**Journal of Mathematical Modeling**
Vol. 8, No. 3, 2020, pp. 241–255

# Vehicular traffic models for speed-density-flow relationship

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**Abstract.** The relationship among vehicles on the road is modeled using fundamental traffic equations. In traffic modeling, a particular speed-density equation usually fits a peculiar dataset. The study seeks to parameterize some existing fundamental models so that a given equation could match different dataset. The new equations are surmisal offshoots from existing equations. The parameterized equations are used in the LWR model and solved using the Lax-Friedrichs differencing scheme. The simulation results illustrate different scenarios of acceleration and deceleration traffic wave profiles. The proposed models appropriately explain the varying transitions of different traffic regimes.

*Keywords*: LWR model, shockwaves, speed-density equation, traffic flow.
*AMS Subject Classification 2010*: 76L05, 53A17, 76M20.

## 1 Introduction

Traffic flow theory expounds on how vehicles and drivers interact with each other and with road infrastructure. Traffic flow models aid to clearly understand and predict this vehicle-driver-infrastructure behavior. The

*Corresponding author.
Received: 10 January 2020 / Revised: 31 March 2020 / Accepted: 11 April 2020.
DOI: 10.22124/jmm.2020.15409.1370

simplest explanation of traffic dynamics is through the speed-density-flow traffic equations. Fundamental diagrams are pictorial representations of the speed-density-flow equations. The mathematical equations derived from these diagrams are called fundamental equations. Fundamental equations are powerful tools for macroscopic traffic analysis. They are often used in the LWR model to explain vehicular traffic.

A research engineer of the traffic bureau at the Ohio State Highway Department was the first to explore the interrelationship among vehicle speed ($u$), traffic density ($k$), and traffic flux ($q$) – in an inquiry to ascertain the wasted time during congestion [11]. Data for the research was collected on a dual-lane using the photographic method. The outcome was a simple model depicting an inverse-linear relationship between vehicle speed and traffic density. His fundamental plots comprised the following: $u_{max}$ as the maximum free-flow speed, and $k_{max}$ as the density at jam traffic. These two parameters $(u_{max}, k_{max})$ materialize in virtually all speed-density models. Also, $k_{crit}, u_{crit}$ and $q_{crit}$ are the critical values for density, speed, and flow respectively. Greenshields categorized vehicular traffic flow into two main regimes, namely: the free-flow regime and the congested regime. In the congested regime, he detected that the relationship between speed and density lacked harmonious uniformity. Modern researchers have attributed this inconsistency to traffic hysteresis [15, 30].

By empirical analysis, Pipes [24] presented a speed-concentration equation in his review of existing car-following models, (Index 6, Table 1). The speed-density curve has a concave shape when the shape parameter $n$ is less than unity, and convex when the shape parameter is greater than unity. The speed and density are related in a linear manner as Greenshields fundamental diagram when the shape parameter equals one. In the traffic sense, $n$ less than one typifies drivers that slow down when traffic concentration is enormous. The instances where drivers reduce their speed at larger headway such as driving at night and through underground passageway is represented by $n$ greater than one. Pipes model had identical properties as Greenberg's single regime equation when $n$ equals 1.71828. The mathematics of these fundamental models are detailed in Table 1.

Some correspondences to Pipes' model are the non-integer deterministic car-following conceptualization [20] and Drew's exponential model [7]. May & Keller's speed-concentration model [20] has two unique parameters $m_1$ and $m_2$. The parameter $m_1$ accounts for the speed of an accompanying vehicle, while $m_2$ accounts for the sensitivity factor for the space-gap between two successive vehicles. When $m_1 = 1$, the model [20] is analogous to Drew's equation. Moreover, the model [20] is identical to the Greenshields framework when $m_1 = m_2 = 1$. The linear equation by May & Keller can be theorized as the generic form of these three models [7, 11, 24].

