quad \text{for } \sigma = 0, \quad \left({}^{\rho}I^{\alpha}_{a+}\Big(\frac{s^{\rho}-a^{\rho}}{\rho}\Big)^{\sigma}\right)(t) = \big({}^{\rho}I^{\alpha}_{a+}1\big)(t) = \frac{\big(\frac{t^{\rho}-a^{\rho}}{\rho}\big)^{\alpha}}{\Gamma(\alpha+1)}, \quad \alpha \ge 0,$$

$$(iii) \quad \text{for } 0 < \alpha < 1, \quad \left({}^{\rho}D^{\alpha}_{a+}\Big(\frac{s^{\rho}-a^{\rho}}{\rho}\Big)^{\alpha-1}\right)(t) = 0.$$

The following lemma has great importance in the proof of our further main results.

Lemma 3. *[23] Suppose that $x > 0$. Then*

$$\Gamma(x) = \lim_{m\to+\infty}\frac{m^x m!}{x(x+1)(x+2)\cdots(x+m)}.$$

We denote $D = [a, a+h]$, $E = \{x : |x(\frac{t^\rho - a^\rho}{\rho})^{1-\gamma} - x_a| \leq b\}$, $D_h = (a, a+h]$ for $h > 0, b > 0$ and $t \in D_h$. Here we choose $I = (a, a+l]$ and $J = [a, a+l]$ such that

$$l = \min\left\{h, \left(\frac{b}{M}\frac{\Gamma(\alpha)}{\mathbb{B}(\alpha,k+1)}\right)^{\frac{1}{\mu+k}}\right\}, \quad \mu = 1 - \beta(1-\alpha).$$

A function $x(t)$ is said to be a solution of the Cauchy-type problem (1), if there exists $l > 0$ such that $x \in C^0(a, a+l]$ satisfies FDE ${}^\rho D_{a+}^{\alpha,\beta} x(t) = f(t,x)$ almost everywhere on I along with the initial value

$$\lim_{t \to a^+} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} x(t) = x_a.$$

To prove the existence of solution of the Cauchy-type problem (1), let us make the following hypotheses:

(H_1) $(t,x) \to f(t, (\frac{t^\rho - a^\rho}{\rho})^{\gamma-1} x(t))$ is defined on $D_h \times E$ satisfies:

(i) $x \to f(t, (\frac{t^\rho - a^\rho}{\rho})^{\gamma-1} x(t))$ is continuous on E for all $t \in D_h$, $t \to f(t, (\frac{t^\rho - a^\rho}{\rho})^{\gamma-1} x(t))$ is measurable on D_h for all $x \in E$;

(ii) for all $t \in D_h$ and $x \in E$, there exist $k > (\beta(1-\alpha) - 1)$ and $M \geq 0$ such that $|f(t, (\frac{t^\rho - a^\rho}{\rho})^{\gamma-1} x(t))| \leq M(\frac{t^\rho - a^\rho}{\rho})^k$ holds.

(H_2) for all $t \in I$ and $x_1, x_2 \in E$, there exists $A > 0$ such that $|f(t, (\frac{t^\rho - a^\rho}{\rho})^{\gamma-1} x_1(t)) - f(t, (\frac{t^\rho - a^\rho}{\rho})^{\gamma-1} x_2(t))| \leq A(\frac{t^\rho - a^\rho}{\rho})^k |x_1 - x_2|$.

Remark 3. In hypothesis (H_1), if $(\frac{t^\rho - a^\rho}{\rho})^{-k} f(t, (\frac{t^\rho - a^\rho}{\rho})^{\gamma-1} x(t))$ is continuous on $D \times E$, one may choose $M = \max\limits_{t \in D} (\frac{t^\rho - a^\rho}{\rho})^{-k} f(t, (\frac{t^\rho - a^\rho}{\rho})^{\gamma-1} x(t))$ continuous on $D_h \times E$ for all $x \in E$.

3 Main results

Here, we state and prove the existence and uniqueness of solution to the Cauchy-type problem (1). We develop the iteration scheme to approximate the solution and discuss its convergence.

