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On the Sin-G class of distributions: theory, model and application

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Abstract. This paper is devoted to the study of the Sin-G class of distributions and one of its special member. We first explore the mathematical properties of the Sin-G class, giving the cumulative and probability density functions and their expansions, quantile function, moments, moment generating function, reliability parameter, Rényi entropy and order statistics. Then, we focus our attention on the special member defined with the Inverse Weibull distribution as baseline, denoted by SinIW. The mathematical and practical aspects of the SinIW distribution are investigated. In order to illustrate the usefulness of the SinIW model, an application to real life data set is carried out.

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1 Introduction

Univariate distributions are usually derived in three ways: by solving differential equations in the style developed by Pearson, by translating and

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scaling or by generating and inverting quantile functions (see [16]). However, there is still a great need for more flexible models in several areas such as genetics, medicine, agronomy, engineering, economics, among others. Thus, the statisticians aim to construct models that are able to provide a better explanation of the phenomenon studied, so we can have a better understanding of the factors involved, as well as in the development of better predictions.

Most of the statistical models proposed in the literature have a large number of parameters in an attempt to make the model more flexible. According to some authors, these estimates are difficult to obtain by means of numerical resources. It is desirable however to develop models that have a small number of parameters and, at the same time, with a large degree of flexibility for modelling the data. In order to reach this aim, few researchers decided to seek new distributions using trigonometric functions. [5] was one of the first to suggest a sine distribution, [10] proposed a distribution that bears his name Gilbert Sine distribution), [22] proposed the Beta Trigonometric (Beta-T) distribution, [2] proposed the *Square Sine* distribution and, more recently, [15] introduced the Sine exponential distribution, special member of the so-called Sin-G class of distributions.

To the best of our knowledge, the Sin-G class has been simultaneously introduced by [15] and [29] via techniques developed by [4]. However, in the present literature, there is no published work on its mathematical and practical properties, in full generality. This surprising gap is one motivation of this paper. By adopting the general setting of the Sin-G class, we derive the mean, variance, general coefficient, skewness, kurtosis, moments, moment generating function, reliability, Rényi entropy and order statistics. Then, in order to illustrate the applicability of the Sin-G class, we consider the special member defined with the Inverse Weibull distribution as baseline, and denoted by SinIW. By applying the established results of the Sin-G class, we derive the main properties of the SinIW distribution. Then, it is considered as a parametric model, with estimation of the parameters via the maximum likelihood method. A Monte Carlo simulation study is performed to assess maximum likelihood estimates. In order to illustrate the potentiality of the SinIW model, we provide an application about the analysis of a real data set.

The rest of the paper is organized as follows. Section 2 presents the Sin-G class of distributions and explores its main properties. The estimation of the parameters for the general model is presented in Section 3. The SinIW distribution is introduced in Section 4, including its mathematical and practical properties. An application to a real data set is given in Section

5. Some concluding remarks ends the study in Section 6.

2 On the Sin-G class

This section is devoted to the definition and the main mathematical properties of the Sin-G class of distributions.

2.1 Definition

The Sin-G class proposed here is characterized by the cumulative distribution function (cdf) given by

$$H_G(x;\boldsymbol{\xi}) = \int_0^{\frac{\pi}{2}G(x;\boldsymbol{\xi})} \cos(t)\mathrm{d}t = \sin\left[\frac{\pi}{2}G(x;\boldsymbol{\xi})\right], \quad x \in \mathbb{R}, \tag{1}$$

where $G(x;\boldsymbol{\xi})$ is an arbitrary baseline cdf of a continuous distribution, which depends on a parameter vector $\boldsymbol{\xi}$. As mention in Introduction, to the best of our knowledge, the Sin-G class finds trace in [15] and [29]. However, there is no published work on its general properties, which is the goal if this section.

The probability density function (pdf) corresponding to the Sin-G class is given by

$$h_G(x;\boldsymbol{\xi}) = \frac{\pi}{2}g(x;\boldsymbol{\xi})\cos\left[\frac{\pi}{2}G(x;\boldsymbol{\xi})\right], \quad x \in \mathbb{R}, \tag{2}$$

where $g(x;\boldsymbol{\xi})$ denotes the pdf corresponding to $G(x;\boldsymbol{\xi})$.

The corresponding hazard rate function (hrf) is given by

$$R_G(x;\boldsymbol{\xi}) = \frac{h_G(x;\boldsymbol{\xi})}{1 - H_G(x;\boldsymbol{\xi})} = \frac{\frac{\pi}{2}g(x;\boldsymbol{\xi})\cos\left[\frac{\pi}{2}G(x;\boldsymbol{\xi})\right]}{1 - \sin\left[\frac{\pi}{2}G(x;\boldsymbol{\xi})\right]}, \quad x \in \mathbb{R}.$$

Hereafter, to lighten the notations, the dependence on the parameter vector $\boldsymbol{\xi}$ will be sometimes omitted and we will simply write $G(x) = G(x;\boldsymbol{\xi})$, $H_G(x;\boldsymbol{\xi}) = H_G(x)$, $h_G(x;\boldsymbol{\xi}) = h_G(x)$ and $R_G(x;\boldsymbol{\xi}) = R_G(x)$.