In the late twentieth century, Jayakrishnan et al. [13] modified the Greenshields equation by introducing the average minimum speed at jam traffic ($u_{min}$). They derived a dynamic assignment model using optimization techniques.

Greenberg [10] derived his logarithmic equation utilizing fluid dynamics construct. He aimed to improve the Greenshields model because the data used for the earlier formulation was relatively smaller. On the contrary, this logarithm model failed to mimic the traffic conditions for a near-empty road. Vehicles speed increase infinitely as the density approaches zero. In the recent past, Ardekani and Ghandehari[2] introduced a minimum density quantity ($k_{min}$) to solve the problem of unbounded velocity. The variable $k_{min}$ is the least density during off-peak traffic.

The maiden conjecture of an exponential speed-density model was postulated by Underwood[27]. He experimented his construct utilizing three different flow cases: the normal flow, the unstable flow, and the forced flow. The same year, Newell [21] proposed an exponential fundamental equation by introducing a constant of proportionality, $\lambda$. This constant was derived from the slope of the velocity-headway plot. The entire model formulation was deduced from the car-following theory.

Again, a bell-shaped curve was educed by a group of Northwestern researchers [6]. This model was reformulated by Papageorgiou et al.[22] using experimental data from Boulevard Peripherique in the Capital of France. In contrast, the numerical quantity in the bell-shaped model was designed to vary in Papageorgiou et al. proposition.

Another radically distinctive representation of this functional equation was derived based on measurement from a loop detector [28]. The author was not only concerned with freeway information but an arterial road, as well as underground tunnels. The resulting fundamental plots of his four-parameter model was as a sigmoid flexure. The model [28] also reverts to the Greenshields type when $A_1 = A_3 = 0$, and $A_2 = u_{max}/k_{max}$. Also, related to the van Aerde model is the fundamental equation by MacNicholas [19]. Although both models have akin speed-density plots, the four-parameter model is difficult to handle. Another offshoot of these equations was a ramification of the Greenshields model through a parametric vector [23]. The Greenshields linear functional is deducted from this five-parameter model when $[E_1, E_2, E_3, E_4, E_5] = [1, 1, 1, 1, 1]$.

Moreover, an exponential KK model was ushered in by Kerner and

Table 1: Some existing speed-density equations.

