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A fitted mesh method for a coupled system of two singularly perturbed first order differential equations with discontinuous source term

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Abstract. In this work, an initial value problem for a weakly coupled system of two singularly perturbed ordinary differential equations with discontinuous source term is considered. In general, the system does not obey the standard maximum principle. The solution to the system has initial and interior layers that overlap and interact. To analyze the behavior of these layers, piecewise-uniform Shishkin meshes and graded Bakhvalov meshes are constructed. A backward finite difference scheme is considered on the meshes and is proved to be uniformly convergent in the maximum norm. Numerical experiments for both the Shishkin and Bakhvalov meshes are provided in support of the theory.

Keywords: Singular perturbation, parameter-uniform convergence, backward difference scheme, Shishkin mesh, Bakhvalov mesh, initial and interior layers.
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1 Introduction

Singularly perturbed initial and boundary value problems arise in various

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fields of applied mathematics and engineering. In particular, a system of first order singularly perturbed ordinary differential equations appears in chemical reactor theory. The presence of small singular perturbation parameter(s) leads to a multi-scale character and prevent us from obtaining the exact solution of these problems. The solution to such problems generally exhibit initial, boundary and interior layer(s) in narrow region(s) where the solution changes rapidly. For the past few decades, various numerical approaches that converges uniformly with respect to the small perturbation parameters have been studied in [1–3, 8, 9, 11–14]. Nagarajan et al. [10] suggested a numerical method to solve a coupled system of two singularly perturbed delay differential equations with given initial conditions on an interval $(0, 2]$ and established the first-order parameter-uniform convergence. Liu and Chen [7] developed an adaptive mesh technique with a suitable choice of monitor function to solve the singularly perturbed initial value problem. These studies worked with the coupling matrix $\boldsymbol{A}$ that was diagonally dominant with strictly positive diagonal entries and nonpositive off-diagonal entries.

Linss and Madden [5] considered a coupled system of singularly perturbed reaction-diffusion equations. The discrete Green's function technique was used to derive the parameter uniform convergence of the central difference scheme on both the Shishkin and Bakhvalov meshes. Kumar and Kumar [4] constructed a numerical scheme for a coupled system of singularly perturbed initial value problem. The local truncation error and barrier function technique was used to obtain the parameter uniform convergence result. The authors considered these systems with the relaxed assumptions on the coupling matrix $\boldsymbol{A}$, but with continuous source term. On the contrary, we consider a system of two singularly perturbed ordinary differential equations having discontinuous source term with prescribed initial condition

$$\begin{cases}(L_1\boldsymbol{u})(x) := -\varepsilon_1 u_1'(x) + a_{11}(x)u_1(x) + a_{12}(x)u_2(x) = f_1(x), \ x \in \Omega_1 \cup \Omega_2,\\ (L_2\boldsymbol{u})(x) := -\varepsilon_2 u_2'(x) + a_{21}(x)u_1(x) + a_{22}(x)u_2(x) = f_2(x), \ x \in \Omega_1 \cup \Omega_2,\end{cases} \tag{1}$$

with the initial conditions

$$u_1(0) = X_1, \qquad u_2(0) = X_2, \tag{2}$$

where ε_1, ε_2 are small perturbation parameters such that $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$ and it is assumed that

$$\begin{cases} a_{11}(x) > 0, \quad a_{22}(x) > 0, & x \in \overline{\Omega},\\ \max\left\{\left\|\frac{a_{12}}{a_{11}}\right\|, \left\|\frac{a_{21}}{a_{22}}\right\|\right\} < \theta < 1, & x \in \overline{\Omega}.\end{cases} \tag{3}$$

Here, $\Omega = (0, 1)$, $\overline{\Omega} = [0, 1]$, $\Omega_1 = (0, \delta)$ and $\Omega_2 = (\delta, 1)$ where $\delta \in \Omega$ and $u_1, u_2 \in C^0(\overline{\Omega}) \cap C^1(\Omega_1 \cup \Omega_2)$. The functions $a_{11}(x)$, $a_{12}(x)$, $a_{21}(x)$

and $a_{22}(x)$ are supposed to be sufficiently smooth for all $x \in \overline{\Omega}$. The source terms $f_1(x)$ and $f_2(x)$ are sufficiently smooth on $\Omega \backslash \{\delta\}$ and has a discontinuity at the point $x = \delta$. This discontinuity leads to the appearance of the interior layers in the solution of the considered problem in addition to the initial layers. The jump at δ in an arbitrary function Ψ is defined as $[\Psi](\delta) = \Psi(\delta+) - \Psi(\delta-)$. The solution satisfies the following interface conditions

$$[u_1](\delta) = 0, \quad [u_2](\delta) = 0. \tag{4}$$

Define the constant α as

$$\alpha = (1-\theta) \min_{x \in \overline{\Omega}} \{a_{11}(x), a_{22}(x)\}. \tag{5}$$

The system of equations (1)-(2) can be represented in vector form as

$$\boldsymbol{L u}(x) := \begin{bmatrix} -\varepsilon_1 & 0 \\ 0 & -\varepsilon_2 \end{bmatrix} \boldsymbol{u}'(x) + \boldsymbol{A}(x)\boldsymbol{u}(x) = \boldsymbol{f}(x), \quad x \in \Omega_1 \cup \Omega_2, \tag{6}$$

with initial condition

$$\boldsymbol{u}(0) = \boldsymbol{X}, \tag{7}$$

satisfying the interface condition

$$[\boldsymbol{u}](\delta) = \boldsymbol{0}, \tag{8}$$

where

$$\boldsymbol{L u}(x) = \begin{bmatrix} (L_1 \boldsymbol{u})(x) \\ (L_2 \boldsymbol{u})(x) \end{bmatrix}, \quad \boldsymbol{u}(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}, \quad \boldsymbol{A}(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix},$$

$$\boldsymbol{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} \quad \text{and} \quad \boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Throughout this paper, $\mathcal{C}$ denotes the general positive constant that is independent of the perturbation parameters ε_1, ε_2 and the mesh parameter $\mathcal{N}$, but it is not necessary that $\mathcal{C}$ takes same value at different places. The norm used to study the convergence of the numerical technique is the maximum norm defined as $\|g(x)\|_{\mathcal{D}} = \max_{x \in \mathcal{D}} \|g(x)\|$ for an arbitrary function $g(x)$ defined on a domain $\mathcal{D}$.