Lemma 4. *Suppose that (H_1) holds. Then $x : J \to \mathbb{R}$ is a solution of the Cauchy-type problem (1) if and only if $x : I \to \mathbb{R}$ is a solution of the integral equation*

$$x(t) = x_a \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{f(s, x(s))}{\Gamma(\alpha)} ds. \quad (3)$$

Proof. Suppose that $x : I \to \mathbb{R}$ is a solution of the Cauchy-type problem (1). Then $|\big(\frac{t^\rho - a^\rho}{\rho}\big)^{1-\gamma}x(t) - x_a| \leq b$ for all $t \in I$. From assumption (H_1), for all $t \in I$, there exist $k > (\beta(1-\alpha) - 1)$ and $M \geq 0$ such that

$$|f(t, x(t))| = \big|f\big(t, \big(\frac{t^\rho - a^\rho}{\rho}\big)^{\gamma-1}\big(\frac{t^\rho - a^\rho}{\rho}\big)^{1-\gamma}x(t)\big)\big| \leq M\big(\frac{t^\rho - a^\rho}{\rho}\big)^k.$$

Then we have,

$$\begin{aligned}\Big|\int_a^t s^{\rho-1}\big(\frac{t^\rho - s^\rho}{\rho}\big)^{\alpha-1}\frac{f(s, x(s))}{\Gamma(\alpha)}ds\Big| &\leq \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} M\frac{\big(\frac{s^\rho - a^\rho}{\rho}\big)^k}{\Gamma(\alpha)}ds\\ &= M\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+k}\frac{\mathbb{B}(\alpha, k+1)}{\Gamma(\alpha)}.\end{aligned}$$

Clearly,

$$\lim_{t\to a+}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{f(s, x(s))}{\Gamma(\alpha)}ds = 0.$$

It follows that

$$x(t) = x_a\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{f(s, x(s))}{\Gamma(\alpha)}ds, \quad t \in I.$$

Since $k > (\beta(1-\alpha) - 1)$, then $x \in C^0(I)$ is a solution of integral equation (3).

On the other hand, we can see that $x : I \to \mathbb{R}$ is a solution of the integral equation (3) implies that x is solution of the Cauchy-type problem (1) defined on J. The proof is complete. $\square$

To prove existence and uniqueness of solution of the Cauchy-type problem (1), we choose a Picard function sequence as follows:

$$\begin{cases}\phi_0(t) &= x_a\left(\dfrac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}, \qquad t \in I,\\ \phi_n(t) &= \phi_0(t) + \displaystyle\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{f(s, \phi_{n-1}(s))}{\Gamma(\alpha)}ds, \ n = 1, 2, \ldots.\end{cases} \tag{4}$$

We state the following existence-uniqueness theorem.

Theorem 1. *Suppose that (H_1)-(H_2) hold. Then the Cauchy-type problem (1) has a unique continuous solution, $\phi(t) = \big(\frac{t^\rho - a^\rho}{\rho}\big)^{\gamma-1}\lim_{n\to\infty}\big(\frac{t^\rho - a^\rho}{\rho}\big)^{1-\gamma}\phi_n(t)$ defined on I, with $\phi_0(t)$ and $\phi_n(t)$ given by (4).*

Lemma 5. *Suppose (H_1) holds. Then ϕ_n is continuous on I and satisfies* $\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\phi_n(t)-x_a\right|\leq b.$

Proof. By assumption (H_1), for all $t\in D_h$ and $|x\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}-x_a|\leq b$, we have

$$\left|f\left(t,\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}x\right)\right|\leq M\left(\frac{t^\rho-a^\rho}{\rho}\right)^k.$$

For $n=1$, we have

$$\phi_1(t)=x_a\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\int_a^t s^{\rho-1}\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\frac{f(s,\phi_0(s))}{\Gamma(\alpha)}ds. \tag{5}$$

Then

$$\begin{aligned}\left|\int_a^t s^{\rho-1}\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\frac{f(s,\phi_0(s))}{\Gamma(\alpha)}ds\right| &\leq \int_a^t s^{\rho-1}\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}M\frac{\left(\frac{s^\rho-a^\rho}{\rho}\right)^k}{\Gamma(\alpha)}ds\\ &= M\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+k}\frac{\mathbb{B}(\alpha,k+1)}{\Gamma(\alpha)}.\end{aligned}$$

Clearly, $\phi_1\in C^0(I)$ and from equation (5), we have

$$\begin{aligned}\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\phi_1(t)-x_a\right| &\leq \left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}M\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+k}\frac{\mathbb{B}(\alpha,k+1)}{\Gamma(\alpha)}\\ &\leq Ml^{\alpha+k+1-\gamma}\frac{\mathbb{B}(\alpha,k+1)}{\Gamma(\alpha)}.\end{aligned} \tag{6}$$

