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Existence of mild solutions of second order evolution integro-differential equations in the Fréchet spaces

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Abstract. In this article, we shall establish sufficient conditions for the existence of mild solutions for second order semilinear integro-differential evolution equations in Fréchet spaces $C(\mathbb{R}_+, E)$, where E is an Banach space. Our approach is based on the concept of a measure of noncompactness and Tykhonoff fixed point theorem. For illustration we give an example.

Keywords: Semilinear integro-differential equation, measure of noncompactness, mild solutions, evolution system, Tykhonoff fixed point theorem.
AMS Subject Classification: 34G20, 35R10.

1 Introduction

The objective of the present work is to give an alternative approach to the existence of mild solutions for the following second order evolution equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}x(t) - A(t)x(t) = f\Big(t, x(t), \displaystyle\int_0^t u(t,s,x(s))ds\Big), \quad t \geq 0, \\ \dfrac{\partial x}{\partial t}(0) = x_0 \in E, \\ x(0) = \widetilde{x}_0 \in E, \end{cases} \tag{1}$$

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where $A(t) : D_t \subset E \to E$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $U(t,s)_{0\leq s\leq t}$ and $f : \mathbb{R}_+ \times E \to E$ is a given function.

Many mathematical models of phenomena occurring in engineering involve semilinear integro-differential equation. Moreover, this type of equations have received a lot of attention in recent years [9, 20]. Useful for the study of abstract second order equations is the existence of an evolution system $U(t,s)$ for the homogenous equation

$$x''(t) = A(t)x(t),\ t \geq 0.$$

The are several techniques to prove the existence of $U(t,s)$ (see Kozak [12]). The study of evolution initial value problems with local or nonlocal conditions have applications in problems in physics and other areas of applied mathematics. Several authors have investigated the problem of nonlocal initial conditions for different classes of abstract differential equations in Banach spaces, for example, we refer the reader to [8, 10, 14, 15, 17, 21, 22] and the references therein. In this paper we prove a theorem on the existence of mild solutions for the equation (1) on the space of all continuous functions on $\mathbb{R}_+$. As several existence results obtained by several authors (see [2–5, 11, 17–19]) in the field of abstract differential equations in Banach spaces, we derive some sufficient conditions for the existence of solutions of second order semilinear functional evolution equations.

The considerations of this paper are based on the notion of measure of noncompactness in the Fréchet spaces of functions continuous on $\mathbb{R}_+$ and Tykhonoff fixed point theorem. Moreover, an application is provided to illustrate the results of this work.

2 Preliminary tools

In what follows, E will represent a Banach space with norm $\|.\|$. Denote by $C(\mathbb{R}_+, E)$ the space of continuous functions $x : \mathbb{R}_+ \to E$. Let $I_n = [0,n]$, $n \in \mathbb{N}$. The space $C(\mathbb{R}_+, E)$ is the locally convex space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$ with the metric

$$d(x,y) = \sup\left\{2^{-n}\frac{\|x-y\|_n}{1+\|x-y\|_n}\ :\ n \in \mathbb{N}\right\},$$

where

$$\|x\|_n := \sup\left\{|x(t)| :\ t \in I_n\right\}.$$

The convergence in $C(\mathbb{R}_+, E)$ is the uniform convergence in the compact intervals, i.e., x_j converge to x in $C(\mathbb{R}_+, E)$ if and only if $\|x_j - x\|_n$ converge to 0 in $(C(I_n), \|.\|_n)$, $\forall n \in \mathbb{N}$. By Arzela-Ascoli theorem, a set $M \subset C(\mathbb{R}_+, E)$ is compact if and only if for each $n \in \mathbb{N}$, M is a compact set in the Banach space $(C(I_n), \|.\|_n)$, see [13].

Next, we present some basic facts concerning measure of noncompactness in $C(\mathbb{R}_+, E)$ (see [1, 11, 16]). Let θ be the zero element of E. Denote by $B(x, r)$ the closed ball centred at x with radius r and by B_r the ball $B(\theta, r)$. If X is a nonempty subset of E we denote by $\overline{X}$, $\text{Conv}(X)$ the closure and convex closure of X, respectively. Finally, let us denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets. Following [1] we accept the following definition of the concept of a measure of noncompactness.