The rest of the paper is organized as follows. In Section 2, the stability result for an analytical solution is derived. A decomposition of the exact solution into smooth and layer components is introduced and finer bounds are obtained on their derivatives. In Section 3, the continuous problem is discretized using a backward difference scheme on a piecewise-uniform Shishkin mesh and graded Bakhvalov mesh. A detailed error analysis of the scheme follows in Section 4. Finally, the numerical test problem is presented in Section 5.

2 Continuous problem

The standard maximum principle does not hold for the system (1)-(2) with the given assumption (3). Therefore, the following stability result for the scalar ordinary differential operator is required.

Lemma 1. *Let $y \in C^0(\overline{\Omega}) \cap C^1(\Omega_1 \cup \Omega_2)$ be the solution of the scalar differential operator $\tilde{L}y(x) := -\mu y'(x) + a(x)y(x)$, $x \in \Omega_1 \cup \Omega_2$, satisfying $y(0) = A_0$ with $a(x) \geq \beta > 0$. Then*

$$\|y\|_{\overline{\Omega}} \leq \max\Big\{\Big\|\frac{\tilde{L}y(x)}{\beta}\Big\|_{\Omega_1\cup\Omega_2}, |y(0)|\Big\}.$$

The next theorem defines the stability result for the continuous problem (1)-(2).

Theorem 1. *The solution $u_i \in C^0(\overline{\Omega}) \cap C^1(\Omega_1 \cup \Omega_2)$ for $i = 1, 2$ of the system (1)-(2) satisfies the following bounds*

$$\|u_i\|_{\overline{\Omega}} \leq \sum_{k=1}^{2} (\Pi^{-1})_{ik} \ \max\left\{\Big\|\frac{f_k}{a_{kk}}\Big\|_{\Omega_1\cup\Omega_2}, |u_k(0)|\right\},$$

where $\Pi = (\kappa_{ij})_{2\times 2}$ is an inverse monotone matrix with the diagonal entries $\kappa_{ii} = 1$ and non-diagonal entries $\kappa_{ij} = -\Big\|\frac{a_{ij}}{a_{ii}}\Big\|$.

Proof. Consider the following decomposition of the solution u_i as the sum $u_i = \chi_i + \psi_i$, where χ_i satisfies the following system of equations

$$\begin{cases} -\varepsilon_i \chi_i'(x) + a_{ii}(x)\chi_i(x) = f_i(x), & x \in \Omega_1 \cup \Omega_2, \\ \chi_i(0) = X_i, \quad [\chi_i](\delta) = 0, \end{cases} \tag{9}$$

and ψ_i satisfies

$$\begin{cases} -\varepsilon_i \psi_i'(x) + a_{ii}(x)\psi_i(x) = -\sum\limits_{\substack{k=1 \\ k\neq i}}^{2} a_{ik}(x)u_k(x), & x \in \Omega_1 \cup \Omega_2, \\ \psi_i(0) = 0, \quad [\psi_i](\delta) = 0. \end{cases} \tag{10}$$

Using triangle inequality and the application of Lemma 1 gives

$$\|u_i\| - \sum_{\substack{k=1 \\ k\neq i}}^{2} \Big\|\frac{a_{ik}}{a_{ii}}\Big\| \|u_k\| \leq \max\Big\{\Big\|\frac{f_i}{a_{ii}}\Big\|_{\Omega_1\cup\Omega_2}, |u_i(0)|\Big\}, \qquad \text{for} \quad i = 1, 2.$$

The inverse monotonicity of Π yields the required stability result. □

To examine the layer part, we introduce the following layer functions

$$B_{\varepsilon_{l_i}}(x) = e^{-\alpha x/\varepsilon_i}, \qquad B_{\varepsilon_{r_i}}(x) = e^{-\alpha(x-\delta)/\varepsilon_i}, \quad \text{for} \ \ i = 1, 2. \tag{11}$$

The reduced system correspond to initial value problem (1)-(2) is given by $\boldsymbol{A}\boldsymbol{u}_0 = \boldsymbol{f}$, $x \in \Omega_1 \cup \Omega_2$. In order to analyze the numerical scheme, more precise bounds are required. This is obtained by decomposing the exact solution $\boldsymbol{u}$ as a sum of the smooth component $\boldsymbol{p}$ and the layer component $\boldsymbol{q}$, that is, $\boldsymbol{u} = \boldsymbol{p} + \boldsymbol{q}$, where the smooth component $\boldsymbol{p}$ is the solution of the following system:

$$\begin{aligned} \boldsymbol{L}\boldsymbol{p}(x) &= \boldsymbol{f}(x), \qquad x \in \Omega_1 \cup \Omega_2, \\ \boldsymbol{p}(0) = \boldsymbol{A}^{-1}(0)\boldsymbol{f}(0), &\quad \boldsymbol{p}(\delta+) = \boldsymbol{A}^{-1}(\delta+)\boldsymbol{f}(\delta+). \end{aligned} \tag{12}$$

and the layer component $\boldsymbol{q}$ is the solution of:

$$\begin{aligned} \boldsymbol{L}\boldsymbol{q}(x) &= \boldsymbol{0}, \qquad x \in \Omega_1 \cup \Omega_2, \\ \boldsymbol{q}(0) = \boldsymbol{u}(0) - \boldsymbol{p}(0), &\quad [\boldsymbol{q}](\delta) = -[\boldsymbol{p}](\delta). \end{aligned} \tag{13}$$