| **Author(s)** | **Relationship Equation** |
|---|---|
| Greenshields[11] | $u = u_{\max}\left(1 - \frac{k}{k_{\max}}\right)$ |
| Greenberg [10] | $u = u_{crit}\ln\left(\frac{k_{\max}}{k}\right)$ |
| Underwood [27] | $u = u_{\max}\exp\left(-\frac{k}{k_{crit}}\right)$ |
| Newell [21] | $u = u_{\max}\left\{1 - \exp\left[-\frac{\lambda}{u_{\max}}\left(\frac{1}{k} - \frac{1}{k_{max}}\right)\right]\right\}$ |
| Drake et al. [6] | $u = u_{\max}\exp\left[-\frac{1}{2}\left(\frac{k}{k_{crit}}\right)^2\right]$ |
| Pipes [24] | $u = u_{\max}\left(1 - \frac{k}{k_{\max}}\right)^n$ |
| May and Keller [20] | $u = u_{\max}\left[1 - \left(\frac{k}{k_{\max}}\right)^{m_1}\right]^{m_2}$ |
| Drew1968 [7] | $u = u_{\max}\left[1 - \left(\frac{k}{k_{\max}}\right)^{\frac{p+1}{2}}\right]$ |
| Papageorgiou et al. [22] | $u = u_{\max}\exp\left[-\frac{1}{\alpha}\left(\frac{k}{k_{crit}}\right)^{\alpha}\right]$ |
| Kerner and Konhäuser [14] | $u = u_{\max}\left[\frac{1}{1+\exp\left(\frac{\frac{k}{k_{crit}}-0.25}{0.06}\right)} - 372\times 10^{-8}\right]$ |
| Del Castillo and Benitez [3] | $u = u_{\max}\left\{1 - \exp\left[\frac{K_w}{u_{max}}\left(1 - \frac{k_{max}}{k}\right)\right]\right\}$ |
| Del Castillo and Benitez [4] | $u = u_{\max}\left\{1 - \exp\left[1 - \exp\left(\frac{K_w}{u_{max}}\left(\frac{k_{max}}{k} - 1\right)\right)\right]\right\}$ |
| Jayakrishnan et al. [13] | $u = u_{\min} + (u_{\max} - u_{\min})\left(1 - \frac{k}{k_{max}}\right)$ |
| van Aerde [28] | $k = \frac{1}{A_1 + uA_3 + \frac{A_2}{u_{\max}-u}}$ |
| Lee et al. [16] | $u = u_{\max}\left(1 - \frac{k}{k_{\max}}\right)\left[1 + E\left(\frac{k}{k_{max}}\right)^{\gamma}\right]^{-1}$ |
| MacNicholas [19] | $u = u_{\max}\left[\frac{k_{\max}^{\psi} - k^{\psi}}{k_{\max}^{\psi} + \vartheta k^{\psi}}\right]$ |
| Ardekani and Ghandehari [2] | $u = u_{crit}\ln\left(\frac{k_{\max}+k_{\min}}{k+k_{\min}}\right)$ |
| Wang et al. [29] | $u = u_{avg} + \frac{u_{\max}-u_{avg}}{\left[1+\exp\left(\frac{k-k_t}{\theta_1}\right)\right]^{\theta_2}}$ |
| Wang et al. [29] | $u = u_{avg} + \frac{u_{\max}-u_{avg}}{\left[1+\exp\left(\frac{k-k_t}{\theta_1(k_t)}\right)\right]^{\theta_2(k_t)}}$ |
| Del Castillo [5] | $u = \frac{1}{k}\left[(u_{\max}k)^{-\theta} + (1-k)^{-\theta}\right]^{-\frac{1}{\theta}}$ |
| Péter and Fazekas [23] | $u = \frac{E_4 u_{\max}}{E_3 + E_2\left(\frac{k}{1-k^{E_5}}\right)^{E_1}}$ |
| Gaddam and Rao [9] | $u = u_{\max}\left[1 - \left(\frac{k}{k_{\max}}\right)^{\gamma_1}\right]^{\gamma_2}\left[1 + E\left(\frac{k}{k_{\max}}\right)^{\gamma_3}\right]^{-1}$ |
| Gaddam and Rao [9] | $u = u_{\max}\left\{\frac{\exp\left[-\left(\frac{k}{k_{crit}}\right)^{1+\delta_1}\right] - \exp\left[-\left(\frac{k_{\max}}{k_{crit}}\right)^{1+\delta_1}\right]}{1-\exp\left[-\left(\frac{k_{\max}}{k_{crit}}\right)^{1+\delta_1}\right]}\right\}^{\delta_2}$ |

Konhäuser [14] in an attempt to parameterize clusters in traffic. Later, Lee at al. [16] introduced a new fundamental expression called the rational model. The authors adopted the same hydrodynamic model as in the KK cluster analysis to deduce their rational model.

The latest development of single-regime speed-density model was through a statistical and theoretical analysis of mixed traffic flow [9]. The authors presented two different speed-density equations: the exponential and rational models. They introduced three shape parameters to correct the inadequacies of the rational model [16].

These multiple speed-density models outlined in Table 1 make it difficult for researchers in selecting the most appropriate equation for their research works. This paper seeks to lessen this burden by parameterizing the main classes of fundamental models. These classes include linear, logarithm, exponential and logistic models. In this paper, a single equation is proposed to represent all speed-density equations that have a linear relationship. With some assumptions, the new model will always revert to one of its classical forms, being [7, 11, 13, 20, 24]. Similar derivations are obtained for logarithm models, exponential models, and logistic models. The proposed equations are compared with their classical forms using traffic wave profiles. As a shortcoming, the theoretical properties of these models are not presented in this research.