By induction hypothesis, for $n=m$ and for all $t\in J$, suppose that $\phi_m\in C^0(J)$ and $|(\frac{t^\rho-a^\rho}{\rho})^{1-\gamma}\phi_m(t)-x_a|\leq b$. We obtain

$$\phi_{m+1}(t)=x_a\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\int_a^t s^{\rho-1}\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\frac{f(s,\phi_m(s))}{\Gamma(\alpha)}ds. \tag{7}$$

From above discussion, we obtain $\phi_{m+1}(t)\in C^0(I)$ and by equation (7),

$$\begin{aligned}&\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\phi_{m+1}(t)-x_a\right|\\ &\quad\leq \left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\int_a^t s^{\rho-1}\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}M\frac{\left(\frac{s^\rho-a^\rho}{\rho}\right)}{\Gamma(\alpha)}ds\\ &\quad= M\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+k+1-\gamma}\frac{\mathbb{B}(\alpha,k+1)}{\Gamma(\alpha)}\\ &\quad\leq Ml^{\alpha+k+1-\gamma}\frac{\mathbb{B}(\alpha,k+1)}{\Gamma(\alpha)}\leq b.\end{aligned}$$

Thus, the result holds for $n=m+1$. As an application of principle of mathematical induction, it is true for all n. Hence the proof is complete. $\square$

Theorem 2. *Suppose* (H_1)-(H_2) *hold. Then the sequence* $\{(\frac{t^\rho-a^\rho}{\rho})^{1-\gamma}\phi_n(t)\}$ *is uniformly convergent on* J.

Proof. For $t\in J$, consider the series

$$\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\phi_0(t)+\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}[\phi_1(t)-\phi_0(t)]+\cdots \\ +\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}[\phi_n(t)-\phi_{n-1}(t)]+\cdots. \tag{8}$$

Using relation (6) in the proof of Lemma 5,

$$\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}|\phi_1(t)-\phi_0(t)|\le M\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+k+1-\gamma}\frac{\mathbb{B}(\alpha,k+1)}{\Gamma(\alpha)},\quad t\in J.$$

From Lemma 5, we have

$$\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}|\phi_2(t)-\phi_1(t)|=AM\frac{\mathbb{B}(\alpha,\alpha+2k+2-\gamma)}{\Gamma(\alpha)} \\ \times\frac{\mathbb{B}(\alpha,k+1)}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{2(\alpha+k+1-\gamma)}.$$

Now suppose for $n=m$

$$\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}|\phi_{m+1}(t)-\phi_m(t)|\le A^mM\left(\frac{t^\rho-a^\rho}{\rho}\right)^{(m+1)(\alpha+k+1-\gamma)}\times P_m,$$

where

$$P_m=\prod_{i=0}^{m}\frac{\mathbb{B}(\alpha,(i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma(\alpha)}. \tag{9}$$

We have

$$\begin{aligned}&\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\Big|\phi_{m+2}(t)-\phi_{m+1}(t)\Big|\\ &\le\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}\int_a^t s^{\rho-1}\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\frac{|f(s,\phi_{m+1}(s))-f(s,\phi_m(s))|}{\Gamma(\alpha)}ds\\ &\le\frac{(\frac{t^\rho-a^\rho}{\rho})^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}A\left(\frac{s^\rho-a^\rho}{\rho}\right)^{k}\\ &\qquad\times\left[\left(\frac{s^\rho-a^\rho}{\rho}\right)^{1-\gamma}|\phi_{m+1}(s)-\phi_m(s)|\right]ds,\end{aligned}$$

which gives

$$\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\Big|\phi_{m+2} - \phi_{m+1}\Big| \le A^{m+1}M\left(\frac{t^\rho - a^\rho}{\rho}\right)^{(m+2)(\alpha+k+1-\gamma)}P_{m+1}.$$

This means the result is true for $n = m+1$. By the principal of mathematical induction, result is true for all n.

$$\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}|\phi_{n+2}(t) - \phi_{n+1}(t)| \le A^{n+1}Ml^{(n+2)(\alpha+k+1-\gamma)}P_{n+1}. \quad (10)$$