Theorem 2. *Let the matrix $\boldsymbol{A}$ satisfies (3). Then for $i = 1, 2$, the smooth component $\boldsymbol{p}$ and its derivatives satisfy the following bounds*

$$\|p_i^{(l)}\|_{\Omega_1\cup\Omega_2} \le \mathcal{C}, \quad l = 0, 1, \qquad \|p_i^{(2)}\|_{\Omega_1\cup\Omega_2} \le \mathcal{C}\varepsilon_i^{-1}.$$

Proof. First we derive the result for $x \in \Omega_1 = (0, \delta)$. The bounds on p_i is an immediate consequence of Theorem 1. Therefore, there exists a constant $\mathcal{C}$ such that

$$\|p_i\| \le \mathcal{C}. \tag{14}$$

Differentiating (12), we have

$$\boldsymbol{L}\boldsymbol{p}' := \begin{bmatrix} -\varepsilon_1 & 0 \\ 0 & -\varepsilon_2 \end{bmatrix} \boldsymbol{p}'' + \boldsymbol{A}\boldsymbol{p}' = \boldsymbol{f}' - \boldsymbol{A}'\boldsymbol{p}, \ x \in \Omega_1, \quad \boldsymbol{p}'(0) = \boldsymbol{0}. \tag{15}$$

Applying Theorem 1 together with (14), it is not hard to see that

$$\|p_i'\| \le \mathcal{C}.$$

Further use of (15) and the estimates on p_i and p_i' gives $\|p_i^{(2)}\| \le \mathcal{C}\varepsilon_i^{-1}$. Following the same steps, the similar results can be obtained for $x \in \Omega_2$. $\square$

Theorem 3. *Let the matrix $\boldsymbol{A}$ satisfies (3). Then for $i = 1, 2$, the layer component $\boldsymbol{q}$ and its derivatives satisfy the following bounds*

$$|q_1^{(l)}(x)| \le \mathcal{C} \begin{cases} \varepsilon_1^{-l} B_{\varepsilon_{l_1}}(x) + \varepsilon_2^{-l} B_{\varepsilon_{l_2}}(x) : & x \in \Omega_1, \\ \varepsilon_1^{-l} B_{\varepsilon_{r_1}}(x) + \varepsilon_2^{-l} B_{\varepsilon_{r_2}}(x) : & x \in \Omega_2, \end{cases} \qquad \textit{for} \ \ l = 0, 1, \tag{16}$$

$$|q_2^{(l)}(x)| \le \mathcal{C} \begin{cases} \varepsilon_2^{-l} B_{\varepsilon_{l_2}}(x): & x \in \Omega_1, \\ \varepsilon_2^{-l} B_{\varepsilon_{r_2}}(x): & x \in \Omega_2, \end{cases} \qquad \text{for} \quad l = 0, 1, \tag{17}$$

$$|q_i^{(2)}(x)| \le \mathcal{C}\varepsilon_i^{-1} \begin{cases} \varepsilon_1^{-1} B_{\varepsilon_{l_1}}(x) + \varepsilon_2^{-1} B_{\varepsilon_{l_2}}(x): & x \in \Omega_1, \\ \varepsilon_1^{-1} B_{\varepsilon_{r_1}}(x) + \varepsilon_2^{-1} B_{\varepsilon_{r_2}}(x): & x \in \Omega_2. \end{cases} \tag{18}$$

Proof. First we derive the result for $x \in \Omega_1 = (0, \delta)$. For $i = 1, 2$, apply the following transformation $q_i(x) = B_{\varepsilon_{l_2}}(x)\hat{q}_i(x)$. Then the components $\hat{q}_i(x)$ satisfy the following system

$$-\varepsilon_i \hat{q}_i' + \Big(a_{ii} - \alpha\frac{\varepsilon_i}{\varepsilon_2}\Big)\hat{q}_i = -\sum_{\substack{k=1 \\ k \ne i}}^{2} a_{ik}\hat{q}_k,$$

with the transformed initial condition

$$\hat{q}_i(0) = \frac{u_i(0) - p_i(0)}{B_{\varepsilon_{l_2}}(0)}.$$

Since $B_{\varepsilon_{l_2}}(0) \ge 1$. Then we have $|\hat{q}_i(0)| \le |u_i(0) - p_i(0)|$. Now by the definition of α we have

$$\Big(a_{ii}(x) - \alpha\frac{\varepsilon_i}{\varepsilon_2}\Big) \ge (a_{ii}(x) - \alpha) > 0.$$

Applying Lemma 1, we get

$$\|\hat{q}_i\| - \sum_{\substack{k=1 \\ k \ne i}}^{2} \Big\|\frac{a_{ik}}{a_{ii} - \alpha}\Big\| \|\hat{q}_k\| \le \mathcal{C}, \qquad \text{for} \quad i = 1, 2.$$

Then, Proposition 2.6 in [5] and the M-matrix criterion implies $\|\hat{q}_i\| \le \mathcal{C}$, that gives

$$|q_i(x)| \le \mathcal{C} B_{\varepsilon_{l_2}}(x). \tag{19}$$

Using (13) and (19), we obtain

$$|q_i'(x)| \le \mathcal{C}\varepsilon_i^{-1} B_{\varepsilon_{l_2}}(x). \tag{20}$$