2.2 Some shape properties

Here, we study the critical points of $h_G(x)$ and $R_G(x)$, as well as some asymptotic properties of these functions. The critical points of $h_G(x)$ are the roof of the equation: $h'_G(x) = 0$, with

$$h'_G(x) = \frac{\pi}{2}g'(x)\left\{\cos\left[\frac{\pi}{2}G(x)\right] - \frac{\pi}{2}g(x)\sin\left[\frac{\pi}{2}G(x)\right]\right\}.$$

So, a root of this equation, say x_*, satisfies $g'(x_*) = 0$ or $\cot\left[\frac{\pi}{2}G(x_*)\right] = \frac{\pi}{2}g(x_*)$. Moreover, x_* corresponds to a local maximum if $\tau(x_*) < 0$, a local minimum if $\tau(x_*) > 0$ or a point of inflection if $\tau(x_*) = 0$, where $\tau(x) = h''_G(x)$. Similarly, the the critical points of $R_G(x)$ are the roof of the equation: $R'_G(x) = 0$, with

$$R'_G(x) = \frac{\pi}{2}g'(x)\left\{\cos\left[\frac{\pi}{2}G(x)\right] - \frac{\pi}{2}g(x)\sin\left[\frac{\pi}{2}G(x)\right]\right\} + \frac{\frac{\pi}{2}g(x)\cos\left[\frac{\pi}{2}G(x)\right]}{1-\sin\left[\frac{\pi}{2}G(x)\right]}.$$

We can also examine the nature of each of the obtained roots, say x_{**}, according to the sign of the function $\upsilon(x) = R''_G(x_{**})$. Some asymptotic results for $h_G(x)$ and $R_G(x)$ are given below. When $G(x) \to 0$, we have

$$h_G(x) \sim \frac{\pi}{2}g(x), \quad R_G(x) \sim \frac{\pi}{2}g(x).$$

When $G(x) \to 1$, we have

$$h_G(x) \sim \frac{\pi^2}{4}g(x)(1-G(x)), \quad R_G(x) \sim 2\frac{g(x)}{1-G(x)}.$$

It is shown in this study that the Sin-G class can deal with general situations in modeling survival data with various shapes of $R_G(x)$.

2.3 Quantile function

The quantile function (qf) of the Sin-G class follows by inverting the Sin-G cdf. It can be expressed in terms of inverse sine function (arcsine function) as

$$Q_G(u) = G^{-1}\left[\frac{2}{\pi}\arcsin(u)\right], \quad u \in (0,1). \tag{3}$$

Hence, the Sin-G class of distributions can be easily simulated and a generator of a random variable X having the Sin-G cdf can be given by considering random variable U following the uniform distribution $U(0,1)$ and $X = Q_G(U)$. Also, the qf can be used to defined the standard quartiles (including the median), the octiles, as well as, several measures of skewness and kurtosis. Another useful function derived to $Q_G(u)$ is the quantile density function obtained by differentiation of $Q_G(u)$ (see [25]). After some algebra, it is given by

$$q_G(u) = \frac{2}{\pi\sqrt{1-u^2}}\frac{1}{g\left\{G^{-1}\left[\frac{2}{\pi}\arcsin(u)\right]\right\}}, \quad u \in (0,1).$$

2.4 Some linear expansions

Useful linear expansions can be derived using the concept of exponentiated distributions. For an arbitrary baseline cdf $G(x)$, with pdf denoted by $g(x)$, the exponentiated-G (exp-G) distribution with power parameter $a > 0$ is characterized by the cdf given by

$$G_a(x) = G(x)^a, \quad x \in \mathbb{R}. \tag{4}$$

The corresponding pdf is given by

$$g_a(x) = ag(x)G(x)^{a-1}, \quad x \in \mathbb{R}.$$

These notations will be adopt in the next. The properties of the exponentiated distributions are well-known for a wide variety of baseline cdfs $G(x)$. We refer to [20], [23], [7] and [17], among others.

The following result presents linear representations of $H_G(x)$ and $h_G(x)$ in terms of exp-G functions.

Proposition 1. *We have the following linear representations:*

$$H_G(x) = \sum_{k=0}^{+\infty} s_k G_{2k+1}(x), \quad h_G(x) = \sum_{k=0}^{+\infty} s_k g_{2k+1}(x), \tag{5}$$

where $s_k = \frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{2}\right)^{2k+1}$, $G_{2k+1}(x)$ *denotes the exp-G cdf with parameter* $(2k+1)$ *and* $g_{2k+1}(x)$ *denotes the corresponding pdf.*

Proof. By writing the sine function via its Taylor series, we have

$$H_G(x) = \sin\left[\frac{\pi}{2}G(x)\right] = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!}\left[\frac{\pi}{2}G(x)\right]^{2k+1} = \sum_{k=0}^{+\infty} s_k G_{2k+1}(x).$$

We obtain the desired linear expansion for $h_G(x)$ by differentiation of the previous function. □

So, from Proposition 1, several mathematical properties of the Sin-G class can be obtained by knowing those of the exp-G distribution. The formulas derived throughout the paper can be easily handled in most computer algebra systems such as MAPLE, MATLAB and MATHEMATICA. These platforms have currently the ability to deal with complex analytic expressions. Established explicit expressions to calculate statistical measures can be more efficient than computing them directly by numerical integration.

In practical terms, we can substitute ∞ in the sums by a large positive integer such as 30 or 40 for most practical purposes.

Hereafter, we consider a random variable X having the Sin-G cdf, i.e., given by (1). Also, we suppose the existence of all the presented integrals and sums. It is also assumed that the required conditions to apply the dominated convergence theorem hold, which is useful to justify the interchange of integral and sum signs.

2.5 On the moments

The m-th moment of X is given by

$$\mu_m = E(X^m) = \int_{-\infty}^{+\infty} x^m h_G(x)dx.$$

A linear representation for μ_m is proposed below. Hereafter, for any integer k, we consider a random variable Y_{2k+1} having the cdf $G_{2k+1}(x)$ (and the pdf $g_{2k+1}(x)$). It follows from (5) and the interchanged of the integral and sum signs that

$$\mu_m = \sum_{k=0}^{+\infty} s_k E\left(Y_{2k+1}^m\right).$$

In particular, the mean of X is given by taking $m = 1$, i.e., $\mu = \mu_1$ and the variance of X is given by $\sigma^2 = \mu_2 - \mu^2$.