Now to prove convergence, we consider

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} MA^{n+1}l^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\mathbb{B}(\alpha,(i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma(\alpha)}.$$

We obtain

$$\begin{aligned}\frac{u_{n+1}}{u_n} &= \frac{MA^{n+2}l^{(n+3)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+2}\frac{\mathbb{B}(\alpha,(i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma(\alpha)}}{MA^{n+1}l^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\mathbb{B}(\alpha,(i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma(\alpha)}}\\ &= Al^{\alpha+k+1-\gamma}\frac{\Gamma((n+3)k+(n+2)(\alpha+1-\gamma)+1)}{\Gamma((n+3)(k+\alpha)+(n+2)(1-\gamma)+1)}.\end{aligned}$$

By using Lemma 3, notation Θ for $\alpha+k+1-\gamma$, we obtain

$$\frac{u_{n+1}}{u_n} = Al^{\Theta}\frac{\lim\limits_{m\to\infty}\frac{m^{(n+3)k+(n+2)(\alpha+1-\gamma)+1}m!}{((n+3)k+(n+2)(\alpha+1-\gamma)+1)\cdots((n+3)k+(n+2)(\alpha+1-\gamma)+m+1)}}{\lim\limits_{m\to\infty}\frac{m^{(n+3)(k+\alpha)+(n+2)(1-\gamma)+1}m!}{((n+3)(k+\alpha)+(n+2)(1-\gamma)+1)\cdots((n+3)(k+\alpha)+(n+2)(1-\gamma)+m+1)}}$$

$$= Al^{\Theta}\left[\lim_{m\to\infty} m^{-\alpha}\frac{((n+3)(k+\alpha)+(n+2)(1-\gamma)+1)\cdots((n+3)(k+\alpha)+(n+2)(1-\gamma)+m+1)}{((n+3)k+(n+2)(\alpha+1-\gamma)+1)\cdots((n+3)k+(n+2)(\alpha+1-\gamma)+m+1)}\right].$$

We observe that $\frac{((n+3)(k+\alpha)+(n+2)(1-\gamma)+1)\cdots((n+3)(k+\alpha)+(n+2)(1-\gamma)+m+1)}{((n+3)k+(n+2)(\alpha+1-\gamma)+1)\cdots((n+3)k+(n+2)(\alpha+1-\gamma)+m+1)}$ is bounded for all m, n. Then $\dfrac{u_{n+1}}{u_n} \to 0$ as $n \to \infty$. Thus, the series $\sum\limits_{n=1}^{\infty} u_n$ is convergent. Hence the series (8) is uniformly convergent for $t \in J$. Therefore the sequence $\{(\frac{t^\rho - a^\rho}{\rho})^{1-\gamma}\phi_n(t)\}$ is uniformly convergent on J. □

Theorem 3. *Suppose that* (H_1) *and* (H_2) *are satisfied. Then*

$$\phi(t) = (\frac{t^\rho - a^\rho}{\rho})^{\gamma-1}\lim_{n\to\infty}(\frac{t^\rho - a^\rho}{\rho})^{1-\gamma}\phi_n(t)$$

is unique continuous solution of the integral equation (3) *defined on* J.

Proof. Since $\phi(t) = (\frac{t^\rho - a^\rho}{\rho})^{\gamma-1} \lim_{n\to\infty} (\frac{t^\rho - a^\rho}{\rho})^{1-\gamma} \phi_n(t)$ on J, and by Lemma 5, we can get $(\frac{t^\rho - a^\rho}{\rho})^{1-\gamma} |\phi(t) - x_0| \le b$. Then

$$|f(t,\phi_n(t)) - f(t,\phi(t))| \le A\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^k |\phi_n(t) - \phi(t)|, \quad t \in I,$$

which gives

$$\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{-k} |f(t,\phi_n(t)) - f(t,\phi(t))| \le A|\phi_n(t) - \phi(t)| \to 0,$$