The bounds obtained on the first component $q_1(x)$ are not sharp enough, for this purpose, consider the first equation of the system (13)

$$\begin{aligned} -\varepsilon_1 q_1' + a_{11} q_1 &= -a_{12} q_2, \\ q_1(0) &= u_1(0) - u_{01}(0), \end{aligned} \tag{21}$$

where u_{01} is the solution of the corresponding reduced problem. Now decomposing $q_1(x)$ similar to (12) and (13) as the sum $q_1 = r_1 + s_1$, where $r_1(x)$ satisfy the following system

$$\begin{aligned} -\varepsilon_1 r_1' + a_{11} r_1 &= -a_{12} q_2, \\ r_1(0) &= -\frac{a_{12}}{a_{11}} q_2(0), \end{aligned} \tag{22}$$

and s_1 satisfy the following equation

$$\begin{aligned} -\varepsilon_1 s_1' + a_{11} s_1 &= 0, \\ s_1(0) &= q_1(0) - r_1(0). \end{aligned} \tag{23}$$

Following the similar arguments used to obtain (19)-(20) with the transformation $s_1(x) = B_{\varepsilon_{l_1}}(x)\hat{s}_1(x)$, we get

$$|s_1^{(l)}(x)| \le \mathcal{C}\varepsilon_1^{-l} B_{\varepsilon_{l_1}}(x), \qquad l = 0, 1. \tag{24}$$

Using the bounds established in (19) and (24), the bounds on r_1 follows from $r_1 = q_1 - s_1$

$$|r_1(x)| \le \mathcal{C} B_{\varepsilon_{l_2}}(x). \tag{25}$$

Differentiating (22), we have

$$-\varepsilon_1 r_1'' + a_{11} r_1' = (-a_{12} q_2)' - a_{11}' r_1, \qquad r_1'(0) = 0. \tag{26}$$

Using the bounds on q_2, q_2' and r_1, we see that

$$|(-a_{12} q_2)' - a_{11}' r_1| \le \mathcal{C}\varepsilon_2^{-1} B_{\varepsilon_{l_2}}(x).$$

Now taking the transformation $r_1'(x) = \varepsilon_2^{-1} B_{\varepsilon_{l_2}}(x)\hat{r}_1(x)$ and repeating the similar steps used to get (19), we obtain

$$|r_1'(x)| \le \mathcal{C}\varepsilon_2^{-1} B_{\varepsilon_{l_2}}(x).$$

Collecting the bounds for $r_1(x)$ and $s_1(x)$, the improved bounds on the layer component $q_1(x)$ are

$$|q_1^{(l)}(x)| \le \mathcal{C}\{\varepsilon_1^{-l} B_{\varepsilon_{l_1}}(x) + \varepsilon_2^{-l} B_{\varepsilon_{l_2}}(x)\}, \qquad l = 0, 1.$$

Finally differentiating (13) and using (16)-(17) provides

$$|q_i^{(2)}(x)| \le \mathcal{C}\varepsilon_i^{-1}\{\varepsilon_1^{-1} B_{\varepsilon_{l_1}}(x) + \varepsilon_2^{-1} B_{\varepsilon_{l_2}}(x)\}.$$

On the similar lines the same result for $x \in \Omega_2 = (\delta, 1)$ can be derived. This completes the bounds on the layer component. $\square$

3 The discrete problem

In this section, piecewise-uniform Shishkin meshes and graded Bakhvalov meshes are constructed. The backward finite difference technique is applied to discretize the continuous problem (1).

3.1 Shishkin Meshes

A piecewise-uniform Shishkin mesh of size $\mathcal{N}$ on $\overline{\Omega}$ is constructed as follows. The interval Ω_1 is subdivided into three subintervals $[0, \sigma_{\varepsilon_{l_1}}]$, $[\sigma_{\varepsilon_{l_1}}, \sigma_{\varepsilon_{l_2}}]$ and $[\sigma_{\varepsilon_{l_2}}, \delta]$. Analogously the domain Ω_2 is subdivided into three subintervals $[\delta, \delta + \sigma_{\varepsilon_{r_1}}]$, $[\delta + \sigma_{\varepsilon_{r_1}}, \delta + \sigma_{\varepsilon_{r_2}}]$ and $[\delta + \sigma_{\varepsilon_{r_2}}, 1]$. Each of the subintervals $[0, \sigma_{\varepsilon_{l_1}}]$, $[\sigma_{\varepsilon_{l_1}}, \sigma_{\varepsilon_{l_2}}]$, $[\sigma_{\varepsilon_{l_2}}, \delta]$, $[\delta, \delta + \sigma_{\varepsilon_{r_1}}]$, $[\delta + \sigma_{\varepsilon_{r_1}}, \delta + \sigma_{\varepsilon_{r_2}}]$ and $[\delta + \sigma_{\varepsilon_{r_2}}, 1]$ are scaled with a uniform mesh of $\frac{\mathcal{N}}{6}$ mesh points. For convenience, we assume that $\mathcal{N}$ is divisible by 3. The interior points of the mesh are given as

$$\Omega^{\mathcal{N}} = \{x_j : 1 \leq j \leq \tfrac{\mathcal{N}}{2} - 1\} \cup \{x_j : \tfrac{\mathcal{N}}{2} + 1 \leq j \leq \mathcal{N} - 1\} = \Omega_1^{\mathcal{N}} \cup \Omega_2^{\mathcal{N}},$$

