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A new approach for solving multi-variable orders differential equations with Prabhakar function

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Abstract. In this paper, we use Chebyshev polynomials to seek the numerical solution of a class of multi-variable order fractional differential equation (MVODEs) that the fractional derivative is described in the Caputo-Prabhakar sense. Using operational matrices, the original equations are transferred to a system of algebraic equations. By solving the system of equations, the numerical solutions are acquired that this system may be solved numerically using an iterative algorithm. The effectiveness and convergence analysis of the numerical scheme is illustrated through four numerical examples.

Keywords: Prabhakar function, multi-variable order, fractional derivative, the fifth-kind Chebyshev polynomials, numerical method.
AMS Subject Classification 2010: 34A08, 34k20, 65N35.

1 Introduction

Fractional calculus becomes a central branch of mathematical analysis and differential equations. Differential equations with fractional order derivatives have important applications in different fields of science, economics and finance, like physics, engineering, possibly including fractal phenomena, including numerical analysis and so on [3, 4, 8–11, 14–17, 23]. In this

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paper, we survey and study a new approach for solving multi variable orders differential equations(MVODEs) with Prabhakar function (the order of derivative (or integral) operator is not a constant but it is a function of space, time or other variables)as followed:

$$\sum_{i=1}^{N} \eta_i(t)\mathbf{D}^{\gamma}_{\rho,\mu_i(t),\omega,a^+}\Theta(t) = f(t,\Theta(t)),\ 0<t<1,$$
$$\Theta(0) = \Theta_0, \tag{1}$$

where N is a positive integer number and $\mu_i(t)$ are bounded in the interval $[0,1]$. $\eta_i(t)$ are known functions. $\Theta(t)$ is a continuously differentiable function. Also the operator $\mathbf{D}^{\gamma}_{\rho,\mu_i(t),\omega,a^+}$ is a differential operator in the sense of the Caputo-Prabhakar fractional derivatives of order $\mu_i(t)$ that the Caputo-Prabhakar fractional derivatives is obtained by modifying the Riemann-Liouville integral operator by replacing its kernel by a Prabhakar function, that this function is defined as follows:

$$E^{\gamma}_{\rho,\mu}(z) = \frac{1}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{n!\Gamma(\rho n+\mu)}z^n, \quad \Re(\rho),\Re(\mu)>0,$$

where, for $\gamma = 1$ we recover the two-parametric Mittag-Leffler function

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty}\frac{1}{\Gamma(\rho n+\mu)}z^n, \quad \Re(\rho),\Re(\mu)>0, \tag{2}$$

and for $\gamma = \mu = 1$ we recover the classical Mittag-Leffler function

$$E_{\rho}(z) = \sum_{n=0}^{\infty}\frac{1}{\Gamma(\rho n+1)}z^n, \quad \Re(\rho).$$

Then the Prabhakar function is a function which extends the well-known two-parameter Mittag-Leffler function ($E_{\rho,\mu}(z)$) which is the most straightforward generalization of the classical Mittag-Leffler function $E_\rho(z)$. One of the reasons that made us interested in the Prabhakar function is related to the description of relaxation and response in anomalous dielectrics of Havriliak-Negami type and a model of complex susceptibility introduced to keep into account the simultaneous non-locality and nonlinearity observed in the response of disordered materials and heterogeneous systems [14–16, 26, 27]. Also there are numerous old and recent studies concerned with the Prabhakar function (see, for example [7, 19, 24, 28]). It is not easy to obtain exact solution of fractional ordinary/ partial/ integro-differential equations with the kernel of the variable order and therefore, a numerical method is used to solve them [2, 12, 13]. Few research has been done in this area for solving variable order of the differential equations using the numerical approximation. A great number of authors have significant contributions which deal with these types of the differential equations of variable order. In fact, different numerical algorithms are developed for obtaining numerical solutions of different types of MVODEs. In this field, kernel method is used in [25, 31], Jacobi polynomials to obtain solution of the multi-variable orders differential equations is used in [18], spectral method is used in [6], Bernstein polynomials to obtain solution the multi-variable orders differential equations is used in [21], finite difference method is used in [30], an improved collocation method is applied in [5].

In this paper, we use the fifth-kind Chebyshev polynomials as basic functions to obtain operational matrices. We transfer the original equations to a system of algebraic equations using operational matrices and collocation method. For this purpose, in Section 2 we recall the fundamental definitions in fractional calculus and introduce the fractional derivative with Prabhakar kernel. In Section 3, we survey the fifth-kind Chebyshev polynomials , also in this section the operational matrix of variable order derivative operators are discussed. In Section 4, we describes the proposed method for solving MVODEs. In Section 5, we investigate the convergence analysis of our proposed method. This section is followed by several illustrative examples in Section 6.