uniformly as $n \to +\infty$ on I. Therefore

$$\begin{aligned}
\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma} \phi(t) &= \lim_{n\to\infty} \phi_n(t) \\
&= x_0 + \frac{(\frac{t^\rho - a^\rho}{\rho})^{1-\gamma}}{\Gamma(\alpha)} \lim_{n\to\infty} \int_a^t s^{\rho-1} \Big(\frac{t^\rho - s^\rho}{\rho}\Big)^{\alpha-1} \Big(\frac{s^\rho - a^\rho}{\rho}\Big)^k \\
&\quad \times \Big(\Big(\frac{s^\rho - a^\rho}{\rho}\Big)^{-k} f(s,\phi_{n-1}(s))\Big) ds \\
&= x_0 + \frac{(\frac{t^\rho - a^\rho}{\rho})^{1-\gamma}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} \Big(\frac{t^\rho - s^\rho}{\rho}\Big)^{\alpha-1} \Big(\frac{s^\rho - a^\rho}{\rho}\Big)^k \\
&\quad \times \lim_{n\to\infty} \Big(\Big(\frac{s^\rho - a^\rho}{\rho}\Big)^{-k} f(s,\phi_{n-1}(s))\Big) ds \\
&= x_0 + \frac{(\frac{t^\rho - a^\rho}{\rho})^{1-\gamma}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} \Big(\frac{t^\rho - s^\rho}{\rho}\Big)^{\alpha-1} f(s,\phi(s)) ds.
\end{aligned}$$

Then ϕ is a continuous solution of integral equation (3) defined on J.

To prove uniqueness of solution, if possible, suppose that $\psi(t)$ defined on I is also solution of the integral equation (3). Then $(\frac{t^\rho - a^\rho}{\rho})^{1-\gamma} |\psi(t)| \le b$ for all $t \in I$ and

$$\psi(t) = x_0 \Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\gamma-1} + \int_a^t s^{\rho-1} \Big(\frac{t^\rho - s^\rho}{\rho}\Big)^{\alpha-1} f(s,\phi(s)) ds, \quad t \in I.$$

It is sufficient to prove that $\phi(t) \equiv \psi(t)$ on I. From (H_1), there exists $k > (\beta(1-\alpha) - 1)$ and $M \ge 0$ such that

$$|f(t,\psi(t))| = |f(t, (\frac{t^\rho - a^\rho}{\rho})^{\gamma-1} (\frac{t^\rho - a^\rho}{\rho})^{1-\gamma} \psi(t))| \le M (\frac{t^\rho - a^\rho}{\rho})^k,$$

for all $t \in I$. Therefore,

$$\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} |\phi_0(t) - \psi(t)| = \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left| \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} f(s,\psi) ds \right|$$
$$\leq \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)} \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} M \left(\frac{s^\rho - a^\rho}{\rho}\right)^k ds$$
$$= \frac{M}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+k+1-\gamma} \frac{\mathbb{B}(\alpha, k+1)}{\Gamma(\alpha)}.$$

Furthermore, we have

$$\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} |\phi_1(t) - \psi(t)|$$
$$= \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)} \left| \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} [f(s,\phi_0(s)) - f(s,\psi(s))] ds \right|$$
$$\leq AM \frac{\mathbb{B}(\alpha, k+1)}{\Gamma(\alpha)} \frac{\mathbb{B}(\alpha, \alpha+2k+2-\gamma)}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{2(\alpha+k+1-\gamma)}.$$

By the induction hypothesis, we suppose that

$$\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} |\phi_n(t) - \psi(t)| \leq A^n M \left(\frac{t^\rho - a^\rho}{\rho}\right)^{(n+1)(\alpha+k+1-\gamma)} P_n.$$

Then

$$\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} |\phi_{n+1}(t) - \psi(t)|$$
$$\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left| \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} [f(s,\phi_n(s)) - f(s,\psi(s))] ds \right|$$
$$\leq A^{n+1} M \left(\frac{t^\rho - a^\rho}{\rho}\right)^{(n+2)(\alpha+k+1-\gamma)} \prod_{i=0}^{n+1} \frac{\mathbb{B}(\alpha, (i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma(\alpha)}$$
$$\leq A^{n+1} M l^{(n+2)(\alpha+k+1-\gamma)} \prod_{i=0}^{n+1} \frac{\Gamma((i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k) + i(1-\gamma)+1)}.$$

By repeating the same arguments used in the proof of Theorem 2, we obtain the convergent series

$$\sum_{n=1}^{\infty} A^{n+1} M l^{(n+2)(\alpha+k+1-\gamma)} \prod_{i=0}^{n+1} \frac{\Gamma((i+1)k + i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k) + i(1-\gamma)+1)}.$$