Definition 1. [1] A function $\mu : \mathcal{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1. The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
2. $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3. $\mu(\overline{X}) = \mu(Conv X) = \mu(X)$.
4. $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
5. If (X_n) is a sequence of nonempty, bounded, closed subsets of $C(\mathbb{R}_+, E)$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and $\lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Next, we consider the measure of noncompactness in $C(\mathbb{R}_+, E)$ defined in [11, 16, 19] as follows. Let

$$\mathcal{M}_r = \{X \subset C(\mathbb{R}_+, E) : X \neq \emptyset \text{ and } \|x(t)\| \leq r(t) \text{ for } x \in X \text{ and } t \geq 0\},$$

where $r : \mathbb{R}_+ \to (0, \infty)$ is a fixed function and let $\mathcal{N}_r$ be the family of all relatively compact subsets of $\mathcal{M}_r$. Fix $X \in \mathcal{M}_{C(\mathbb{R}_+,E)}$ and a positive number $T > 0$. For $x \in X$ and $\epsilon > 0$, denote by $w^T(x, \epsilon)$ the modulus of continuity of the function x on the interval $[0, T]$, i.e.,

$$w^T(x, \epsilon) = \sup\{\|x(t) - x(s)\| : t, s \in [0, T], |t - s| \leq \epsilon\}.$$

Further, let us put

$$\begin{aligned} w^T(X, \epsilon) &= \sup\left\{w^T(x, \epsilon) : \ x \in X\right\}, \\ w_0^T(X) &= \lim_{\epsilon\to 0} w^T(X, \epsilon). \end{aligned}$$

Now, let μ be a regular measure of noncompactness in E and let

$$\mu^T(X) = \sup\{\mu(X(t)) : t \in [0,T]\}. \tag{2}$$

Let us take a function $R : \mathbb{R}_+ \to (0,\infty)$ such that $R(t) \geq r(t)$ for $t \geq 0$. Define the mapping γ_R on the family $\mathcal{M}_r$ by

$$\gamma_R(X) = \sup\left\{\frac{1}{R(t)}\Big(w_0^T(X) + \mu^T(X)\Big) : T \geq 0\right\}. \tag{3}$$

The properties of γ_R is given by the following theorem

Theorem 1. *[19] The mapping $\gamma_R : \mathcal{M}_r \to \mathbb{R}_+$ satisfies the conditions*

(1) The family $ker\gamma_R = \{X \in \mathcal{M}_r : \gamma_R(X) = 0\} = \mathcal{N}_r$.

(2) $\gamma_R(Conv(X)) = \gamma_R(X)$.

(3) If (X_n) is a sequence of closed sets from $\mathcal{M}_r$ such that

$$X_{n+1} \subset X_n, \ n = 0, 1, ...,$$

and if $\lim_{n\to\infty} \gamma_R(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Let d be the metric associated with the norm $\|.\|$ in E and $X \subset E$ be a bounded subset. By $d(x, X)$, we denote the distance between point x and the set X.

Definition 2. Let $X, Y \subset E$ be two nonempty and bounded sets. The number

$$d_{\mathcal{H}}(X,Y) = \max\left\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,Y)\right\},$$

is called the Hausdorff distance between A and B.

Lemma 1. *[1] If μ is a regular measure of noncompactness, then*

$$|\mu(X) - \mu(Y)| \leq \mu(B(\theta,1))d_{\mathcal{H}}(X,Y),$$

for any bounded subsets X, Y of E, d_H is the Hausdorff distance between X and Y.

The following lemmas borrowed from [1,19] will be needed in the proof of our existence result of solution of (1) .

Lemma 2. *[19] If all functions belonging to X are equicontinuous on compact subsets of $\mathbb{R}_+$, then*

$$\mu\Big(\int_0^t X(s)ds\Big) \leq \int_0^t \mu(X(s))ds.$$

Lemma 3. *[11] (Cauchy's formula) If $f : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function then*

$$\int_a^t \int_a^{s_1} ... \int_a^{s_n} f(s_{n+1})ds_{n+1}ds_n...ds_1 = \frac{1}{n!}\int_a^t f(s)(t-s)^n ds \ \text{ for each } t \geq a.$$

Our consideration are based on following Tichonov fixed point theorem.