The m-th factorial moment is given by

$$\mu_{(m)} = E\left[X(X-1)(X-m+1)\right] = \int_{-\infty}^{+\infty} x(x-1)(x-m+1)h_G(x)dx.$$

By using the Taylor series, we have

$$\mu_{(m)} = \sum_{k=0}^{m} s_*(m,k)\mu_k,$$

where $s_*(m,k) = \frac{1}{k!}[x(x-1)\ldots(x-m+1)]^{(k)} \mid_{x=0}$.

The m-th central moment of X is given by

$$\mu'_m = E[(X-\mu)^m] = \int_{-\infty}^{+\infty} (x-\mu)^m h_G(x)dx.$$

By using the binomial theorem, it can be expressed via moments as

$$\mu'_m = \sum_{r=0}^{m} \binom{m}{r} (-1)^{m-r} \mu^{m-r} \mu_r,$$

where $\binom{m}{r} = \frac{m!}{r!(m-r)!}$. We rediscover the variance of X by taking $m = 2$, i.e., $\sigma^2 = \mu_2'$. Also, the skewness and kurtosis coefficients of X are, respectively given by

$$S = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3'}{\sigma^3}, \quad K = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \frac{\mu_4'}{\sigma^4}.$$

The moment generating function (mgf) of X is given by

$$M_X(t) = E(\exp(tX)) = \int_{-\infty}^{+\infty} \exp(tx) h_G(x) dx.$$

It follows from (5) and the interchanged of the integral and sum signs that

$$M_X(t) = \sum_{k=0}^{+\infty} s_k M_{Y_{2k+1}}(t),$$

where $M_{Y_{2k+1}}(t)$ denotes the mgf of Y_{2k+1}.

The characteristic function of X as well as incomplete moments of X can be expressed in a similar manner.

2.6 Reliability parameter

We now investigate the reliability of the Sin-G class following the concept described in [14]. Let $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ be two parameter vectors. Let X_1 be a random variable having the cdf given by (1) with baseline cdf given by $G(x; \boldsymbol{\xi}_1)$, with pdf denoted by $h_1(x)$, and X_2 be a random variable having the cdf given by (1) with baseline cdf given by $G(x; \boldsymbol{\xi}_2)$, with cdf denoted by $H_2(x)$. We suppose that X_1 and X_2 are independent. Then, a measure of reliability is given by the parameter R defined by

$$R = P(X_2 < X_1) = \int_{-\infty}^{+\infty} h_1(x) H_2(x) dx.$$

An alternative linear representation of R is given below. It follows from (5) and the interchanged of the integral and sum signs that

$$\begin{aligned} R &= \int_{-\infty}^{+\infty} \left[\sum_{k=0}^{+\infty} s_k g_{2k+1}(x; \boldsymbol{\xi}_1) \right] \times \left[\sum_{q=0}^{+\infty} s_q G_{2q+1}(x; \boldsymbol{\xi}_2) \right] dx \\ &= \sum_{k=0}^{+\infty} \sum_{q=0}^{+\infty} s_k s_q \int_{-\infty}^{+\infty} g_{2k+1}(x; \boldsymbol{\xi}_1) G_{2q+1}(x; \boldsymbol{\xi}_2) dx \\ &= \sum_{k=0}^{+\infty} \sum_{q=0}^{+\infty} s_k s_q (2k+1) \int_{-\infty}^{+\infty} g(x; \boldsymbol{\xi}_1) G(x; \boldsymbol{\xi}_1)^{2k} G(x; \boldsymbol{\xi}_2)^{2q+1} dx. \end{aligned}$$

The integral can be computed numerically for given $G(x; \boldsymbol{\xi}_1)$ and $G(x; \boldsymbol{\xi}_2)$. One can observe that, if there exists $\upsilon > 0$ such that $G(x; \boldsymbol{\xi}_2) = G(x; \boldsymbol{\xi}_1)^{\upsilon}$ (corresponding to the exponentiated case with parameter $\upsilon > 0$), then R is reduced to

$$R = \sum_{k=0}^{+\infty} \sum_{q=0}^{+\infty} s_k s_q \frac{2k+1}{2k+1+\upsilon(2q+1)+1}.$$

One can notice that R does not depend on the baseline distribution. We can check that, in the identically distributed case, i.e., $\upsilon = 1$, we have $R = \frac{1}{2}$.

2.7 Rényi entropy

The entropy of a distribution is a measure of uncertainty; the greater the entropy, the higher the disorder and less likely to observe a given event; the lower the entropy, the lower its disorder and the higher the probability of observing a particular event. In this section, we focus on one of the most useful entropy: the Rényi entropy (see [27]). For the Sin-G class, the Rényi entropy is given by

$$\begin{aligned} \mathfrak{L}_{R,G}(\gamma) &= \frac{1}{1-\gamma} \log \left[\int_{-\infty}^{+\infty} h_G(x)^{\gamma} dx \right] \\ &= \frac{1}{1-\gamma} \log \left[\int_{-\infty}^{+\infty} \left(\frac{\pi}{2}\right)^{\gamma} g(x)^{\gamma} \left\{ \cos\left[\frac{\pi}{2} G(x)\right] \right\}^{\gamma} dx \right], \end{aligned}$$

where $\gamma > 0$ and $\gamma \neq 1$. We now derive an alternative sum expression for $\mathfrak{L}_{R,G}(\gamma)$. By considering the Taylor series of the function $\left\{ \cos\left[\frac{\pi}{2} s\right] \right\}^{\gamma}$ at the point $s = \frac{1}{2}$, we can write

$$\left\{\cos\left[\frac{\pi}{2}s\right]\right\}^{\gamma} = \sum_{k=0}^{+\infty}\sum_{r=0}^{k} a_k \binom{k}{r}(-1)^{k-r}\left(\frac{1}{2}\right)^{k-r} s^r,$$

where $a_k = \dfrac{1}{k!}\left[\left\{\cos[\frac{\pi}{2}s]\right\}^{\gamma}\right]^{(k)} |_{s=\frac{1}{2}}$.