Therefore, $A^{n+1}Ml^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}$ converges to zero as $n\to\infty$. We observe that $\lim_{n\to\infty}\big(\frac{t^\rho-a^\rho}{\rho}\big)^{1-\gamma}\phi_n(t)=\big(\frac{t^\rho-a^\rho}{\rho}\big)^{1-\gamma}\psi(t)$ uniformly on J. Thus $\phi(t)\equiv\psi(t)$ on I. $\square$

Proof of Theorem 1:

Proof. In the light of Lemma 4 and from Theorem 3, one can easily deduce that

$$\phi(t)=\big(\frac{t^\rho-a^\rho}{\rho}\big)^{\gamma-1}\lim_{n\to\infty}\big(\frac{t^\rho-a^\rho}{\rho}\big)^{1-\gamma}\phi_n(t),$$

is the unique continuous solution of the Cauchy-type problem (1) defined on I. Thus the proof is ended here. $\square$

4 An example

We consider the following singular Cauchy-type problem

$$\begin{cases}\big({}^\rho D_{a+}^{\alpha,\beta}\big)x(t)=f(t,x(t)), \quad t>a,\\ \lim\limits_{t\to a}\big(\frac{t^\rho-a^\rho}{\rho}\big)x(t)=x_a, \quad \gamma=\alpha+\beta(1-\alpha),\end{cases} \tag{11}$$

where

$$\begin{cases}f(t,x(t))=\dfrac{\big(\frac{t^\rho-a^\rho}{\rho}\big)^{-\frac{5}{8}}\big(1+\big|\sin\big(\frac{t^\rho-a^\rho}{\rho}\big)\big|\big)}{64\big(1+\sqrt{\big(\frac{t^\rho-a^\rho}{\rho}\big)}\big)\sin\big(\frac{t^\rho-a^\rho}{\rho}\big)}, & t\neq a,\\ f(t,x(t))=0, & t=a.\end{cases}$$

One can easily see that, f is singular at $t=a$, and is a continuous function for $t\in(a,b]=(1,2]$.

Let us choose $\mu=\frac{5}{6}$, $b=4, \rho=0.5>0$, $k=-\frac{5}{8}>-\frac{5}{6}$. Then in correspondence with Cauchy-type problem (1), we have $\alpha=\frac{1}{4}$, $\beta=\frac{1}{2}$ gives $\gamma=\frac{5}{8}$. Thus

$$l=\min\left\{0.828428, \left(\frac{4}{M}\frac{\Gamma(0.625)}{\Gamma(0.375)}\right)^{4.8}\right\},$$

where

$$M=\max_{t\in[1,2],x\in[1,4]}\frac{\big(1+\big|\sin\big(\frac{t^\rho-a^\rho}{\rho}\big)\big|\big)}{64\big(1+\sqrt{\big(\frac{t^\rho-a^\rho}{\rho}\big)}\big)\big(\sin\big(\frac{t^\rho-a^\rho}{\rho}\big)\big)}\approx 0.019281,$$

with

$$\begin{cases} \phi_0(t) = x_a\big(\frac{t^\rho - a^\rho}{\rho}\big)^{-\frac{3}{4}}, \qquad t \in (1, l], \\ \phi_n(t) = \phi_0(t) + \frac{1}{\Gamma(\frac{1}{4})}\displaystyle\int_a^t s^{-0.5}\big(\frac{t^{0.5} - a^{0.5}}{0.5}\big)^{-\frac{3}{4}} f(s, \phi_{n-1}(s))ds, \ n = 1, 2, \ldots. \end{cases}$$

We observe that, all the conditions of Theorem 1 are satisfied. Therefore, the Cauchy-type problem (1) has the unique continuous solution $\phi(t)$,

$$\phi(t) = \big(\frac{t^\rho - a^\rho}{\rho}\big)^{-\frac{3}{4}} \lim_{n\to+\infty} \big(\frac{t^\rho - a^\rho}{\rho}\big)^{\frac{3}{4}} \phi_n(t),$$

on $[1, 4]$.

5 Concluding remarks

The existence and uniqueness of solution for a general class of FDEs is obtained with the help of Picard successive approximations. The function $f(t, x)$ considered without assuming the monotonic property and the iterative scheme is developed for approximating the unique solution. With the help of traditional convergence criteria, the ratio test, the uniform convergence of solution of the Cauchy-type problem is established. Through our results, we demonstrated the remedy for disclosing a definite interval for the existence of a solution which could not be determined by fixed point theory. Our results essentially improved the existing results in the literature.