2 Preliminaries

In this section, we recall some definitions and lemmas of fractional integral and differential operators which are used in the next sections.

Definition 1. *[22, 29]* For $0 < \alpha \leq 1$ and $f \in L^1[a, b]$, $0 < t < b \leq \infty$, the left-and right-sided Riemann-Liouville fractional integrals and derivatives of order α are defined as:

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau,$$
$$I_{b^-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b f(\tau)(\tau-t)^{\alpha-1} d\tau,$$

$$D_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_a^t f(\tau)(t-\tau)^{-\alpha} d\tau,$$
$$D_{b^-}^{\alpha} f(t) = -\frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_t^b f(\tau)(\tau-t)^{-\alpha} d\tau.$$

Also, for the absolutely continuous function f, the left-and right-sided Caputo fractional derivatives of order α are defined as follows:

$$
\begin{aligned}
{}^{C}D_{a^+}^{\alpha} f(t) &= I_{a^+}^{1-\alpha} \frac{d}{dt} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} \frac{d}{d\tau} f(\tau) d\tau, \\
{}^{C}D_{b^-}^{\alpha} f(t) &= -I_{b^-}^{1-\alpha} \frac{d}{dt} f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\tau-t)^{-\alpha} \frac{d}{d\tau} f(\tau) d\tau.
\end{aligned}
$$

Definition 2. *[20]* For $m-1 < \Re(\mu) \leq m$ and $f \in L^1[0,b]$, $0 < t < b \leq \infty$, the left-and right-sided Prabhakar fractional integrals are defined as follows:

$$
\begin{aligned}
(\mathbf{E}_{\rho,\mu,\omega,a^+}^{\gamma} f)(t) &= \int_a^t (t-\tau)^{\mu-1} E_{\rho,\mu}^{\gamma}(\omega(t-\tau)^{\rho}) f(\tau) d\tau, \\
(\mathbf{E}_{\rho,\mu,\omega,b^-}^{\gamma} f)(t) &= \int_t^b (\tau-t)^{\mu-1} E_{\rho,\mu}^{\gamma}(\omega(\tau-t)^{\rho}) f(\tau) d\tau,
\end{aligned}
$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C}$ and $E_{\rho,\mu}^{\gamma}$ is Prabhakar function(2).

Definition 3. *[20]* For the function $f \in L^1[0,b]$, the left-and right-sided Prabhakar fractional derivatives are defined as:

$$
\begin{aligned}
(D_{\rho,\mu,\omega,a^+}^{\gamma} f)(t) &= \frac{d^m}{dt^m} \mathbf{E}_{\rho,m-\mu,\omega,a^+}^{-\gamma} f(t), \\
(D_{\rho,\mu,\omega,b^-}^{\gamma} f)(t) &= (-1)^m \frac{d^m}{dt^m} \mathbf{E}_{\rho,m-\mu,\omega,b^-}^{-\gamma} f(t),
\end{aligned}
$$

where $m-1 < \Re(\mu) \leq m$. For the absolutely continuous function f, the left-and right-sided Caputo-Prabhakar fractional derivatives are also defined as follows:

$$
\begin{aligned}
{}^{C}\mathbb{D}_{\rho,\mu,\omega,a^+}^{\gamma} f(t) &= \mathbf{E}_{\rho,m-\mu,\omega,a^+}^{-\gamma} \frac{d^m}{dt^m} f(t), \\
{}^{C}\mathbb{D}_{\rho,\mu,\omega,b^-}^{\gamma} f(t) &= (-1)^m \mathbf{E}_{\rho,m-\mu,\omega,b^-}^{-\gamma} \frac{d^m}{dt^m} f(t).
\end{aligned}
$$

Lemma 1. *[22] Let $\rho, \mu, \gamma, \omega, \nu \in \mathbb{C}$ with $\Re(\rho), \Re(\mu), \Re(\nu) > 0$. Then:*

$$
\int_0^x (x-t)^{\mu-1} E_{\rho,\mu}^{\gamma}(\omega(x-t)^{\rho}) t^{\nu-1} dt = \Gamma(\nu) x^{\mu+\nu-1} E_{\rho,\mu+\nu}^{\gamma}(\omega x^{\rho}). \tag{1}
$$