The rest of the paper is organized as follows. In section two, five new fundamental equations are presented. Namely: the generalized linear equation, the generalized logarithm equation, the simplified exponential equation, the composite exponential equation, and the simplified logistic equation. The mathematics of the Lax-Friedrichs numerical scheme is detailed in section three. In this same section, a comparative analysis of these models using traffic waves are also presented. In section four, we present a general summary of the entire work.

## 2 Proposed speed-density equations

In this section, five speed-density equations are hypothesized from existing equations. These proposed equations are the generalized linear equation, the generalized logarithm equation, the simplified exponential equation, the composite exponential equation, and the simplified logistic equation. In some cases, new parameters are infused into some existing models to obtain the new equation. These parameters aim to scale and shape these equations to appropriately fit realistic data.

The relationship between density and speed can be expressed in a linear form. Some of these equations that express such relationship are the Greenshields' model [11], the Pipes' model [24], May and Keller model [20], Drew model [7], and the model by Jayakrishnan et al. [13]. Here, these linear equations are conflated to obtain the generalized linear speed-density equations. The generalized linear model is expressed as:

$$u = u_{\min} + (u_{\max} - u_{\min}) \left[1 - \left(\frac{k}{k_{\max}}\right)^{\beta_1}\right]^{\beta_2}. \tag{1}$$

The values of $\beta_1$ and $\beta_2$ are significantly important for macroscopic traffic modeling. These equations [7, 11, 13, 20, 24] are deducible from the generalized linear model under the following conditions.

- The Greenshields' model [11] is obtained when $u_{\min} = 0, \beta_1 = \beta_2 = 1$.
- The Pipes' model [24] is obtained when $u_{\min} = 0, \beta_1 = 1$.
- The May & Keller model [20] is obtained when $u_{\min} = 0$.
- Drew's model [7] is obtained when $u_{min} = 0, \beta_2 = 1$.
- The Jayakrishnan et al. model [13] is obtained when $\beta_1 = \beta_2 = 1$.

Also, two equations were studied to derive the generalized logarithm model. These include the models by Greenberg [10] and Ardekani & Ghandehari [2]. Merging these two equations and introducing the shape parameter $\phi$, we obtained the generalized logarithm equation as:

$$u = u_{crit} \ln \left(\frac{k_{\max} + k_{\min}}{k + k_{\min}}\right)^{\phi}. \tag{2}$$

With this reformulation, speed converges to $u_{crit} \ln(1 + k_{\max}/k_{\min})$ as $k$ approaches 0. Greenberg's model is obtained when $k_{\min} = 0$ and $\phi = 1$. With $\phi = 0$, we obtain the model by Ardekani & Ghandehari.

Moreover, the models by [6, 22, 27] are put together as a singleton. This is again a novel speed-density equation. Because of the simplicity of the proposed equation, compared to other exponential models, it is captioned as the simplified exponential model. It is mathematically expressed as:

$$u = u_{\max} \exp \left[-\left(\frac{k}{k_{crit}}\right)^{\alpha_1}\right]^{\alpha_2}. \tag{3}$$

Del Castillo and his collaborator Benitez also brought forth two similar models to explain the interrelationship among speed, density, and flow [3, 4]. Their first paper on the general theory produced a kinematic wave exponential model and a maximum sensitivity model in their empirical analysis. This model is also known as the double exponential equation. In this research, this equation has been generalized as the composite exponential model (4):

$$u = u_{\max}\left\{1 - \exp\left[1 - \exp\left(\frac{\varphi_1}{u_{\max}}\left(\frac{k_{\max}}{k} - 1\right)^{\varphi_2}\right)\right]\right\}^{\varphi_3}. \tag{4}$$

Though the exponential curve was somewhat different from the maximum sensitivity curve, the parameter to account for the kinematic wave speed $K_w$ was present in each formulation. $K_w$ is unrestricted in this current formulation. That is to say $\varphi_1 = 1$.