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References

[1] S. Abbas, M. Benchohra, J.E. Lagreg and Y. Zhou, *A survey on Hadamard and Hilfer fractional differential equations: Analysis and stability,* Chaos Solitons Fractals **102** (2017) 47–71.

[2] R.P. Agarwal, M. Benchohra and S. Homani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions,* Acta. Appl. Math. **109** (2010) 973–1033.

[3] R. Almeida, A.B. Malinowska and T. Odzijewicz, *Fractional differential equations with dependence on the Caputo-Katugampola derivative,* J. Comput. Nonlinear Dynam. **11**(6) (2016), ID 061017, 11 pages.

[4] S.P. Bhairat, *Existence and continuation of solution of Hilfer fractional differential equations*, J. Math. Model. **7**(1) (2019) 1–20.

[5] S.P. Bhairat, *On stability of generalized Cauchy-type problem,* Dyn. Contin. Discrete Impuls. Syst. Ser. A- Math. Anal., Special Issue–RAMSA–2017, **27** (4) (2020) 235-244.

[6] S.P. Bhairat and D.B. Dhaigude, *Existence of solution of generalized fractional differential equation with nonlocal initial conditions,* Math. Bohem. **144**(2) (2019) 203–220.

[7] S.P. Bhairat and D.B. Dhaigude, *Existence and stability results for fractional differential equations involving generalized Katugampola derivative,* Studia U.B.B. Mathematica, **65**(1) (2020) 29–46.

[8] S.P. Bhairat, *On existence and approximation of solution of Hilfer fractional differential equations,* 2019, Submitted.

[9] D.B. Dhaigude and S.P. Bhairat, *Existence and uniqueness of solution of Cauchy-type problem for Hilfer fractional differential equations,* Commun. Appl. Anal. **22**(1) (2018) 121–134.

[10] D.B. Dhaigude and S.P. Bhairat, *Local existence and uniqueness of solution of Hilfer-Hadamard fractional differential equations,* Nonlinear Dyn. Syst. Theory **18**(2) (2018) 144–153.

[11] K.M. Furati, M.D. Kassim and N.E-.Tatar, *Existence and uniqueness for a problem involving Hilfer fractional derivative,* Comput. Math. Appl., **64** (2012) 1616–1626.

[12] Y. Gambo, F. Jarad, D. Baleanu and T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives, Adv. Differ. Eq. **10** (2014) 1–12.

[13] R. Hilfer, *Applications of fractional calculus in physics*, World Scientific, Singapore, 2000.

[14] M.D. Kassim, K.M. Furati and N.E-. Tatar, *On a differential equation involving Hilfer-Hadamard fractional derivative*, Abst. Appl. Anal., Special Issue (1085-3375), 2012, 17 pages.

[15] U.N. Katugampola, *New approach to a generalized fractional integral*, Appl. Math. Comput. **218** (2011) 860–865.

[16] U.N. Katugampola, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. Appl. **6** (2014) 1–15.

[17] U.N. Katugampola, *Existence and uniqueness results for a class of generalized fractional differential equations*, arXiv:1411.5229v2 [math.CA], 2016.

[18] A.A. Kilbas, *Hadamard-type fractional calculus*, J. Korean Math. Soc. **38**(6) (2001) 1191–1204.

[19] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies 204, Editor: Jan Van Mill, Elsevier, Amsterdam, The Netherlands, 2006.

[20] V. Lakshmikantham, *Theory of fractional dynamic systems*, Cambridge Scientific Publishers, 2009.

[21] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity,* an Introduction to Mathematical Model, Imperical College Press, World Scientific Publishing, London, 2010.

[22] D.S. Oliveira and E. Capelas de Oliveira, *Hilfer-Katugampola fractional derivative*, arXiv:1705.07733v1 [math.CA], 2017.

[23] I. Podlubny, *Fractional differential equations,* in: Mathematics in Science and Engineering, Vol. 198, Acad. Press, San-Diego, 1999.

[24] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional intergrals and derivatives: theory and applications*, Gorden and Breach, Amsterdam, 1987, (Engl. Trans. from Russian, 1993).

[25] X. Yang and Y. Liu, *Picard iterative processes for initial value problems of singular fractional differential equations*, Adv. Differ. Eq. **102** (2014) 17 pages.