After some algebra and the interchange of the integral and sum signs, we obtain

$$\mathfrak{L}_{R,G}(\gamma) = \frac{1}{1-\gamma}\left\{\gamma\log\left(\frac{\pi}{2}\right) + \log\left[\sum_{k=0}^{+\infty}\sum_{r=0}^{k} a_k\binom{k}{r}(-1)^{k-r}\left(\frac{1}{2}\right)^{k-r}\mathrm{U}_r\right]\right\},$$

where $\mathrm{U}_r = \int_{-\infty}^{+\infty} g(x)^{\gamma} G(x)^r dx$. In most of the cases, U_r can be computed at least numerically.

2.8 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Here, we investigate some of distributional properties of the i-th order statistic from the Sin-G class. Let $X_1, \ldots, X_n$ be n random variables, i.i.d., having the common Sin-G cdf given by (1). Then, the pdf of the i-th order statistic is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} h_G(x) H_G(x)^{i-1}\left[1 - H_G(x)\right]^{n-i}, \qquad x \in \mathbb{R},$$

where $B(i, n-i+1) = \dfrac{(i-1)!(n-i)!}{n!}$. By using the binomial formula and the linear representations given by (5), we get

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j h_G(x) H_G(x)^{j+i-1} \\ &= \frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\left[\sum_{k=0}^{+\infty} s_k g_{2k+1}(x)\right] \\ &\quad\times\left[\sum_{q=0}^{+\infty} s_q G_{2q+1}(x)\right]^{j+i-1}. \end{aligned}$$

For the expansion of the last term, one can use Taylor series. A more elegant approach is to use [11, Result 0.314]. This result says that, for a positive integer v, a sequence of real numbers $(a_k)_{k\in\mathbb{N}}$ and $y \in \mathbb{R}$, we have

$$\left[\sum_{m=0}^{+\infty} a_m y^m\right]^v = \sum_{m=0}^{+\infty} d_{v,m} y^m,$$

where $d_{v,0} = a_0^v$ and, for any $m \geq 1$,

$$d_{v,m} = \frac{1}{ma_0}\sum_{\ell=1}^{m}[\ell(v+1)-m]a_\ell d_{v,m-\ell}.$$

Therefore, by taking $y = G(x)^2$, we have

$$\begin{aligned}\left[\sum_{q=0}^{+\infty} s_q G_{2q+1}(x)\right]^{j+i-1} &= G(x)^{j+i-1}\left[\sum_{q=0}^{+\infty} s_q \left\{G(x)^2\right\}^q\right]^{j+i-1}\\ &= \sum_{q=0}^{+\infty} d_{j+i-1,q} G_{2q+j+i-1}(x),\end{aligned}$$

where $d_{j+i-1,0} = s_0^{j+i-1}$ and, for any $q \geq 1$,

$$d_{j+i-1,q} = \frac{1}{qs_0}\sum_{\ell=1}^{q}[\ell(j+i)-q]s_\ell d_{j+i-1,q-\ell}.$$

By putting the above equalities together, we get

$$\begin{aligned}&f_{i:n}(x)\\ &= \frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}\sum_{k=0}^{+\infty}\sum_{q=0}^{+\infty}\binom{n-i}{j}(-1)^j s_k d_{j+i-1,q} g_{2k+1}(x) G_{2q+j+i-1}(x)\\ &= \sum_{k=0}^{+\infty}\sum_{q=0}^{+\infty}\omega_{k,q,n,i}\, g_{2k+2q+j+i}(x),\end{aligned}$$

where

$$\omega_{k,q,n,i} = \frac{1}{B(i,n-i+1)}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j s_k d_{j+i-1,q}\frac{2k+1}{2k+2q+j+i}.$$

From this linear representation, combination of exp-G pdfs, several mathematical quantities can be expressed. For instance, by introducing a random variable $Y_{2k+2q+j+i}$ having the pdf $g_{2k+2q+j+i}(x)$, the m-th moment of $X_{i:n}$ can be expressed as

$$E(X_{i:n}^m) = \int_{-\infty}^{+\infty} x^m f_{i:n}(x)dx = \sum_{k=0}^{+\infty}\sum_{q=0}^{+\infty}\omega_{k,q,n,i} E(Y_{2k+2q+j+i}^m).$$

3 Estimation

Under some regularity conditions, the maximum likelihood estimates (MLEs) of the model parameters can be obtained by equating the derivative of the log-likelihood function with respect to each parameter to zero. In this section, we determine the MLEs of the Sin-G model parameters from complete samples only. Let $\tilde{x} = x_1, \ldots, x_n$ be a random sample from a random variable X having the Sin-G cdf given by (1), where $\boldsymbol{\xi}$ denotes a vector of unknown parameters in the baseline distribution $G(x; \boldsymbol{\xi})$. Let $\boldsymbol{\Theta} = (\boldsymbol{\xi})^\top$ be the $p \times 1$ parameter vector. Then, the total log-likelihood function for $\boldsymbol{\Theta}$ is given by

$$\ell(\boldsymbol{\Theta}) = n \log\left(\frac{\pi}{2}\right) + \sum_{i=1}^{n} \log\left[g(x_i; \boldsymbol{\xi})\right] + \sum_{i=1}^{n} \log\left\{\cos\left[\frac{\pi}{2} G(x_i; \boldsymbol{\xi})\right]\right\}.$$

We assume that the following standard regularity conditions for the log-likelihood $\ell(\boldsymbol{\Theta})$ hold: i) The support of X does not depend on unknown parameters; ii) The parameter space of X, say $\boldsymbol{\Psi}$, is open and $\ell(\boldsymbol{\Theta})$ has a global maximum in $\boldsymbol{\Psi}$; iii) For almost all $\tilde{x}$, the fourth-order log-likelihood derivatives with respect to the model parameters exist and are continuous in an open subset of $\boldsymbol{\Psi}$ that contains the true parameter; iv) The expected information matrix is positive definite and finite; v) The absolute values of the third-order log-likelihood derivatives with respect to the parameters are bounded by expected finite functions of X.