The models by Wang el al. [29] are called the logistic speed-density equations. From Table 1, the parameters of the logistic model are: $u_{avg}$, the mean speed during stop and go flow; $k_t$, the switching point from free-flow to cluster traffic; $\theta_1$, a scale parameter; and $\theta_2$, a parameter controlling the skewness of a curve. The pictorial form of this five-parameter model is double curved as the letter $S$. Since $\theta_1$ and $\theta_2$ were not easily observable from field data, the scale parameter and the lopsided parameter were made functions of the transition term. To ease computational difficulties, these terms have been replaced with autonomous parameters. The result is a simplified logistic model (5):

$$u = u_{\max}\left[1 + \exp\left(k - k_t\right)^{\eta_1}\right]^{\eta_2}, \qquad \eta_2 < 0. \tag{5}$$

There has been adequate parameterization in this present surmises to ensure that all the proposed models are coherent to macroscopic flow phenomena.

# 3 Analysis and results

## 3.1 Numerical scheme

Two renowned papers 'on kinematic waves ... ' [18] and 'shockwaves on the highway' [25] produced an outstanding traffic flow model called the LWR equation. The model equation

$$k_t(x,t) + q_x(x,t) = f(x,t), \tag{6}$$

was derived from the theory of fluid mechanics. The authors likened the flow of vehicular traffic to the flow of water in a pipe; thus, the birthing of aggregate traffic models. Before that, only microscopic flow quantities were studied. With the LWR formulation, $f(x,t)$ is used as a source term. Surprisingly, the LWR model can also be used to model the dynamics of air traffic flow [26]. The quantities $k(x,t)$ and $q(x,t)$ are the respective density and flow rate at the location $x$ and time $t$. $u(x,t)$ is the vehicular speed. These variables are related by the equation

$$q(x,t) = k(x,t) \cdot u(x,t). \tag{7}$$

Equation (7), the flow function, is often used in (6) for macroscopic traffic investigation. A more detailed flow characteristics are evident when the speed-density equation is used in the LWR model for traffic analysis [1, 8, 12]. With the absence of the source term $f(x,t)$ from equation (6), we obtain the homogeneous LWR model. This homogeneous model is solved numerically using the Lax-Friedrichs scheme. The partial differential equation $k_t(x,t) + q_x(x,t) = 0$ is discretized and solved as a Riemann problem using the Lax-Friedrichs finite difference method. Thence

$$k_j^{n+1} = \frac{1}{2}\left(k_{j+1}^n + k_{j-1}^n\right) - \frac{l}{2h}\left[f(k_{j+1}^n) - f(k_{j-1}^n)\right]. \tag{8}$$

The discretization index, $n$ represents the temporal step size, while $j$ represents the spatial step size. The variables $h$ and $l$ are the grid point for total distance and time respectively. Equation (8) is the non-conservative form of the LWR model. This method may sometimes fail to converge to a discontinuous solution. Therefore, the Lax scheme is further expressed in its conservative form

$$k_j^{n+1} = k_j^n - \frac{l}{h}\left[F\left(k_{j+1}^n, k_j^n\right) - F(k_j^n, k_{j-1}^n)\right],$$

to prevent solution divergence, where

$$F(k_{j+1}, k_j) = h/2l\,(k_j - k_{j+1}) + 0.5[f(k_j) + f(k_{j+1})],$$

and $F$ is the numerical flux function [17].

For consistency of the scheme, the following should be satisfied:

$$F(k,k) = \frac{h}{2l}(k-k) + \frac{1}{2}\left[f(k) + f(k)\right] = f(k),$$

and

$$\begin{aligned} |F(x,y) - F(k,k)| &= \\ &\left|\frac{h}{2l}(x-k) - \frac{h}{2l}(y-k) + \frac{1}{2}\left[f(x) - f(k)\right] + \frac{1}{2}\left[f(y) - f(k)\right]\right| \\ &\le (h/l + L_c)\max\{|x-k|, |y-k|\}, \end{aligned}$$

where $L_c$ is a constant for the actual flow rate $f$. It is Lipschitz continuous.