Theorem 2. *[6] Let K be a closed convex subset of locally convex Hausdorff space E. Assume that $F : K \to K$ is continuous and that $F(K)$ is relatively compact in E. Then F has at least one fixed point in K.*

In what follows we give some definitions of concept of evolution operator when developed by Kozak [12].

Definition 3. [7] A two parameters family of bounded linear operators $U(t,s)$ $(0 \leq s \leq t)$, on E is called an evolution system if the following conditions are satisfied

1. $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$, for $0 \leq s \leq r \leq t$,
2. $(t,s) \mapsto U(t,s)$ is strongly continuous for $0 \leq s \leq t$.

Definition 4. A family U of bounded operators $U(t;s) : E \to E$ such that for $(t,s) \in \triangle := \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : s \leq t\}$, is called an evolution operator of the equation (1) if the following conditions hold:

1. For any $x \in E$ the map $(t,s) \to U(t,s)x$ is continuously differentiable and
   (a) for any $t \geq 0, U(t,t) = 0$,
   (b) for all $(t,s) \in \triangle$ and for any $x \in E$ we have $\frac{\partial}{\partial t}U(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}U(t,s)x|_{t=s} = -x$
2. For all $(t,s) \in \triangle$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}U(t,s)x \in D(A(t))$, the map $(t,s) \to U(t,s)x$ is of class C^2 and

(a) $\frac{\partial^2}{\partial t^2}U(t,s)x = A(t)U(t,s)x$,

(b) $\frac{\partial^2}{\partial s^2}U(t,s)x = U(t,s)A(s)x$,

(c) $\frac{\partial^2}{\partial t \partial s}U(t,s)x|_{t=s} = 0$.

3. For all $(t,s) \in \triangle$, then $\frac{\partial}{\partial s}U(t,s)x \in D(A(t))$, there exist $\frac{\partial^3}{\partial t^2 \partial s}U(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}U(t,s)x$ and

   (a) $\frac{\partial^3}{\partial t^2 \partial s}U(t,s)x = A(t)\frac{\partial}{\partial t}U(t,s)x$. Moreover the map $(t,s) \mapsto A(t)\frac{\partial}{\partial t}U(t,s)x$ is continuous,

   (b) $\frac{\partial^3}{\partial s^2 \partial t}U(t,s)x = \frac{\partial}{\partial t}U(t,s)A(s)x$.

Definition 5. A continuous function $x : \mathbb{R}_+ \to E$ is said to be a mild solution of (1) if x satisfies to

$$\begin{aligned} x(t) &= -\frac{\partial}{\partial s}U(t,0)x_0 + U(t,0)\widetilde{x}_0 \\ &\quad + \int_0^t U(t,s)f\Big(s, x(s), \int_0^s u(s,\tau,x(\tau))d\tau\Big)ds. \end{aligned} \tag{4}$$

In what follows, we define operators $F, S : C(\mathbb{R}_+, E) \to C(\mathbb{R}_+, E)$ by the formulas

$$\begin{aligned} (Sx)(t) &= f\Big(t, x(t), \int_0^t u(t,s,x(s))ds\Big), \\ (Fx)(t) &= -\frac{\partial}{\partial s}U(t,0)x_0 + U(t,0)\widetilde{x}_0 + \int_0^t U(t,s)(Sx)(s)ds. \end{aligned}$$

3 Existence of mild solutions

In this section by using the usual technique of measure of noncompactness and its application in differential equations in Banach space (see [11]), we give an existence result for the problem (1). The following hypotheses well be needed in the sequel.

(H_A) There are two constants $M \geq 1$ and $\widetilde{M} \geq 0$ such that

$$\|U(t,s)\|_{B(E)} \leq M \text{ and } \left\|\frac{\partial}{\partial s}U(t,s)\right\|_{B(E)} \leq \widetilde{M} \text{ for } (t,s) \in \triangle.$$

(H_f) (i) $(t,x,y) \mapsto f(t,x,y)$ satisfies the Carathodory condition, i.e. $f(.,x,y)$ is measurable for $(x,y) \in E \times E$ and $f(t,.,.)$ is continuous for a.e. $t \in \mathbb{R}_+$

(ii) $m : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|f(t,x,y)\| \leq m(t)\Big(\|x\| + \|y\|\Big)$ for a.e. $t \geq 0$ and all $x,\ y \in E$.