The k-th component of the score function $U(\boldsymbol{\Theta}) = (U_{\boldsymbol{\xi}})^\top$ is given by

$$U_{\boldsymbol{\xi}_k} = \sum_{i=1}^{n} \frac{\partial g(x_i; \boldsymbol{\xi})/\partial \boldsymbol{\xi}_k}{g(x_i; \boldsymbol{\xi})} - \frac{\pi}{2} \sum_{i=1}^{n} \frac{\partial G(x_i; \boldsymbol{\xi})}{\partial \boldsymbol{\xi}_k} \tan\left[\frac{\pi}{2} G(x_i; \boldsymbol{\xi})\right].$$

For interval estimation, we require the $p \times p$ observed information matrix. The (k, l)-th element of this matrix is given by $-U_{\boldsymbol{\xi}_k, \boldsymbol{\xi}_l}$, where

$$\begin{aligned} U_{\boldsymbol{\xi}_k, \boldsymbol{\xi}_l} = & \sum_{i=1}^{n} \frac{\partial^2 g(x_i; \boldsymbol{\xi})/\partial \boldsymbol{\xi}_k \partial \boldsymbol{\xi}_l}{g(x_i; \boldsymbol{\xi})} - \frac{\pi}{2} \sum_{i=1}^{n} \frac{\partial^2 G(x_i; \boldsymbol{\xi})}{\partial \boldsymbol{\xi}_k \partial \boldsymbol{\xi}_l} \tan\left[\frac{\pi}{2} G(x_i; \boldsymbol{\xi})\right] \\ & - \sum_{i=1}^{n} \frac{[\partial g(x_i; \boldsymbol{\xi})/\partial \boldsymbol{\xi}_k][\partial g(x_i; \boldsymbol{\xi})/\partial \boldsymbol{\xi}_l]}{g(x_i; \boldsymbol{\xi})^2} \\ & - \frac{\pi^2}{4} \sum_{i=1}^{n} \left\{\sec\left[\frac{\pi}{2} G(x_i; \boldsymbol{\xi})\right]\right\}^2 \left\{\frac{\partial g(x_i; \boldsymbol{\xi})}{\partial \boldsymbol{\xi}_k}\right\} \left\{\frac{\partial g(x_i; \boldsymbol{\xi})}{\partial \boldsymbol{\xi}_l}\right\}. \end{aligned}$$

4 Particular case: the SinIW distribution

In this section, we examine a particular distribution of the Sin-G class defined with the Inverse Weibull (IW) cdf $G(x)$ as baseline. From a mathematical point of view, if a random variable Y has the Weibull distribution with parameters $\alpha > 0$ and $\theta > 0$, then $X = \frac{1}{Y}$ has the IW distribution with parameters α and θ, and its cdf is given by

$$G(x) = \exp(-\alpha x^{-\theta}), \quad x > 0. \tag{6}$$

The corresponding pdf takes the form

$$g(x) = \alpha\theta x^{-\theta-1}\exp(-\alpha x^{-\theta}), \quad x > 0.$$

Also, the corresponding hrf is given by

$$h(x) = \alpha\theta\frac{x^{-\theta-1}}{\exp(\alpha x^{-\theta})-1}, \quad x > 0$$

and presents unimodal form. The IW distribution has received much attention in the literature due to undeniable merits. For instance, in [13], the IW distribution is introduced to describe the degeneration phenomena of mechanical components. The IW distribution also provides a good fit to several kinds of data such as the breakdown times of an insulating fluid, subject to the action of constant tension (see [24]). This distribution is also called Reverse Weibull distribution (see [28]), Additional Weibull distribution (see [9]) and the Reciprocal Weibull distribution (see [19]). Some order statistics properties of the IW distribution are derived in [1]. In [6], the maximum likelihood and least square estimations of parameters are discussed. Some authors have introduced variations of the IW distribution such as [12] who proposed a generalization (exponentiated form) of the IW distribution: the generalized model, also known as reversed Weibull, has the IW distribution as particular case, and is thus more flexible.

The new proposed distribution is created by applying the Sin-G transform class to the IW distribution. It is called the Sine Inverse Weibull distribution, and denoted by SinIW for the purpose of this study. Thus, the cdf of the SinIW distribution is defined from (1) by taking $G(x)$ given by (6), i.e.,

$$H_{IW}(x) = \sin\left[\frac{\pi}{2}\exp(-\alpha x^{-\theta})\right], \quad x > 0.$$

The corresponding pdf is given by

$$h_{IW}(x) = \frac{\pi}{2}\alpha\theta x^{-\theta-1}\exp(-\alpha x^{-\theta})\cos\left[\frac{\pi}{2}\exp(-\alpha x^{-\theta})\right], \quad x > 0.$$

The corresponding hrf is given by

$$R_{IW}(x) = \frac{\frac{\pi}{2}\alpha\theta x^{-\theta-1}\exp(-\alpha x^{-\theta})\cos\left[\frac{\pi}{2}\exp(-\alpha x^{-\theta})\right]}{1-\sin\left[\frac{\pi}{2}\exp(-\alpha x^{-\theta})\right]}, \quad x > 0.$$

Figure 1 displays some plots of the SinIW pdf and cdf for some selected values of α and θ. Also, the hrf and the survival function (sf), i.e., $S_{IW}(x) = 1 - H_{IW}(x)$, for selected parameter values are shown in Figure 2. The importance of the hrf is to be quite flexible for modeling survival data. Indeed, for selected parameter values, the SinIW hrf can have, for example, decreasing and inverted-bathtub forms. As the IW distribution is very useful in modeling failure rates, the new distribution also proved to be very flexible in the modeling of these type of data (as this will be developed in Section 5). Then, the SinIW model has greater applicability to problems in the biological area, since most of these problems have unimodal failure rate.