2.1 Variable-order fractional calculus

In this subsection, we replace the fractional order μ with a bounded function $m-1 < \mu(t) \leq m$, $m \in \mathbb{N}$ and consider the definitions of derivative and integral as follows:

Definition 4. The right-sided Prabhakar fractional integrals of order $\mu(t)$ are defined by:

$$(\mathfrak{E}^{\gamma}_{\rho,\mu(t),\omega,a^+}f)(t) = \int_a^t (t-\tau)^{\mu(t)-1}E^{\gamma}_{\rho,\mu(t)}(\omega(t-\tau)^{\rho})f(\tau)d\tau, \qquad (2)$$

$$(\mathfrak{E}^{\gamma}_{\rho,\mu(t),\omega,b^-}f)(t) = \int_t^b (\tau-t)^{\mu(t)-1}E^{\gamma}_{\rho,\mu(t)}(\omega(\tau-t)^{\rho})f(\tau)d\tau, \qquad (3)$$

$$t>0,\ m-1<\mu(t)\leq m,\ m\in\mathbb{N}.$$

Definition 5. The right-sided Prabhakar fractional derivatives of order $\mu(t)$ are defined by:

$$(\mathfrak{D}^{\gamma}_{\rho,\mu(t),\omega,a^+}f)(t) = \frac{d^m}{dt^m}\mathfrak{E}^{-\gamma}_{\rho,m-\mu(t),\omega,a^+}f(t),$$

$$(\mathfrak{D}^{\gamma}_{\rho,\mu(t),\omega,b^-}f)(t) = (-1)^m\frac{d^m}{dt^m}\mathfrak{E}^{-\gamma}_{\rho,m-\mu(t),\omega,b^-}f(t),$$

$$t>0,\ m-1<\mu(t)\leq m,\ m\in\mathbb{N}.$$

Also, for the absolutely continuous function f, the left-sided and the right-sided Caputo-Prabhakar fractional derivatives are defined as follows:

$$\mathbf{D}^{\gamma}_{\rho,\mu(t),\omega,0^+}f(t) = \mathfrak{E}^{-\gamma}_{\rho,m-\mu(t),\omega,a^+}\frac{d^m}{dt^m}f(t), \qquad (4)$$

$$\mathbf{D}^{\gamma}_{\rho,\mu(t),\omega,0^+}f(t) = (-1)^m\mathfrak{E}^{-\gamma}_{\rho,m-\mu(t),\omega,b^-}\frac{d^m}{dt^m}f(t). \qquad (5)$$

$$t>0,\ m-1<\mu(t)\leq m,\ m\in\mathbb{N}.$$

3 The fifth-kind Chebyshev polynomials

The well-known $i\,th$ degree fifth-kind Chebyshev polynomials can be defined on the interval $[-1,1]$ and can be determined with the following formula [1]:

$$\chi_i(t) = \frac{1}{\sqrt{\varepsilon_i}}\overline{H}_i^{-3,2,-1,1}(t),$$

where

$$\varepsilon_i = \begin{cases} \frac{\pi}{2^{2i+1}}, & i\,even, \\ \frac{\pi(i+2)}{i2^{2i+1}}, & i\,odd, \end{cases}$$

and

$$\overline{H}_i^{r,s,\nu,w}(t) = \Big(\prod_{k=0}^{\lfloor \frac{i}{2} \rfloor -1} \frac{(2k + (-1)^{k+1} + 2)w + s}{(2k + (-1)^{k+1} + 2\lfloor \frac{i}{2} \rfloor \nu)w + r} \Big) H^{r,s,\nu,w}(t),$$

$$H_i^{r,s,\nu,w}(t) = \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} \Big(\begin{pmatrix} \lfloor \frac{i}{2} \rfloor \\ j \end{pmatrix} \Big(\prod_{k=0}^{\lfloor \frac{i}{2} \rfloor -j-1} \frac{(2k + (-1)^{k+1} + 2\lfloor \frac{i}{2} \rfloor)\nu + r}{(2k + (-1)^{k+1} + 2)w + s} \Big) t^{i-2j} \Big).$$

$\chi_i(t)$ are orthonormal on $[-1, 1]$, i.e.,

$$\int_0^1 \frac{t^2}{\sqrt{t-t^2}} \chi_m(t)\chi_n(t)dt = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