(iii) There exists a locally integrable function $h_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any nonempty sets X, Y containing continuous functions $x, y : \mathbb{R}_+ \to E$ respectively which are uniformly bounded on compact subintervals of $\mathbb{R}_+$, the inequality

$$\chi\Big(f([0,t]\times X\times Y\Big) \leq h_1(t)\sup\Big\{\chi(X(s))+\chi(Y(s)) :\ 0 \leq s \leq t\Big\}$$

hold for a.e $t \in \mathbb{R}_+$.

(H_u) (i) $u(t,s,x) : \mathbb{R}_+ \times \mathbb{R}_+ \times E \to E$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times E$.

(ii) $q : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $m(t)\max\Big(1, q(t)\Big) \in L^1_{loc}(\mathbb{R}_+)$, nondecreasing and

$$\left\|\int_0^t u(t,s,x(s))ds\right\| \leq q(t)\phi(\|x(t)\|),$$

where $\phi : \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing with $\phi(0) = 0$.

(iii) There exists $h_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $t \mapsto \int_0^t h_2(t,s)ds$ is essentially bounded function on compact intervals of $\mathbb{R}_+$ and

$$\chi(\int_0^t u(t,s,X(s))ds) \leq h_2(t)\chi(X)$$

for a.e. $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$ and all bounded subset X of E.

Remark 1. If the functions f and u are compact, or satisfies Lipschitz-type condition, then conditions $H_f(iii)$ and $H_u(iii)$ are not necessary.

The precise definition of the function $p(t)$ used in our considerations will be given further on. For our further purposes we will also need the important following lemma.

Theorem 3. *Let E be a separable Banach space. Assume that the assumptions $(H_A), (H_f)$ and (H_u) are satisfied. If there exists r_0 such that*

$$\widetilde{M}\|x_0\| + M\|\widetilde{x}_0\| + M(r_0 + \phi(r_0))\sup_{t\geq 0}\int_0^t m(s)\max(1, q(s))ds \leq r_0,$$

then for $x_0, \widetilde{x}_0 \in E$, the Eq. (1) has at least one mild solution x in $C(\mathbb{R}_+, E)$.

Proof. Consider the operator F defined by formula

$$\begin{aligned}(Fx)(t) &= -\frac{\partial U}{\partial s}(t,0)x_0 + U(t,0)\widetilde{x}_0 \\ &\quad + \int_0^t U(t,s) f\Big(s, x(s), \int_0^s u(s,\tau,x(\tau))d\tau\Big)ds, \quad t \geq 0. \qquad (5)\end{aligned}$$

Let r_0 be a number satisfying assumption of Theorem 3 and define the set $B_{r_0} \subset C(\mathbb{R}_+, E)$ by $B_{r_0} = \{x \in C(\mathbb{R}_+, E) : \ \|x\| \leq r_0\}$. For $x \in B_{r_0}$, applying assumptions H_f(i) and H_u(ii) we have

$$\begin{aligned}\|(Fx)(t)\| &\leq \Big\|-\frac{\partial U}{\partial s}(t,0)x_0 + U(t,0)\widetilde{x}_0\Big\| \\ &\quad + \Big\|\int_0^t U(t,s) f(s, x(s), \int_0^s u(s,\tau,x(\tau))d\tau)ds\Big\| \\ &\leq \Big\|-\frac{\partial U}{\partial s}(t,0)x_0 + U(t,0)\widetilde{x}_0\Big\| \\ &\quad + \int_0^t \|U(t,s)\| \|f(s, x(s), \int_0^s u(s,\tau,x(\tau))d\tau)\| ds \\ &\leq \widetilde{M}\|\widetilde{x}_0\| + M\|x_0\| + M\int_0^t m(s)\Big[\|x(s)\| + q(s)\phi(\|x(s)\|)\Big]ds \\ &\leq \widetilde{M}\|\widetilde{x}_0\| + M\|x_0\| + \int_0^t Mm(s)\max(1,q(s))\Big(r_0 + \phi(r_0)\Big)ds \\ &\leq r_0. \qquad (6)\end{aligned}$$