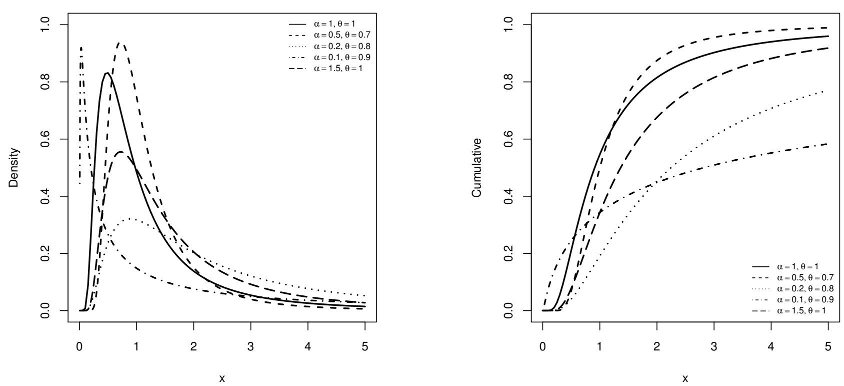

Figure 1: Plots of the SinIW pdf and cdf for some parameter values.

Using similar arguments to those used for the Sin-G cdf and pdf expansions, the SinIW cdf and pdf can be rewritten as a sum of exponentiated IW functions as follows:

$$H_{IW}(x) = \sum_{k=0}^{+\infty} s_k G_{2k+1}(x), \quad h_{IW}(x) = \sum_{k=0}^{+\infty} s_k g_{2k+1}(x). \tag{7}$$

Using (7), it is possible to simply obtain some measures of the SinIW distribution by using those of exponentiated IW distribution. Properties

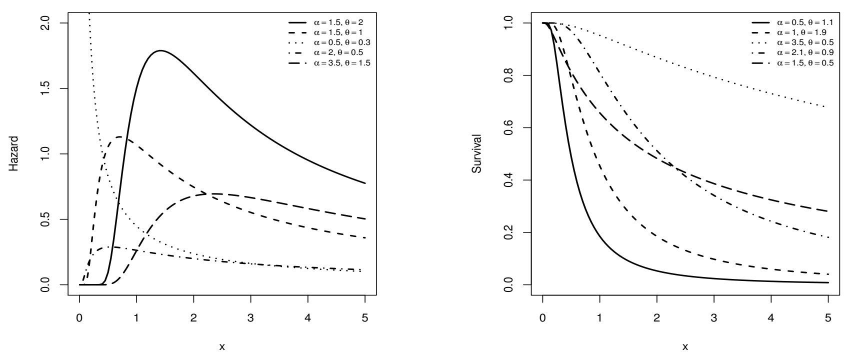

Figure 2: Plots of the SinIW hrf and sf for some parameter values.

such as moments, central moments, variance, skewness, kurtosis and mgf are summarized in Table 1, where $\Gamma(x)$ denotes the gamma function defined by $\Gamma(x) = \int_0^{+\infty} y^{x-1} \exp(-y)dy$, $x > 0$, and, for the sake of space, we have set

$$b_{k,r}(m) = \binom{m}{r}(-1)^{m-r}\mu^{m-r}s_k(2k+1)^{\frac{r}{\theta}}\alpha^{\frac{r}{\theta}}\Gamma\left(1-\frac{r}{\theta}\right)$$

(assuming that $\theta > r$) and

$$\mathrm{U}_r = (\alpha\theta)^{\gamma-1}(r+\gamma)^{-\frac{(\theta+1)(\gamma-1)}{\theta}-1}\alpha^{-\frac{(\theta+1)(\gamma-1)}{\theta}}\Gamma\left(\frac{(\theta+1)(\gamma-1)}{\theta}+1\right).$$

These explicit expressions were obtained from the results found in Section 2.

4.1 Maximum likelihood estimation

In this section, we derive the MLEs of the unknown parameters α and θ of the SinIW model. Let $x_1, \ldots, x_n$ be a sample of size n from the SinIW distribution. Then, the log-likelihood function for the vector of parameter $\mathbf{\Theta} = (\alpha, \theta)^\top$ can be expressed as

$$\ell(\mathbf{\Theta}) = n\log\left(\frac{\pi}{2}\right) + n\log(\alpha\theta) - (\theta+1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\alpha x_i^{-\theta} + \sum_{i=1}^{n}\log\left\{\cos\left[\frac{\pi}{2}\exp(-\alpha x_i^{-\theta})\right]\right\}.$$

Table 1: Some structural properties for the SinIW distribution.

Measures	**Explicit expressions**
qf	$\left\{-\frac{1}{\alpha}\log\left[\frac{2}{\pi}\arcsin(u)\right]\right\}^{-\frac{1}{\theta}}$
Moments	$\sum_{k=0}^{+\infty} s_k(2k+1)^{\frac{m}{\theta}}\alpha^{\frac{m}{\theta}}\Gamma\left(1-\frac{m}{\theta}\right)$
Central moments	$\sum_{k=0}^{+\infty}\sum_{r=0}^{m} b_{k,r}(m)$
Variance	$\sum_{k=0}^{+\infty}\sum_{r=0}^{2} b_{k,r}(2)$
Skewness	$\frac{\sum_{k=0}^{+\infty}\sum_{r=0}^{3} b_{k,r}(3)}{\left[\sum_{k=0}^{+\infty}\sum_{r=0}^{2} b_{k,r}(2)\right]^{\frac{3}{2}}}$
Kurtosis	$\frac{\sum_{k=0}^{+\infty}\sum_{r=0}^{4} b_{k,r}(4)}{\left[\sum_{k=0}^{+\infty}\sum_{r=0}^{2} b_{k,r}(2)\right]^{2}}$
mgf	$\sum_{k=0}^{+\infty}\sum_{m=0}^{+\infty} s_k\frac{t^m}{m!}(2k+1)^{\frac{m}{\theta}}\alpha^{\frac{m}{\theta}}\Gamma\left(1-\frac{m}{\theta}\right)$
Rényi entropy	$\frac{1}{1-\gamma}\left\{\gamma\log\left(\frac{\pi}{2}\right)+\log\left[\sum_{k=0}^{+\infty}\sum_{r=0}^{k} a_k\binom{k}{r}(-1)^{k-r}\left(\frac{1}{2}\right)^{k-r}\mathrm{U}_r\right]\right\}$