In order to obtain these polynomials on the interval $[0, 1]$ we introduce the change of variable $t \to 2t - 1$ and substitute $2t - 1$ to $\chi_i(t)$ then, the fifth-kind Chebyshev polynomials can be defined as:

$$\mathcal{L}_i(t) = \chi_i(2t - 1).$$

The functions $\mathcal{L}_i(t)$ with respect to the following weight function are orthonormal

$$\mathbb{W}(t) = \frac{(2t-1)^2}{\sqrt{t-t^2}}, \tag{1}$$

and the orthogonality of the functions $\mathcal{L}_i(t)$ is shown as follows:

$$\int_0^1 \frac{(2t-1)^2}{\sqrt{t-t^2}} \mathcal{L}_m(t)\mathcal{L}_n(t)dt = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

We can rewrite $\mathcal{L}_i(t)$ as:

$$\mathcal{L}_i(t) = \sum_{l=0}^{i} \sigma_{l,i} t^l,$$

where

$$\sigma_{l,i} = \frac{2^{2l+\frac{3}{2}}}{\sqrt{\pi}(2l)!} \begin{cases} 2\sum_{k=\lfloor \frac{l+1}{2} \rfloor}^{\frac{i}{2}} \frac{(-1)^{\frac{i}{2}+k-l} k\delta_k (2k+l-1)!}{(2k-l)!}, & i\, even, \\ \frac{1}{\sqrt{i(i+2)}} \sum_{k=\lfloor \frac{l}{2} \rfloor}^{\frac{i-1}{2}} \frac{(-1)^{\frac{i+1}{2}+k-l}(2k+1)^2(2k+l)!}{(2k-l+1)!}, & i\, odd, \end{cases}$$

and

$$\delta_k = \begin{cases} \frac{1}{2}, & k = 0, \\ 1, & k > 0. \end{cases}$$

We can define the shifted fifth-kind Chebyshev vector as follows:

$$\varphi(t) = [\mathcal{L}_0(t), \mathcal{L}_1(t), \ldots, \mathcal{L}_n(t)]^T = \mathbf{A}T_n(t), \tag{2}$$

where

$$T_n = [1, t, t^2, \ldots, t^n]^T,$$
$$\mathbf{A} = \begin{bmatrix} \sigma_{0,0} & 0 & 0 & \ldots & 0 \\ \sigma_{0,1} & \sigma_{1,1} & 0 & \ldots & 0 \\ \sigma_{0,2} & \sigma_{1,2} & \sigma_{2,2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{0,n} & \sigma_{1,n} & \sigma_{2,n} & \ldots & \sigma_{n,n} \end{bmatrix},$$

and $\sigma_{0,2i} = \sqrt{\frac{2}{\pi}}$. Also the matrix $\mathbf{A}$ is invertible.

4 Operational matrix of variable order derivative operators

In this section, we obtain an approximation for the function $\Theta(t)$ in terms of the shifted fifth-kind polynomials. First we need to find an approximation for the Caputo-Prabhakar fractional derivative. So we can express the function $\Theta(t)$ in the following series:

$$\Theta(t) = \sum_{i=0}^{\infty} a_i \mathcal{L}_i(t). \tag{1}$$

Also, we can approximate $\Theta(t)$ by the first $n+1$ terms of the shifted fifth-kind Chebyshev polynomials as:

$$\Theta(t) \simeq \Theta_n = \sum_{i=0}^{n} a_i \mathcal{L}_i(t) = \Delta^T \varphi(t),$$

where the shifted fifth-kind Chebyshev vector $\varphi(t)$ and the shifted fifth-kind Chebyshev coefficient vector Δ are given by:

$$\varphi(t) = [\mathcal{L}_0(t), \mathcal{L}_1(t), \ldots, \mathcal{L}_n(t)]^T,$$
$$\Delta = [a_0, a_1, \ldots, a_n]^T.$$

Also, the coefficient vector a_i can be determined by the inner product as [1]:

$$a_i = \int_0^1 \frac{(2t-1)^2}{\sqrt{t-t^2}} \mathcal{L}_i(t)\Theta(t)dt, \tag{2}$$

where a_i are bounded. We derive the operational matrices of derivative operators of order $\mu_i(t)$ for vector $\varphi(t)$ as:

$$\mathbf{D}^{\gamma}_{\rho,\mu_i(t),\omega,0^+}\varphi(t) = \mathbf{D}^{\gamma}_{\rho,\mu_i(t),\omega,0^+}\Big[\mathbf{A}T_n\Big] = \mathbf{A}\mathbf{D}^{\gamma}_{\rho,\mu_i(t),\omega,0^+}[1,t,t^2,\dots,t^n]^T.$$