From the estimate (6), we deduce that F transforms B_{r_0} into itself. In what follows we will estimate the modulus of continuity of the function Fx. To do this let us fix $x \in C(\mathbb{R}_+, E)$ such that $\|x(t)\| \leq r_0$. Fix an arbitrary $T \geq 0$ and $\epsilon \geq 0$ and let $t_1, t_2 \in [0,T]$ such that $|t_1 - t_2| \leq \epsilon$. Without loss of generality, we may assume that $t_1 \leq t_2$. Then, in view of our assumptions we get:

$$\begin{aligned}&\|(Fx)(t_2) - (Fx)(t_1)\| \\ &\leq \Big\|\Big(U(t_2,0) - U(t_1,0)\Big)\widetilde{x}_0\Big\| + \Big\|\Big(\frac{\partial U}{\partial s}(t_1,0) - \frac{\partial U}{\partial s}(t_2,0)\Big)x_0\Big\| \\ &\quad + \Big(r_0 + \phi(r_0)\Big)\int_{t_1}^{t_2} \|U(t_2,s)\| m(s)\max(1,q(s))ds \\ &\quad + \Big(r_0 + \phi(r_0)\Big)\int_0^{t_1} \|(U(t_2,0) - U(t_1,0)\| m(s)\max(1,q(s))ds\end{aligned}$$

$$
\begin{aligned}
&\leq w_1^T(U(.,0),\epsilon)\|\widetilde{x}_0\| + w_2^T(U(.,0),\epsilon)\|x_0\| \\
&\quad + \nu^T(U,\epsilon)\Big(r_0+\phi(r_0)\Big)\int_0^{t_1} m(s)\max(1,q(s))ds \\
&\quad + M\Big(r_0+\phi(r_0)\Big)\int_{t_1}^{t_2} m(s)\max(1,q(s))ds. \qquad (7)
\end{aligned}
$$

Putting

$$
\begin{aligned}
\Delta^T(U,\epsilon) = {} & w_1^T(U(.,0),\epsilon)\|\widetilde{x}_0\| + w_2^T(U(.,0),\epsilon)\|x_0\| \\
& + \nu^T(U,\epsilon)\Big(r_0+\phi(r_0)\Big)\int_0^{t_1} m(s)\max(1,q(s))ds \\
& + M\Big(r_0+\phi(r_0)\Big)\int_{t_1}^{t_2} m(s)\max(1,q(s))ds.
\end{aligned}
$$

where

$$
\begin{aligned}
w_1^T(U,\epsilon) &= \sup\{\|U(t_2,0)-U(t_1,0)\| : t_1,t_2\in[0,T], |t_1-t_2|\leq\epsilon\}, \\
w_2^T(U,\epsilon) &= \sup\{\|\frac{\partial U}{\partial s}(t_2,0)-\frac{\partial U}{\partial s}(t_1,0)\| : t_1,t_2\in[0,T], |t_1-t_2|\leq\epsilon\}, \\
\nu^T(U,\epsilon) &= \sup\{\|U(t_2,s)-U(t_1,s)\| : 0\leq t_1\leq t_2\leq T, |t_2-t_1|\leq\epsilon\}.
\end{aligned}
$$

Then, we have

$$
\|(Fx)(t_2)-(Fx)(t_1)\| \leq \Delta^T(U,\epsilon), \text{for } x \text{ such that } \|x(t)\|\leq r_0. \qquad (8)
$$

Under assumptions, we have $\lim_{\epsilon\to 0}\Delta^T(U,\epsilon)=0$. Next, define the subset

$$
\Omega = \Big\{x\in C(\mathbb{R}_+,E) : \|x(t)\|\leq r_0 \ \text{ and } \ w^T(x,\epsilon)\leq\Delta^T(x,\epsilon) \text{ for } T\geq 0\Big\}.
$$

It is easy to see that Ω is a bounded, closed and convex subset of $C(\mathbb{R}_+,E)$. Next, we show that F is continuous on the set Ω. Let $(x_n)_n\subset\Omega$ be a sequence converging to x and fix $T>0$. We show that $\|Fx_n - Fx\|$ converges uniformly to 0 on $[0,T]$.