Clearly $x_{\frac{\mathcal{N}}{2}} = \delta$, the point of discontinuity. The mesh points of the discrete interval are given by $\overline{\Omega}^{\mathcal{N}} = \{x_j : 0 \leq j \leq \mathcal{N}\}$. The transition parameters in $\overline{\Omega}$ are defined as

$$\begin{aligned}
\sigma_{\varepsilon_{l_2}} &= \min\left\{\tfrac{2\delta}{3}, \tfrac{\varepsilon_2}{\alpha}\ln\mathcal{N}\right\}, \qquad \sigma_{\varepsilon_{l_1}} = \min\left\{\tfrac{\sigma_{\varepsilon_{l_2}}}{2}, \tfrac{\varepsilon_1}{\alpha}\ln\mathcal{N}\right\},\\
\sigma_{\varepsilon_{r_2}} &= \min\left\{\tfrac{2(1-\delta)}{3}, \tfrac{\varepsilon_2}{\alpha}\ln\mathcal{N}\right\}, \qquad \sigma_{\varepsilon_{r_1}} = \min\left\{\tfrac{\sigma_{\varepsilon_{r_2}}}{2}, \tfrac{\varepsilon_1}{\alpha}\ln\mathcal{N}\right\}.
\end{aligned}$$

Note that the mesh is uniform if we choose the transition parameters as $\sigma_{\varepsilon_{l_2}} = \frac{2\delta}{3}$, $\sigma_{\varepsilon_{l_1}} = \frac{\sigma_{\varepsilon_{l_2}}}{2}$, $\sigma_{\varepsilon_{r_2}} = \frac{2(1-\delta)}{3}$ and $\sigma_{\varepsilon_{r_1}} = \frac{\sigma_{\varepsilon_{r_1}}}{2}$. The step size in each of the six subintervals of the domain $\overline{\Omega}^{\mathcal{N}}$ are given by

$$H_1 = \frac{6\sigma_{\varepsilon_{l_1}}}{\mathcal{N}}, \quad H_2 = \frac{6(\sigma_{\varepsilon_{l_2}} - \sigma_{\varepsilon_{l_1}})}{\mathcal{N}}, \quad H_3 = \frac{6(\delta - \sigma_{\varepsilon_{l_2}})}{\mathcal{N}},$$

$$H_4 = \frac{6\sigma_{\varepsilon_{r_1}}}{\mathcal{N}}, \quad H_5 = \frac{6(\sigma_{\varepsilon_{r_2}} - \sigma_{\varepsilon_{r_1}})}{\mathcal{N}}, \quad H_6 = \frac{6(1 - \delta - \sigma_{\varepsilon_{r_2}})}{\mathcal{N}}.$$

3.2 Bakhvalov Meshes

Let $M : [a, b] \to \mathbb{R}$ be a strictly positive function. A mesh $\omega : a = x_0 < x_1 < \cdots < x_{\mathcal{N}} = b$ is said to equidistribute the monitor function M if:

$$\int_{x_{j-1}}^{x_j} M(s)ds = \frac{1}{\mathcal{N}} \int_a^b M(s)ds \quad \text{for} \quad j = 1, \ldots, \mathcal{N}.$$

The Bakhvalov mesh is constructed by equidistributing the following monitor function

$$Ba(s) := \max_{1\le i\le 2}\left\{1, \frac{M_i}{\varepsilon_i}e^{-\alpha s/\sigma_i\varepsilon_i}, \frac{M_i}{\varepsilon_i}e^{-\alpha(s-\delta)/\sigma_i\varepsilon_i}\right\},$$

where $\sigma_i \ge 1$ and M_i are some positive constants.

3.3 Finite difference scheme

The mesh function $\boldsymbol{U}(x_j) = (U_1(x_j), U_2(x_j))^T$ for all $x_j \in \Omega_1^{\mathcal{N}} \cup \Omega_2^{\mathcal{N}}$ on a piecewise-uniform Shishkin mesh $\overline{\Omega}^{\mathcal{N}}$, the initial value problem (1)-(2) is discretized using a standard backward difference technique as follows

$$\boldsymbol{L}^{\mathcal{N}}\boldsymbol{U}(x_j) := \begin{bmatrix} -\varepsilon_1 & 0 \\ 0 & -\varepsilon_2 \end{bmatrix} D^-\boldsymbol{U}(x_j) + \boldsymbol{A}(x)\boldsymbol{U}(x_j) = \boldsymbol{f}(x_j), \tag{27}$$

$$\boldsymbol{U}(x_0) = \mathbf{X}, \tag{28}$$

and at the point of discontinuity, the scheme is defined as

$$\boldsymbol{L}^{\mathcal{N}}\boldsymbol{U}(x_{\frac{\mathcal{N}}{2}}) := \begin{bmatrix} -\varepsilon_1 & 0 \\ 0 & -\varepsilon_2 \end{bmatrix} \boldsymbol{U}(x_{\frac{\mathcal{N}}{2}}) + \boldsymbol{A}(x_{\frac{\mathcal{N}}{2}})\boldsymbol{U}(x_{\frac{\mathcal{N}}{2}}) = \boldsymbol{f}(x_{\frac{\mathcal{N}}{2}-1}), \tag{29}$$

where

$$D^-U_i(x_j) = \frac{U_i(x_j) - U_i(x_{j-1})}{x_j - x_{j-1}} \quad \text{for} \quad i = 1, 2.$$

Note that $\boldsymbol{L}^{\mathcal{N}}\boldsymbol{U}(x_j) = ((L_1^{\mathcal{N}}\boldsymbol{U})(x_j), (L_2^{\mathcal{N}}\boldsymbol{U})(x_j))^T$ with

$$(L_i^{\mathcal{N}}\boldsymbol{U})(x_j) := -\varepsilon_i D^-U_i(x_j) + a_{i1}(x_j)U_1(x_j) + a_{i2}(x_j)U_2(x_j),$$

for $i = 1, 2$. To prove the stability of the approximate solution, the following lemma is needed.