We use relations (2), (4) and Lemma 1, we obtain:

$$\begin{aligned}\mathbf{D}^{\gamma}_{\rho,\mu_i(t),\omega,0^+}\varphi(t) &= \mathbf{A}\mathfrak{E}^{-\gamma}_{\rho,1-\mu_i(t),\omega,0^+}\frac{d}{dt}[1,t,t^2,\dots,t^n]^T \\ &= \mathbf{A}\Big[0, t^{-\mu_i(t)-1}E^{-\gamma}_{\rho,2-\mu_i(t)}(\omega t^{\rho}), 2t^{-\mu_i(t)}E^{-\gamma}_{\rho,3-\mu_i(t)}(\omega t^{\rho}), \\ &\dots, n!t^{n-\mu_i(t)-2}E^{-\gamma}_{\rho,n-\mu_i(t)+1}(\omega t^{\rho})\Big]^T = \mathbf{A}\Psi_i(t)T_n,\end{aligned}$$

where $\Psi_i(t)$ are $(n+1)\times(n+1)$ matrices as $\Psi_i(t) = [\psi_{il,j}]_{(n+1)\times(n+1)}$, and

$$\psi_{il,j} = \begin{cases} j!t^{j-\mu_i(t)-2}E^{-\gamma}_{\rho,j-\mu_i(t)+1}(\omega t^{\rho}), & l = j > 0, \\ 0, & otherwise. \end{cases}$$

According to the relation (2) we have $T_n(t) = \mathbf{A}^{-1}\varphi(t)$, then

$$\mathbf{D}^{\gamma}_{\rho,\mu_i(t),\omega,0^+}\varphi(t) = \mathbf{A}\Psi_i(t)\mathbf{A}^{-1}\varphi(t).$$

We rewrite $\mathbf{A}\Psi_i(t)\mathbf{A}^{-1}\varphi(t) = \Omega_i\varphi(t)$, where Ω_i are the operational matrices for variable orders derivatives based on the fifth-type Chebyshev polynomials.

Then by employing the suggested method on Eq.(1) with initial condition $\Theta(0) = \Theta_0$, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\eta_i(t)\mathbf{D}^{\gamma}_{\rho,\mu_i(t),\omega,a^+}\Delta^T\varphi(t) = \sum_{i=1}^{N}\eta_i(t)\Delta^T\Omega_i\varphi(t) = f(t,\Delta^T\varphi(t)),\ 0<t<1, \\ \Delta^T\varphi(0) = \Theta_0. \qquad (3)\end{aligned}$$

Suppose $t_j = j/(n+1)$, $j = 1,2,\dots,n$ be the collocation points and $\mathfrak{R}$ be the residual function which is calculated as:

$$\mathfrak{R}(t,a_0,\dots,a_n) = \sum_{i=1}^{N}\eta_i(t)\Delta^T\Omega_i\varphi(t) - f(t,\Delta^T\varphi(t)). \qquad (4)$$

To calculate solution of Eq.(3) we are solving below system:

$$\begin{aligned}\mathfrak{R}(t_j,a_0,\dots,a_n) = 0, \\ \Delta^T\varphi(0) = \Theta_0. \qquad (5)\end{aligned}$$

By solving the above system we can obtain coefficients a_i and the unknown function $\Theta(t)$ can be calculated.

5 Error analysis

In this section, we present the convergence analysis of our proposed method in theorems as follows.

Theorem 1. *[1] Assume that a function $\Theta(t) \in L^2[0,1]$ with $|\Theta^{(3)}(t)| \leq L$ has the expansion as (1). If we define $E_n(t) = \sum_{i=n+1}^{\infty} a_i \mathcal{L}_i(t)$, that $E_n(t)$ be the global error. Then E_n can be estimated as:*

$$|E_n(t)| = |\Theta(t) - \Theta_n(t)| < \frac{3L}{n}.$$

Theorem 2. *Assume that a function $\Theta(t) \in L^2[0,1]$ satisfies in Theorem 1 so that the function $\Theta(t)$ has a finite approximation as $\Theta_n(t) = \sum_{i=0}^{n} a_i \mathcal{L}_i(t)$ and the polynomials $\mathcal{L}_i(t)$ are bounded on $[0,1]$. Then we have:*

$$\sup_{t\in[0,1]} |\Theta(t) - \Theta_n(t)| \leq \frac{3L}{n} + \varepsilon_n \parallel \tilde{\Delta} - \Delta \parallel_2,$$

where $\varepsilon_n = \sum_{i=0}^{n} \sqrt{\frac{2(i+2)^2}{\pi}}$.