$$
\|(Sx_n)(t)-(Sx)(t)\| \leq 2M(r_0+\phi(r_0)m(t)\max(1,q(t)) \in L^1_{loc}(\mathbb{R}_+).
$$

Then, by Lebesgues dominated convergence theorem we have

$$
\lim_{n\to\infty}\sup_{t\leq T}\|(Fx_n)(t)-(Fx)(t)\| \leq M\lim_{n\to\infty}\int_0^T \|(Sx_n)(s)-(Sx)(s)\|ds = 0.
$$

In order to complete our proof, let us defined the sequence (Q_n) of subsets of $C(\mathbb{R}_+, E)$ by

$$\begin{cases} Q_0 = \Omega, \\ Q_n = \mathrm{Conv}(F(Q_{n-1})) \text{ for } n \in \mathbb{N} \setminus \{0\}. \end{cases} \tag{9}$$

Observe that all subsets of this sequence are nonempty, closed and convex. Moreover, $Q_{n+1} \subset Q_n$ for $n \in \mathbb{N}$. The equicontinuity of the set Ω on compact intervals, implies that

$$w_0^T(Q_n) = 0 \text{ for } n \in \mathbb{N} \text{ and } T \geq 0. \tag{10}$$

Next, define the sequence $(z_n) \in C(\mathbb{R}_+, \mathbb{R}_+)$ by $z_n(t) = \mu(Q_n(t))$. Obviously $0 \leq z_{n+1}(t) \leq z_n(t)$, for $n = 0, 1, \ldots$. Thus the sequence converges uniformly to the function $z_\infty(t)$. By Lemma 1 and Eq. (8) we get

$$|z_n(t) - z_n(s)| \leq \mu(B(\theta, 1))\Delta^T(U, |t - s|)$$

which implies the continuity of z_n on $\mathbb{R}_+$. Using Lemma 2, (C)(i), and the properties of the measure of noncompactness μ, we obtain

$$\begin{aligned} z_n(t) &= \mu\Big(\mathrm{Conv}(FQ_n)(t)\Big) \\ &\leq \mu\Big(\int_0^t U(t,s) f\Big(s, Q_{n-1}(s), \int_0^s u(s, \tau, Q_{n-1}(\tau)) d\tau\Big) ds\Big) \\ &\leq M \int_0^t h_1(s)\Big(\mu(Q_{n-1}(s)) + h_2(s)\mu(Q_{n-1}(s))\Big) ds \\ &\leq M \int_0^t h_1(s)\Big(z_{n-1}(s) + h_2(s) z_{n-1}(s) d\tau\Big) ds \\ &\leq M \int_0^t h(s) z_{n-1}(s) ds, \end{aligned} \tag{11}$$

where $h(s) = h_1(s)\max(1, h_2(s))$. Denote $\widetilde{h}(t) = \operatorname{ess\,sup}\{h(s) : 0 \leq s \leq t\}$.

The inequality (11) will have the form

$$z_n(t) \leq M\widetilde{h}(t) \int_0^t z_{n-1}(s) ds. \tag{12}$$

Denote by G the operator acting from $C(\mathbb{R}_+, \mathbb{R}_+)$ into itself, defined as follows

$$(Gx)(t) = \int_0^t x(s) ds.$$

Hence, in view of (12) we get $z_n(t) \leq (M\widetilde{h}(t))^n (G^n) z_0(t)$, where G^n denotes the n-th iteration of the operator G. Further, taking into account the linearity of the operator G, and Lemma 3 we have the following inequality

$$(G^n)z_0(t) \quad = \quad \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} z_0(s_n) ds_n \dots ds_1. \tag{13}$$

Putting

$$g(t) = \begin{cases} 1, & \text{for} \quad t \in [0,1], \\ t^n, & \text{for} \quad t > 1, \end{cases}$$

we get

$$(G^n)z_0(t) \leq \frac{g(t)}{n!} \int_0^t z_0(s) ds.$$

Since the integral from (13) has n components, it follows that

$$z_n(t) \leq \frac{(M\widetilde{h}(t))^n g(t)}{n!} \int_0^t z_0(s) ds. \tag{14}$$