The elements of the score vector $U(\Theta)$ are given by

$$U_\alpha = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i^{-\theta} + \frac{\pi}{2}\sum_{i=1}^{n} x_i^{-\theta}\exp(-\alpha x_i^{-\theta})\tan\left[\frac{\pi}{2}\exp(-\alpha x_i^{-\theta})\right]$$

and

$$U_\theta = \frac{n}{\theta} - \sum_{i=1}^{n}(1-\alpha x_i^{-\theta})\log(x_i) - \frac{\alpha\pi}{2}\sum_{i=1}^{n} x_i^{-\theta}\log(x_i)\exp(-\alpha x_i^{-\theta})\tan\left[\frac{\pi}{2}\exp(-\alpha x_i^{-\theta})\right].$$

The MLEs of α and θ are obtained by solving simultaneously the equations: $U_\alpha = 0$ and $U_\theta = 0$. Then, under standard regularity assumptions, all the guaranties of convergence of the MLEs hold, including asymptotic normality, allowing the interval and test estimations. One of the main tool to construct these mathematical object is the observed information matrix given by

$$J(\Theta) = -\begin{pmatrix} U_{\alpha,\alpha} & U_{\alpha,\theta} \\ . & U_{\theta,\theta} \end{pmatrix}$$

where $U_{\alpha,\alpha} = \frac{\partial^2}{\partial\alpha^2}\ell(\boldsymbol{\Theta})$, $U_{\alpha,\theta} = \frac{\partial^2}{\partial\alpha\partial\theta}\ell(\boldsymbol{\Theta})$ and $U_{\theta,\theta} = \frac{\partial^2}{\partial\theta^2}\ell(\boldsymbol{\Theta})$. These components are expressible by using any symbolic software.

4.2 Numerical evaluation of the MLE bias

Using the SinIW R package developed by [30], we produce samples for many parameter combinations of the SinIW distribution, and calculated the MLEs for each sample (via the [26] standard optim implementation of the BFGS algorithm). This allows us to test difficulties in parameter estimation such as sharpness or flatness of the likelihood function, and provides estimates for the size and direction (underestimate or overestimate) of the MLEs bias.

The simulation uses samples with sizes 10, 100 and 1000, and explored the parameter space $(\alpha, \theta) \in \Theta = \{0.5, 1, 1.5\} \times \{0.5, 0.85, 1\}$. We produced 10 000 replicas for each combination of parameters values and sample size and took the mean of the results. Table 2 summarizes the experiment, showing average estimate for each parameter, bias for each parameter and relative (percentual) bias for each parameter. One can see that α is consistently underestimated and θ is consistently overestimated by the maximum likelihood method.

Table 3 presents some useful statistics in order to understand the overall simulation process. Variance estimates are obtained by inverting the estimated Heassian matrix, and the distance between the empirical density and the estimated density is represented by the Kolmogorov-Smirnov (KS) statistic. On this table the sample size effect is clear: the variances and the KS statistic are smaller for the larger sample size.

5 Application

Here, we use the SinIW model in an application to a real data set. We shall compare to exponentiated Weibull distribution introduced by [18], the Beta exponential distribution proposed by [21] and the Weibull distribution

Table 2: Numerical evaluation of the maximum likelihood bias for the SinIW distribution.

n	α	θ	$\hat{\alpha}$	$\hat{\theta}$	$b(\alpha)$	$b(\theta)$	$b(\alpha)/\alpha$	$b(\theta)/\theta$
10	0.50	0.70	0.461	0.878	-0.039	0.178	-24.442	15.106
		0.85	0.471	1.017	-0.029	0.167	-22.762	11.292
		1.00	0.487	1.159	-0.013	0.159	-18.688	8.709
	1.00	0.70	0.691	1.072	-0.309	0.372	-59.571	30.398
		0.85	0.748	1.240	-0.252	0.390	-46.977	27.261
		1.00	0.801	1.395	-0.198	0.395	-37.616	24.050
	1.50	0.70	0.931	1.268	-0.569	0.568	-74.834	40.873
		0.85	1.019	1.456	-0.480	0.697	-60.864	37.803
		1.00	1.105	1.623	-0.395	0.625	-49.046	34.463
100	0.50	0.70	0.492	0.754	-0.008	0.054	-2.482	6.661
		0.85	0.496	0.889	-0.004	0.039	-1.685	3.942
		1.00	0.506	1.022	-0.006	0.022	0.279	1.727
	1.00	0.70	0.696	0.919	-0.304	0.219	-44.529	23.513
		0.85	0.743	1.073	-0.257	0.223	-35.519	20.425
		1.00	0.789	1.210	-0.211	0.210	-27.573	16.984
	1.50	0.70	0.899	1.075	-0.601	0.375	-67.739	34.501
		0.85	0.973	1.241	-0.527	0.391	-54.999	31.108
		1.00	1.042	1.392	-0.458	0.392	-44.815	27.800
1000	0.50	0.70	0.495	0.743	-0.005	0.043	-1.119	5.750
		0.85	0.499	0.878	-0.001	0.028	-0.256	3.168
		1.00	0.509	1.010	-0.009	0.010	-1.730	0.978
	1.00	0.70	0.698	0.908	-0.302	0.208	-43.390	22.897
		0.85	0.746	1.057	-0.254	0.207	-34.194	19.590
		1.00	0.790	1.195	-0.209	0.195	-26.571	16.311
	1.50	0.70	0.897	1.057	-0.603	0.357	-67.342	33.772
		0.85	0.972	1.223	-0.528	0.373	-54.478	30.450
		1.00	1.039	1.373	-0.460	0.373	-44.345	27.109

introduced by [31]. The corresponding pdfs are, respectively, given by

$$f_{EW}(x) = \alpha \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} \left[1 - \exp\left(-\frac{x}{\lambda}\right)^{k}\right] \exp\left(-\frac{x}{\lambda}\right)^{k}, \quad x > 0,$$