Proof. Suppose that $\tilde{\Theta}_n(t)$ be an approximate solution of Eq.(1), that $\tilde{\Theta}_n(t)$ is obtained as follows:

$$\tilde{\Theta}_n(t) = \sum_{i=0}^{n} \tilde{a}_i \mathcal{L}_i(t) = \tilde{\Delta}^T \varphi(t) = \tilde{\Delta}^T \mathbf{A} T_n,$$
$$\tilde{\Delta} = [\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_n]^T,$$

and $\Theta_n(t) = \sum_{i=0}^{n} a_i \mathcal{L}_i(t) = \Delta^T \mathbf{A} T_n, \Delta = [a_0, a_1, \ldots, a_n]^T$. Then

$$\begin{aligned} |\Theta(t) - \Theta_n(t)| &\leq |\Theta(t) - \tilde{\Theta}_n(t)| + |\tilde{\Theta}_n(t) - \Theta_n(t)| \\ &= |\Theta(t) - \tilde{\Theta}_n(t)| + |\sum_{i=0}^{n} \tilde{a}_i \mathcal{L}_i(t) - \sum_{i=0}^{n} a_i \mathcal{L}_i(t)| \\ &= |\Theta(t) - \tilde{\Theta}_n(t)| + |\sum_{i=0}^{n} (\tilde{a}_i - a_i) \mathcal{L}_i(t)|. \end{aligned}$$

Using the Theorem 1, we have:

$$|\Theta(t) - \Theta_n(t)| \leq \frac{3L}{n} + |\sum_{i=0}^{n} (\tilde{a}_i - a_i) \mathcal{L}_i(t)|. \tag{1}$$

Also, using Cauchy-Schwarz inequality for part two of the Eq. (1), we can write

$$|\sum_{i=0}^{n}(\tilde{a}_i - a_i)\mathcal{L}_i(t)| \leq \Big(\sum_{i=0}^{n}|\tilde{a}_i - a_i|^2\Big)^{\frac{1}{2}} \times \Big(\sum_{i=0}^{n}|\mathcal{L}_i(t)|^2\Big)^{\frac{1}{2}}.$$

According to [1], we have

$$|\mathcal{L}_i(t)| \leq \sqrt{\frac{2}{\pi}}(i+2) \Longrightarrow \sum_{i=0}^{n}|\mathcal{L}_i(t)|^2 \leq \sum_{i=0}^{n}\frac{2}{\pi}(i+2)^2,$$

and hence

$$\Big(\sum_{i=0}^{n}|\mathcal{L}_i(t)|^2\Big)^{\frac{1}{2}} \leq \sum_{i=0}^{n}\sqrt{\frac{2(i+2)^2}{\pi}} = \varepsilon_n,$$

Therefore, we get

$$|\sum_{i=0}^{n}(\tilde{a}_i - a_i)\mathcal{L}_i(t)| \leq \varepsilon_n \parallel \tilde{\Delta} - \Delta \parallel_2 . \tag{2}$$

Using (1) and (2), we get

$$\sup_{t\in[0,1]}|\Theta(t) - \Theta_n(t)| \leq \frac{3L}{n} + \varepsilon_n \parallel \tilde{\Delta} - \Delta \parallel_2 .$$

The proof is complete. □

6 Numerical examples

In this section, four numerical examples are provided to show the effectiveness of the present method and compare the approximate solution with the exact solution. In this section, the absolute error are defined as: $e(t) = |\Theta(t) - \Delta^T\varphi(t)|,\ t \in [0,1]$.

Example 1. Let $\mu(t) = t$, $\eta_1(t) = 1,\ \Theta(0) = 0$ and

$$f(t,\Theta(t)) = \sum_{p=0}^{n}\frac{(-t)^p B(t)t^{tp}}{(1-t)^{p+1}\Gamma(tp+2)}\Big(\frac{2t^2}{tp+2} - t\Big),\ t \in [0,1].$$

Its analytical solution is $\Theta(t) = t^2 - t$. The figures of approximate solution and its absolute error are shown in Fig. 1 that it indicates the accuracy of the presented method.