### 3.2 Simulation results

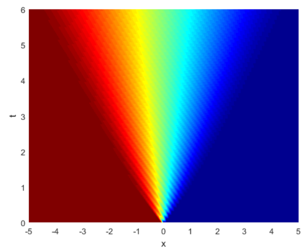
The waves dissolution and its formation are illustrated using the following initial density profiles.

| | Shockwave | Rarefaction wave |
|---|---|---|
| Upstream density | $k_u = 0.57veh/m$ | $k_u = 0.80veh/m$ |
| Downstream density | $k_d = 0.89veh/m$ | $k_d = 0.20veh/m$ |

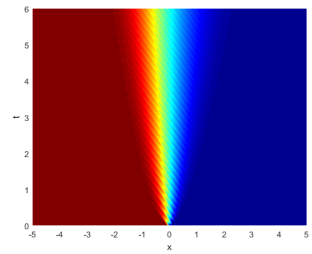
For the numerical analysis, a 1000 meter road stretch is divided into 100 cells. For this simulation, density and velocity are both standardized such that $k, u \in [0, 1]$. The basic parameter values used for the simulation are: $u_{\max} = 1.00m/s$, $u_{crit} = 0.81m/s$, $k_{\max} = 1.00veh/m$, $k_{crit} = 0.455veh/m$. The minimum velocity during jam traffic is given as $u_{min} = 0.071m/s$. $k_t = 0.61$. $k_{min} = 0.132veh/m$ is the least density during off-peak flow. The values for the scale and shape parameters are deduced from their respective literature. These values are chosen in connection to a realistic flow.

There are four illustrations each for wave dissolution (Figures 1-2). Figures 3 to 5 are used to exemplify queue formation. In the figures, $x$ is the spatial distance and $t$ is the total simulation time. The linear models and logarithm models are compared with some of its corresponding classical forms using rarefaction wave profiles. These are respectively illustrated using (Figure 1) and (Figure 2). The exponential models and logistic models are also compared with some of its corresponding classical forms using shockwave profiles. These are also detailed using Figures 3 to 5. These plots differ depending on the value of either the shape or scale parameter. From these plots, all the proposed models have the potency to replicate varying wave characteristics.

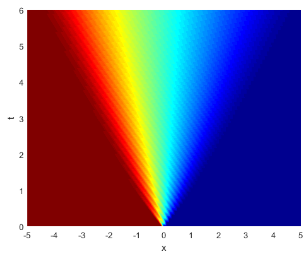
Given the point of discontinuity as $x = 0$ (being an exemplary position of a traffic light), it can be observed from (Figures 1-2) that the jam dissolves gradually over time. This is because of the relatively low traffic density downstream. For these rarefaction waves, it takes more than six minutes for the waves to fade away totally. The time for total wave dissolution depends also on the scale and shape parameters. Comparing the simulation results of Greenshields and Greenberg models to these new proposals (generalized linear model and generalized logarithm model), we noticed that the former models have rigid dissolution fans as opposed to these new offshoots. The new postulations can capture diverse jam dissipation. The profiles for the generalized linear model with $\beta_1 = 1.55$ dissolves faster than the simulation results of the same model with the same parameter equals 0.11.



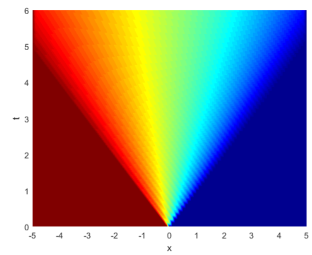
Greenshields.



Generalized linear with $\beta_1 = 0.55$, $\beta_2 = 1.8$.



Generalized linear with $\beta_1 = 1.55$, $\beta_2 = 1.8$.