Lemma 2. *Let Y be the solution of the following discrete system $(\tilde{L}^{\mathcal{N}}Y)(x_j) := -\mu(D^-Y)(x_j) + a(x_j)Y(x_j)$, $Y(x_0) = A_0$ with $a(x_j) \ge \beta > 0$. Then*

$$\|Y\|_{\overline{\Omega}^{\mathcal{N}}} \le \max\left\{\left\|\frac{\tilde{L}^{\mathcal{N}}Y}{\beta}\right\|_{\Omega_1^{\mathcal{N}}\cup\Omega_2{}^{\mathcal{N}}}, |Y(x_0)|\right\}.$$

Lemma 3. *Let the coupling matrix $\boldsymbol{A}$ satisfies (3). Then the solution $\boldsymbol{U}(x_j)$ for $i = 1, 2$ of the discrete system (27)-(29) satisfy the following bounds*

$$\|U_i\|_{\overline{\Omega}^{\mathcal{N}}} \le \sum_{k=1}^{2}(\Pi^{-1})_{ik} \ \max\left\{\left\|\frac{f_k}{a_{kk}}\right\|_{\Omega_1^{\mathcal{N}}\cup\Omega_2{}^{\mathcal{N}}}, |U_k(x_0)|\right\}.$$

4 Error Analysis

The error analysis for the approximate solution obtained by the backward difference scheme (27)-(29) is carried out and the parameter-uniform convergence result is derived in the end of the section.

Let $\boldsymbol{\mathcal{R}} = \boldsymbol{u} - \boldsymbol{U}$ represents the error of the numerical solution for the difference scheme (27)-(28), where $\boldsymbol{\mathcal{R}} = (\mathcal{R}_1, \mathcal{R}_2)$ and $\mathcal{R}_i = u_i - U_i$; $i = 1, 2$. We introduce an operator $(\mathcal{L}_i^{\mathcal{N}} Y)(x_j) := -\varepsilon_i (D^- Y)(x_j) + a_{ii}(x_j) Y(x_j)$. Splitting the error $\mathcal{R}_i$ into two parts as the sum $\mathcal{R}_i = \chi_i + \psi_i$ Here, χ_i satisfy the following system of equations

$$(\mathcal{L}_i^{\mathcal{N}} \chi_i)(x_j) := (L_i^{\mathcal{N}} \boldsymbol{\mathcal{R}})(x_j), \quad x_j \in \Omega^{\mathcal{N}}, \quad \chi_i(x_0) = 0, \ [\chi_i](x_{\frac{\mathcal{N}}{2}}) = 0, \tag{30}$$

and ψ_i satisfy the following system

$$(\mathcal{L}_i^{\mathcal{N}} \psi_i)(x_j) = -\sum_{\substack{k=1 \\ k \neq i}}^{2} a_{ik}(x_j) \mathcal{R}_k(x_j), \quad x_j \in \Omega^{\mathcal{N}}, \quad \psi_i(x_0) = 0, \ [\psi_i](x_{\frac{\mathcal{N}}{2}}) = 0. \tag{31}$$

Applying Lemma 3, we obtain

$$\|\psi_i\|_{\Omega^{\mathcal{N}}} \leq \sum_{\substack{k=1 \\ k \neq i}}^{2} \left\| \frac{a_{ik}}{a_{ii}} \right\| \|\mathcal{R}_k\|_{\Omega^{\mathcal{N}}}.$$

By the application of the triangle inequality, we have

$$\|\mathcal{R}_i\|_{\Omega^{\mathcal{N}}} - \sum_{\substack{k=1 \\ k \neq i}}^{2} \left\| \frac{a_{ik}}{a_{ii}} \right\| \|\mathcal{R}_k\|_{\Omega^{\mathcal{N}}} \leq \|\chi_i\|_{\Omega^{\mathcal{N}}}.$$

Using the inverse monotonicity of the defined matrix Π, we get

$$\|u_i - U_i\|_{\Omega^{\mathcal{N}}} \leq C \|\chi_i\|_{\Omega^{\mathcal{N}}}, \quad i = 1, 2. \tag{32}$$

Since

$$(L_i^{\mathcal{N}} \boldsymbol{\mathcal{R}})(x_j) = (L_i^{\mathcal{N}} \boldsymbol{u})(x_j) - (L_i^{\mathcal{N}} \boldsymbol{U})(x_j) = \varepsilon_i (D^- u_i(x_j) - u_i'(x_j)).$$

Applying Lemma 2 we obtain

$$\|\chi_i\|_{\Omega^{\mathcal{N}}} \leq C \max_{x_j \in \Omega^{\mathcal{N}}} \varepsilon_i |(D^- u_i(x_j) - (u_i')(x_j)|, \quad i = 1, 2.$$

Then, the Taylor expansion with an integral form of remainder and the bounds on $u_i''(x)$ gives

$$\begin{aligned}\varepsilon_i\Big|\Big(D^- - \frac{d}{dx}\Big)u_i(x_j)\Big| &\leq \int_{x_{j-1}}^{x_j} \varepsilon_i|u_i''(s)|ds \\ &\leq \int_{x_{j-1}}^{x_j} \Big(1+\varepsilon_1^{-1}B_{\varepsilon_{l_1}}(s)+\varepsilon_2^{-1}B_{\varepsilon_{l_2}}(s)\Big)ds, \quad i=1,2.\end{aligned}$$

Hence

$$\|\chi_i\|_{\Omega^{\mathcal{N}}} \leq C\vartheta(\Omega^{\mathcal{N}}), \tag{33}$$

where

$$\vartheta(\Omega^{\mathcal{N}}) = \int_{x_{j-1}}^{x_j} \Big(1+\varepsilon_1^{-1}B_{\varepsilon_{l_1}}(s)+\varepsilon_2^{-1}B_{\varepsilon_{l_2}}(s)\Big)ds.$$

Thus for a piecewise-uniform Shishkin mesh constructed in section 3.1, we have

$$\int_{x_{j-1}}^{x_j} \Big(1+\varepsilon_1^{-1}B_{\varepsilon_{l_1}}(s)+\varepsilon_2^{-1}B_{\varepsilon_{l_2}}(s)\Big)ds \leq \mathcal{C}(\mathcal{N}^{-1}\ln\mathcal{N}).$$