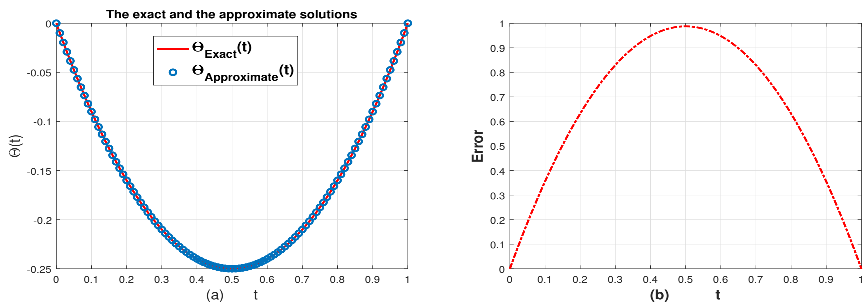

Figure 1: (a) The exact and the approximate solutions (b) The absolute error $(n = 5)$, for Example 1.

Example 2. Let $\mu_1(t) = 1 - 0.5e^t$, $\eta_1(t) = 1$, $\Theta(0) = 1$ and

$$f(t, \Theta(t)) = e^t\Big(1 + \sum_{p=0}^{n} \frac{(\mu_1(t))^p B(\mu_1(t)) t^{\mu_1(t)p+1}\tau}{(1-\mu_1(t))^{p+1}\Gamma(\mu_1(t)p+2)}\Big) - \Theta(t),\ \ t \in [0,1],$$

where

$$\tau = Hypergeometric\,{}_1F_1[\mu_1(t)p + 1, \mu_1(t)p + 2, -t].$$

Its analytical solution is $\Theta(t) = e^t$. The figures of approximate solution and its absolute error are shown in Fig. 2 and absolute errors for various n are shown in Table 1 that it improves the accuracy exponentially.

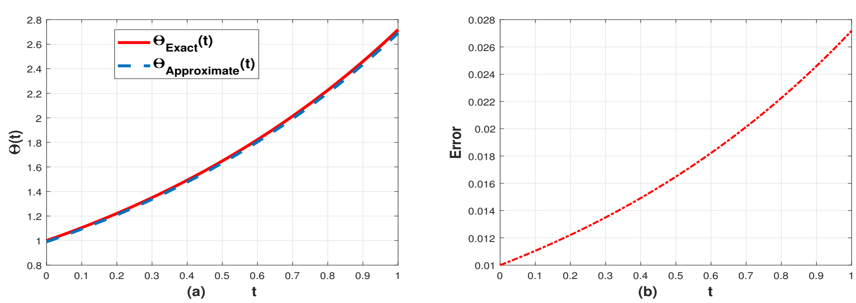

Figure 2: (a) The exact and the approximate solutions (b) The absolute error $(n = 5)$, for Example 2.

Table 1: The absolute error for various n.

t	$n=3$	$n=5$	$n=7$	$n=9$
0.1	0.1665	0.3555	0.5211	0.7119
0.2	0.296	0.632	0.9264	.2656
0.3	0.3885	0.8295	1.2159	1.6611
0.4	0.444	0.948	1.3896	1.8984
0.5	0.4625	0.9875	1.4475	1.9775
0.6	0.444	0.948	1.3896	1.8984
0.7	0.3885	0.8295	1.2159	1.6611
0.8	0.296	0.632	0.9264	1.2656
0.9	0.1665	0.41712	0.5211	0.7119

Example 3. Let $\mu_1(t) = t,\ \mu_2(t) = 1-0.5e^t,\ \eta_1(t) = 1,\ \eta_2(t) = \sin t,\ \Theta(0) = 0$ and

$$\begin{aligned} f(t,\Theta(t)) &= \sum_{p=0}^{n} \frac{6(-\mu_1(t))^p B(\mu_1(t)) t^{\mu_1(t)p+3}}{(1-\mu_1(t))^{p+1}\Gamma(\mu_1(t)p+4)}\Big), \\ &+ 6\sin t \sum_{p=0}^{n} \frac{6(-\mu_2(t))^p B(\mu_2(t)) t^{\mu_2(t)p+3}}{(1-\mu_2(t))^{p+1}\Gamma(\mu_2(t)p+4)}\Big), \\ &+ t^3\cos t - \cos t\Theta(t),\ t \in [0,1], \end{aligned} \tag{1}$$

where the exact solution of this example is $\Theta(t) = t^3$. Figure 3 shows comparison between approximate solution, exact solution and absolute error that this figure indicates that approximate solution obtained by the suggested method on Eq. (1) with initial condition is more close to exact solution. Also, The absolute errors of the presented algorithm are shown in Table 2.