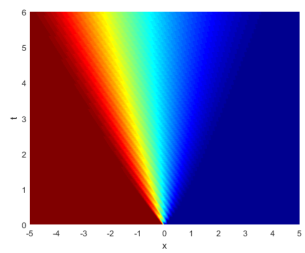
Generalized linear with $\beta_1 = 2.55$, $\beta_2 = 1.8$.

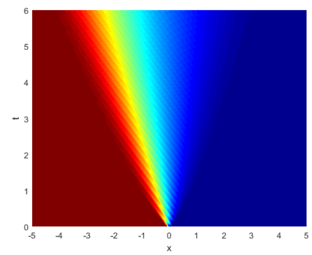
Figure 1: Rarefaction wave profiles using the generalized linear model.

These characterizations are made possible by the introduction of these new parameters. Note that a different set of values will lead to various unique rarefaction fans. These fans are again illustrated using the generalized logarithm model (Figures 2). The value of $\phi$ determines the speed of the dissolution fans.

The backward moving wave is illustrated using the exponential and the logistic models (Figure 3-5). Different shockwave structures are illustrated using these proposed equations. Though the existing models can capture the traffic shockwave phenomenon, the length of these queues was often left to chance. From these plots, we realize that it is possible to model different transitions between stop traffic and averagely moving traffic. Each set of parameter values will yield different results.

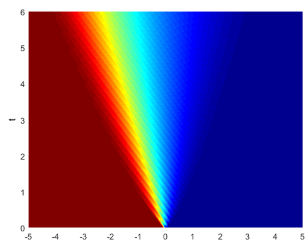
The results from all these models are consistent with real traffic flow. Although the existing speed-density models are well able to characterize these wave profiles, the proposed equations have extra parametric terms that could be used to accurately model certain nonlinear traffic features.

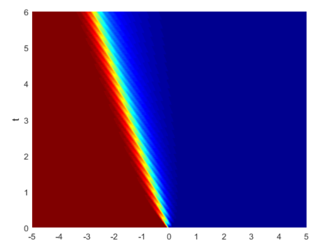




Greenberg.

Generalized logarithm with $\phi = 0.0314$.

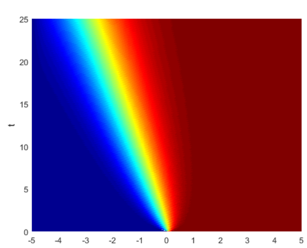


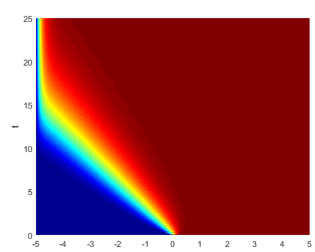


Generalized logarithm with $\phi = 1.314$.

Generalized logarithm with $\phi = 2.314$.

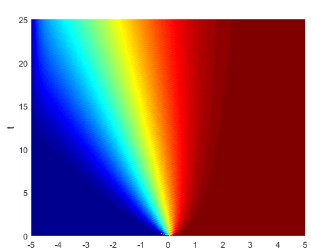
Figure 2: Rarefaction wave profiles using the generalized logarithm model.

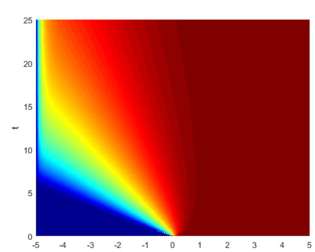




Underwood.

Simp. Exp. with $\alpha_1 = 1.7$, $\alpha_2 = 0.8$.

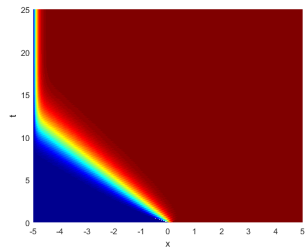


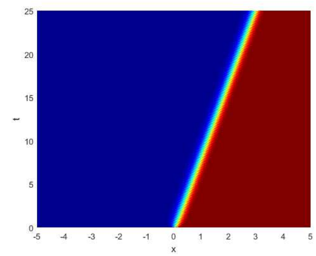


Simp. Exp. with $\alpha_1 = 1.7$, $\alpha_2 = 1.8$.