Table 3: General goodness of fit statistics for the simulations related to the SinIW distribution.

n	α	θ	$Var(\alpha)$	$Var(\theta)$	KS Distance
10	0.50	0.70	0.169	0.202	0.189
		0.85	0.171	0.235	0.187
		1.00	0.175	0.269	0.184
	1.00	0.70	0.212	0.245	0.194
		0.85	0.223	0.285	0.190
		1.00	0.233	0.322	0.189
	1.50	0.70	0.256	0.288	0.198
		0.85	0.273	0.333	0.195
		1.00	0.291	0.373	0.193
100	0.50	0.70	0.056	0.053	0.084
		0.85	0.056	0.063	0.077
		1.00	0.507	0.073	0.073
	1.00	0.70	0.057	0.064	0.092
		0.85	0.067	0.076	0.086
		1.00	0.069	0.086	0.081
	1.50	0.70	0.072	0.074	0.102
		0.85	0.078	0.087	0.096
		1.00	0.081	0.097	0.089
1000	0.50	0.70	0.086	0.016	0.055
		0.85	0.018	0.019	0.047
		1.00	0.021	0.023	0.041
	1.00	0.70	0.022	0.020	0.064
		0.85	0.022	0.023	0.057
		1.00	0.024	0.027	0.051
	1.50	0.70	0.022	0.023	0.076
		0.85	0.026	0.027	0.068
		1.00	0.027	0.030	0.061

$$f_{BE}(x) = \frac{\lambda}{B(a,b)} \exp(-b\lambda x) \left[1 - \exp(-\lambda x)\right]^{a-1}, \quad x > 0,$$

and

$$f_W(x) = \alpha \lambda^{\alpha} x^{\alpha-1} \exp\left[-(\lambda x)^{\alpha}\right], \quad x > 0.$$

Also, let let mention that all the introduced parameters are strictly positive. The considered data set is from [3]. It contains 72 observations representing the survival times in days of guinea pigs injected with different doses of tubercle bacilli. The data are listed in Table 4. Table 5 displays some descriptive statistics of the data.

Table 6 presents the MLEs of the parameters (standard errors in parentheses). We see that the SinIW model, when compared to others, provided better statistics, specially with the BIC and Anderson-Darling (A^*) (see [8]) and Cramér - von Mises (W^*). Thus, we conclude that the SinIW model is quite flexible in the modeling of the proposed data. Also, Figure 3 suggests an excellent fit to the data distribution to the adequacy of the data.

Table 4: Guinea Pigs Data.

12	15	22	24	24	32	32	33	34	38	38	43	44	48
52	53	54	54	55	56	57	58	58	59	60	60	60	60
61	62	63	65	65	67	68	70	70	72	73	75	76	76
81	83	84	85	87	91	95	96	98	99	109	110	121	127
129	131	143	146	146	175	175	211	233	258	258	263	297	341
341	376												

Table 5: Descriptive statistics.

Min.	Q_1	Median	Mean	Q_3	Max.	Var.
12.00	54.75	70.00	99.82	112.80	376.00	6580.122

Table 6: Estimates of the considered models for Guinea Pigs Data.

Distributions	Estimates			AIC	BIC	CAIC	HQIC	A^*	W^*
SinIW (α, θ)	115.12	1.09	–	787.66	792.21	787.83	789.47	0.81	0.14
	(41.96)	(0.09)	–						
EW (α, k, λ)	10.04	0.57	13.58	786.73	793.56	787.09	789.45	0.89	0.17
	(6.39)	(0.12)	(10.49)						
BE (a, b, λ)	3.64	0.30	0.05	788.26	795.09	788.62	790.98	1.12	0.21
	(1.17)	(0.09)	(0.01)						
W (α, λ)	1.39	0.01	–	798.29	802.84	798.47	800.11	2.39	0.43
	(0.12)	(0.00)	–						

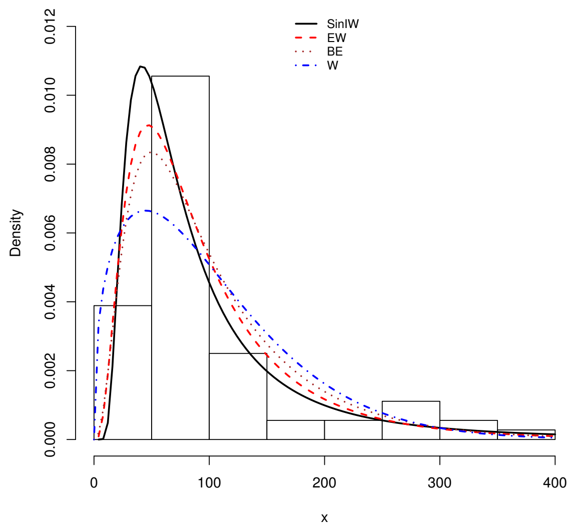

Figure 3: Some estimated fitted densities of the data.

6 Concluding remarks

We proposed a new class of trigonometric distribution, the Sin-G class, and a new distribution in this class, the Sine Inverse Weibull distribution, denoted by SinIW distribution. We obtain the probability density function, cumulative density function and their expansions, quantile function, moments, moment generating function, reliability parameter, Rényi entropy and order statistics. By considering the SinIW model, the parameters are estimated via the maximum likelihood method. Plots of the estimated pdf and cdf indicate that SinIW model is superior to the other considered models. In particular, In Figure 3, we can see that this model can help in the analysis of survival data, as well as in other areas of knowledge.

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