In the following example the proposed method is compared with results available in the literature [18] in order to show the performance and accuracy of the proposed method.

Example 4. We consider the following multi-variable orders differential equations (MVODE):

$$\begin{aligned} \mathbf{D}^{\gamma}_{\rho,\mu(t),\omega,0^+}\Theta(t) &= -\sin(t)\mathbf{D}^{\gamma}_{\rho,\mu_1(t),\omega,0^+}\Theta(t) - \cos t\Theta(t) + \frac{6t^{3-\mu(t)}}{\Gamma(4-\mu(t))} \\ &+ \frac{6\sin(t)t^{3-\mu_1(t)}}{\Gamma(4-\mu_1(t))} + t^3\cos(t), \end{aligned} \tag{2}$$

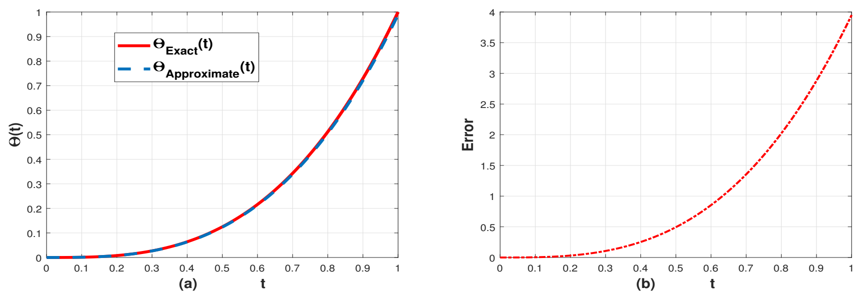

Figure 3: (a) The exact and the approximate solutions (b) The absolute error ($n = 5$), for Example 3

Table 2: The absolute error for various n.

t	$n=3$	$n=5$	$n=7$	$n=9$
0.1	0.00185	0.00001	0.00579	0.00791
0.2	0.0148	0.00008	0.04632	0.06328
0.3	0.04995	0.00027	0.15633	0.21357
0.4	0.1184	0.00064	0.37056	0.50624
0.5	0.23125	0.00125	0.72375	0.98875
0.6	0.3996	0.00216	1.25064	1.70856
0.7	0.63455	0.00343	1.98597	2.71313
0.8	0.9472	0.00512	2.96448	4.04992
0.9	1.34865	0.00729	4.22091	5.76639

with the initial condition $\Theta(0) = \Theta'(0) = 0$, where, for this problem $\mu(t) = 2 - \sin^2(t)$ and $\mu_1(t) = 1 - \frac{e^{-t^3}}{6}$. For this problem the exact solution is $\Theta(t) = t^3$. The approximate solutions of Eq. (2) for $\mu(t)$, $\mu_1(t)$ are demonstrated in Fig. 4. The results in Table 3 express the performance of the Caputo-Prabhakar fractional derivative in conjunction with variable order Caputo derivative [18] in the table captions.

7 Conclusion

In this paper, a new kind of Chebyshev polynomials called Chebyshev polynomials of the fifth-kind was employed for treating some types of multi variable orders differential equations with non-local and non-singular kernel that they have been solved using operational matrices based on the fifth-

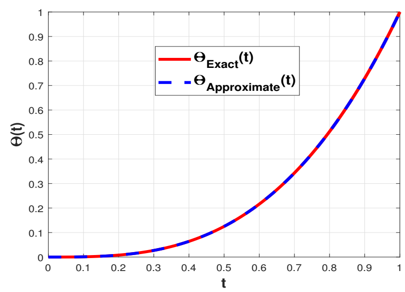

Figure 4: The exact and the approximate solutions ($n = 5$), for Example 4.

Table 3: The absolute error for various values t and $n = 4$.

t	$Caputo - Prabhakar$	$Caputo$
0.1	0.000995510109830	$4.46691e-17$
0.2	0.007964080878637	$1.2837e-16$
0.3	0.026878772965398	$2.15106e-16$
0.4	0.063712647029093	$2.91434e-16$
0.5	0.124438763728696	$3.33067e-16$
0.6	0.215030183723187	$3.33067e-16$
0.7	0.241459967671543	$2.77556e-16$
0.8	0.309701176232740	$3.33067e-16$
0.9	0.425726870065757	$4.44089e-16$

kind orthonormal Chebyshev polynomials. The derivative is described in the Caputo-Prabhakar sense. By applying collocation method, we obtained the approximate solution. The convergence of the presented method was discussed.

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