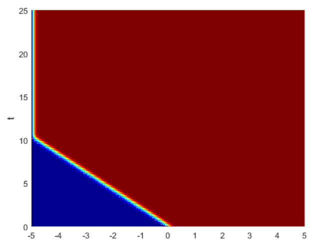
Simp. Exp. with $\alpha_1 = 2.7, \alpha_2 = 0.8$.

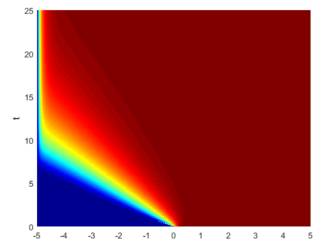
Figure 3: Shockwave profiles using the simplified exponential (Simp. Exp.) model.





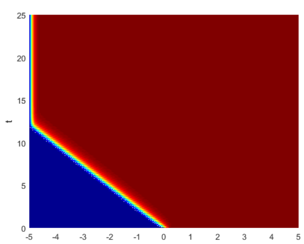
DelCastillo & Benitez with $k_w = 0.43$. Comp. Exp. with $\varphi_2 = 0.41$, $\varphi_3 = 0.6$.

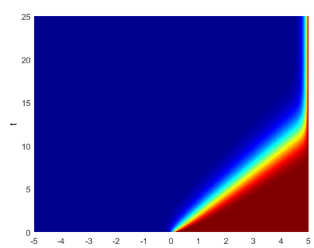




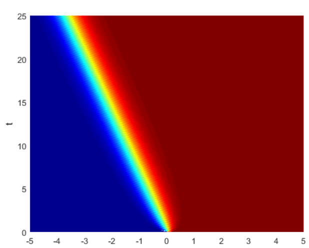
Comp. Exp. with $\varphi_2 = 0.41, \varphi_3 = 1.6$. Comp. Exp. with $\varphi_2 = 0.41$, $\varphi_3 = 4.6$.

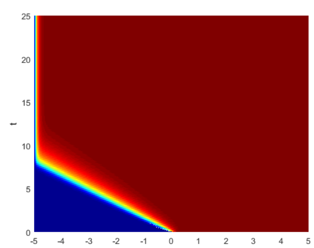
Figure 4: Shockwave profiles using the composite exponential (Comp. Exp.) model.





Wang et al. with $\theta_1 = 0.037$, $\theta_2 = 0.117$. Simp. Log. $\eta_1 = 0.3$, $\eta_2 = -1.117$.





Simp. Log. $\eta_1 = 3.0$, $\eta_2 = -1.117$. Simp. Log. $\eta_1 = 6.0$, $\eta_2 = -1.117$.

Figure 5: Shockwave profiles using the simplified logistic (Simp. Log.) model.

## 4 Conclusion

Single-regime fundamental models are hypothetical descriptions of the pragmatic complexity of traffic. These models are also called speed-density equations or simply fundamental equations. In this treatise, we presented a thorough review of existing single-regime speed-density models. The paper adds to these classical equations; five new propositions. In specifics, the Greenshields' types of models were coalesced as the generalized linear model. A similar derivation was used to obtain the generalized logarithm equation. However, the existing exponential and logistic equations were remodeled to ease theoretical and analytic computations. Finally, a composite exponential model was deduced from the Del Castillo family of models.

The fundamental equations were used in the LWR to examine their respective degrees of precision in mimicking vehicular traffic waves. The equations were expressed as a Riemann problem and solved using the Lax-Friedrichs numerical scheme. These equations were compared with some existing fundamental models using acceleration and deceleration wave profiles. The proposed speed-density models characterized the underlying flow properties of vehicular traffic. Moreover, the proposed models possess unique tweaking ability to control transitions of different traffic regimes.

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