From (32) and (33), we obtain

$$\|u_i - U_i\|_{\Omega^{\mathcal{N}}} \leq \mathcal{C}(\mathcal{N}^{-1}\ln\mathcal{N}).$$

For the graded Bakhvalov mesh defined in Section 3.2, we have [6]

$$\int_{x_{j-1}}^{x_j} \Big(1+\varepsilon_1^{-1}B_{\varepsilon_{l_1}}(s)+\varepsilon_2^{-1}B_{\varepsilon_{l_2}}(s)\Big)ds \leq \mathcal{C}\mathcal{N}^{-1} \text{ for } x \in \Omega^{\mathcal{N}},$$

and therefore

$$\|u_i - U_i\|_{\Omega^{\mathcal{N}}} \leq \mathcal{C}\mathcal{N}^{-1}.$$

At the point of discontinuity $x_{\frac{\mathcal{N}}{2}} = \delta$, the local truncation error is given by:

$$\begin{aligned}(L_1^{\mathcal{N}}(\boldsymbol{U}-\boldsymbol{u}))(\delta) &= (L_1^{\mathcal{N}}\boldsymbol{U})(\delta) - (L_1^{\mathcal{N}}\boldsymbol{u})(\delta), \\ &= f_1(\delta - H_3) + \varepsilon_1 D^- u_1(\delta) - a_{11}(\delta)u_1(\delta) - a_{12}(\delta)u_2(\delta), \\ &= f_1(\delta - H_3) + \frac{\varepsilon_1}{H_3}(u_1(\delta) - u_1(\delta - H_3)) \\ &\quad - a_{11}(\delta)u_1(\delta) - a_{12}(\delta)u_2(\delta).\end{aligned}$$

Using the bounds on u_1, we obtain

$$\|(L_1^{\mathcal{N}}(\boldsymbol{U}-\boldsymbol{u}))(\delta)\| \leq \mathcal{C}(\mathcal{N}^{-1}\ln\mathcal{N}).$$

Similarly, we can prove that

$$\|(L_2^{\mathcal{N}}(\boldsymbol{U}-\boldsymbol{u}))(\delta)\| \leq \mathcal{C}(\mathcal{N}^{-1}\ln\mathcal{N}).$$

With the suitable construction of the barrier function, the following main result can be concluded.

Theorem 4. *Let $\boldsymbol{u}$ be the solution of the continuous problem (1)-(2) and $\boldsymbol{U}$ be the solution of discrete problem (27)-(29). Then for the Shishkin mesh*

$$\|\boldsymbol{U}-\boldsymbol{u}\|_{\overline{\Omega}^{\mathcal{N}}} \leq \mathcal{C}\mathcal{N}^{-1}\ln\mathcal{N},$$

and for the Bakhvalov mesh we have

$$\|\boldsymbol{U}-\boldsymbol{u}\|_{\Omega^{\mathcal{N}}} \leq \mathcal{C}\mathcal{N}^{-1}.$$

5 Numerical experiments

In this section two text examples are presented to illustrate the theoretical result obtained in Section 4.

Example 1. Consider the following singularly perturbed initial value problem with discontinuous source term

$$-\varepsilon_1 u_1'(x) + (2+x)u_1(x) + (1+x/2)u_2(x) = f_1(x),$$

$$-\varepsilon_2 u_2'(x) + (1+x)u_1(x) + (2+5x)u_2(x) = f_2(x),$$

with initial conditions

$$u_1(0) = 1, \quad u_2(0) = 3/2,$$

where

$$f_1(x) = \begin{cases} 1, & 0 \leq x \leq 0.5, \\ 0.8, & 0.5 < x \leq 1, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} x+2, & 0 \leq x \leq 0.5, \\ 3/2, & 0.5 < x \leq 1. \end{cases}$$

Example 2. Consider the following singularly perturbed initial value problem with discontinuous source term

$$-\varepsilon_1 u_1'(x) + (2+x)u_1(x) + u_2(x) = f_1(x),$$

$$-\varepsilon_2 u_2'(x) + (1+x)u_1(x) + (2+x)u_2(x) = f_2(x),$$

with initial conditions

$$u_1(0) = 1, \quad u_2(0) = 3/2,$$

where

$$f_1(x) = \begin{cases} 0.2, & 0 \leq x \leq 0.5, \\ x+2, & 0.5 < x \leq 1, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} e^x, & 0 \leq x \leq 0.5, \\ 1, & 0.5 < x \leq 1. \end{cases}$$

By the definition of α in (5), set $\alpha = 0.4$ for Example 1 and $\alpha = 0.6$ for Example 2. Take $\sigma = 1$ and $M_i = \sigma/\alpha$. Due to the fact that the exact solution of the considered problem is not known, the maximum nodal errors and the order of convergence are computed using the double mesh principle defined in [13]. For this, an approximate solution $U_i^{\mathcal{N}}$ is calculated on the mesh $x_j \in \overline{\Omega}^{\mathcal{N}}$ along with the approximate solution $U_i^{2\mathcal{N}}$ on the mesh $\overline{\Omega}^{\mathcal{N}}$ consisting of $2\mathcal{N}$ mesh intervals, that is, $x_{2j} = x_j,\ j = 0, \dots, \mathcal{N}$ and $x_{2j+1} = (x_j + x_{j+1})/2,\ j = 0, \dots, \mathcal{N}$. For the different values of the perturbation parameters ε_1, ε_2 and the mesh parameter $\mathcal{N}$, we calculate the estimate

$$\mathcal{R}^{\mathcal{N}}_{\varepsilon_1,\varepsilon_2} = \|U_i^{2\mathcal{N}} - U_i^{\mathcal{N}}\|_{\overline{\Omega}^{\mathcal{N}}},$$

and

$$\mathcal{R}^{\mathcal{N}} = \max_{T_{\varepsilon}}\{\mathcal{R}^{\mathcal{N}}_{\varepsilon_1,\varepsilon_2}\},$$

where the small perturbation parameters takes the values in the set

$$T_{\varepsilon} = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_2 = 2^0, 2^{-2}, \dots, 2^{-30}, \varepsilon_1 = \varepsilon_2, 2^{-2}\varepsilon_2, \dots, 2^{-40}\}.$$

Further the numerical convergence rate for Shishkin Mesh is defined as

$$D^{\mathcal{N}} = \frac{\ln \mathcal{R}^{\mathcal{N}} - \ln \mathcal{R}^{2\mathcal{N}}}{\ln(2\ln \mathcal{N}) - \ln(\ln(2\mathcal{N}))},$$

and for Bakhvalov mesh, it is given by

$$D^{\mathcal{N}} = \frac{\ln \mathcal{R}^{\mathcal{N}} - \ln \mathcal{R}^{2\mathcal{N}}}{\ln 2}.$$

The uniform error estimate $\mathcal{R}^{\mathcal{N}}$ and the uniform rate of convergence $D^{\mathcal{N}}$ are described in Table 1 and Table 2 for Example 1 and Example 2, respectively. From Table 1 and Table 2, it is observed that the applied difference scheme gives an almost first order of convergence for a piecewise-uniform Shishkin mesh and the first order of convergence on a graded Bakhvalov mesh. Figure 1 and Figure 2 describe the presence of initial layer at point $x = 0$ and the interior layer to the right hand side of the point of discontinuity $x = \delta$ for some particular values of ε_1, ε_2 and $\mathcal{N}$ for Example 1 and Example 2, respectively.

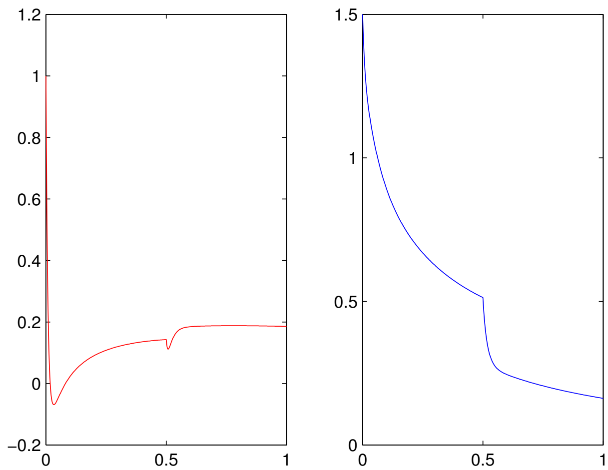

Figure 1: The appearance of initial layer at $x = 0$ and the interior layer to the right hand side of the point of discontinuity $x = \delta$ with $\varepsilon_1 = 2^{-6}$, $\varepsilon_2 = 2^{-4}$ and $\mathcal{N} = 384$.

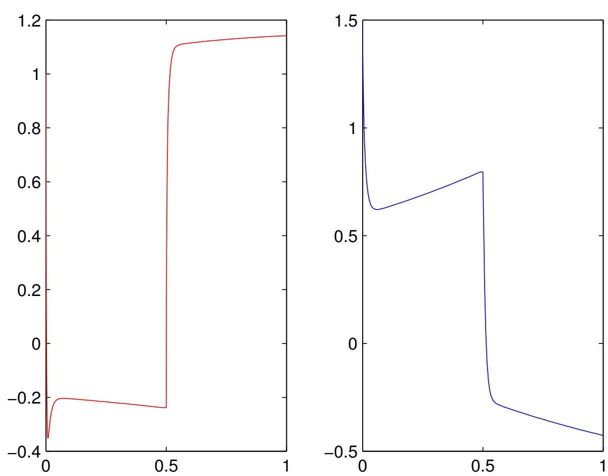

Figure 2: The appearance of initial layer at $x = 0$ and the interior layer to the right hand side of the point of discontinuity $x = \delta$ with $\varepsilon_1 = 2^{-8}$, $\varepsilon_2 = 2^{-6}$ and $\mathcal{N} = 192$.

Table 1: The errors $\mathcal{R}^{\mathcal{N}}$ and the convergence rate $D^{\mathcal{N}}$

	Shishkin Mesh		Bakhvalov Mesh	
$\mathcal{N}$	$\mathcal{R}^{\mathcal{N}}$	$D^{\mathcal{N}}$	$\mathcal{R}^{\mathcal{N}}$	$D^{\mathcal{N}}$
96	5.69E-02	0.74	1.73E-02	0.94
192	3.78E-02	0.84	9.04E-03	0.97
384	2.34E-02	0.89	4.62E-03	0.98
768	1.39E-02	0.94	2.34E-03	0.99
1536	7.99E-02	0.96	1.18E-03	1.00
3072	4.46E-02	-	5.89E-04	-

Table 2: The errors $\mathcal{R}^{\mathcal{N}}$ and the convergence rate $D^{\mathcal{N}}$

	Shishkin Mesh		Bakhvalov Mesh	
$\mathcal{N}$	$\mathcal{R}^{\mathcal{N}}$	$D^{\mathcal{N}}$	$\mathcal{R}^{\mathcal{N}}$	$D^{\mathcal{N}}$
96	8.85E-02	0.63	3.43E-02	0.88
192	6.26E-02	0.80	1.86E-02	1.16
384	3.98E-02	0.86	8.31E-03	0.84
768	2.41E-02	0.92	4.63E-03	1.06
1536	1.40E-02	0.95	2.21E-03	0.90
3072	7.86E-03	-	1.19E-03	-

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