Now we apply the measure of noncompactness γ_R defined in $C(\mathbb{R}_+, E)$ by (3). Let us put

$$R(t) = r_0 \Big(1 + \int_0^t z_0(s) ds \Big) g(t) \exp(M\widetilde{h}(t)).$$

Observe that $R(t) \geq r_0$. Then, keeping in mind the measure μ^T defined by formula (2), in view of (14) we obtain

$$\mu^T(Q_n) = \sup_{t \leq T} \{ z_n(t) \} \leq \frac{(M\widetilde{h}(T))^n g(T)}{n!} \int_0^T z_0(s) ds.$$

Hence, in view of the estimate

$$\sup \left\{ \frac{t^n}{n! \exp(t)} \quad : t \geq 0 \right\} \leq \frac{n^n}{n! \exp(n)},$$

for $n \in N$, we derive the following evaluation

$$\frac{\mu^T(Q_n)}{R(T)} \leq \frac{(M\widetilde{h}(T))^n}{r_0 n! \exp(M\widetilde{h}(T))} \leq \frac{n^n}{r_0 n! \exp(n)} \ .$$

Linking the last inequality with (10) and taking into account that

$$\frac{n^n}{r_0 n! \exp(n)} \simeq \frac{1}{\sqrt{2\pi n}} \ ,$$

we infer that $\lim_{n\to\infty} \gamma_R(Q_n) = \gamma_R(Q_\infty) = 0$. Under the properties of γ_R given by Theorem 1, Q_∞ is nonempty subset, closed and convex. By applying the Tichonov fixed point theorem for $F : Q_\infty \to Q_\infty$ we conclude that F has at least a fixed point $x \in Q_\infty$. Obviously, the function x is a mild solution of (1). This completes the proof. □

4 Application

Consider the following second order partial differential equation with local conditions, denote that

$$\begin{cases} \dfrac{\partial^2 x(t,\xi)}{\partial t^2} = Lx(t,\xi) + \dfrac{x(t,\xi)}{1+e^{-t}} \\ \qquad\qquad + \sqrt{|x(t,\xi)|}\displaystyle\int_0^t \frac{(e^{-t})x(s,\xi)}{(1+e^{-(t-s)})(1+|x(s,\xi)|)}ds, \quad t>0 \\ \dfrac{\partial x}{\partial t}(0,\xi) = x_0(\xi),\ \xi \in [0,1]^n, \\ x(0,\xi) = \widetilde{x}_0(\xi),\ \xi \in [0,1]^n. \end{cases} \tag{15}$$

Let $([0,1],\mathcal{A},P)$ be a complete probability measure space. Let also $E = (L^2([0,1]),\mathcal{A},P)$ be the space of $\mathcal{A}$-measurable maps with the norm defined by, for $t>0$ fixed, we have

$$\|x(t)\|_2 = \Big(\int_0^1 |x(t,\zeta)|dP(\zeta)\Big)^{\frac{1}{2}},$$

where $dP(\zeta) = d\zeta$. Denote by L the Laplace operator $\displaystyle\sum_{i=1}^n \frac{\partial^2}{\partial\zeta_i^2}$. Then L generates a compact, analytic semigroup $U(.)$ of uniformly bounded linear operators. Let $(t,x) \in \mathbb{R}_+ \times L^2([0,1]^n)$, $\zeta \in [0,1]^n$. Bay using the Jensen's inequality it is not difficult to see that

$$\|f(t,x(t,\xi),\int_0^t u(t,s,x(s,\xi))ds)\|_2 \le \frac{1}{1+t^2}\Big[\|x(t)\|_2 + \sqrt{\|x(t)\|_2}\Big].$$

Put

$$\begin{cases} p(t) = \frac{1}{1+e^{-t}}, \\ q(t) = 1, \\ \phi(s) = \sqrt{s}, \quad \text{for all} \quad s \ge 0. \end{cases}$$

Denote by $\nu(x,\epsilon)$ the Kolmogorov modulus of continuity of x

$$\nu(x,\epsilon) = \sup\{\|x(t;\zeta+h) - x(t,\zeta)\|_2, \quad |h| \le \epsilon\}.$$

Next, for X bounded subset of $L^2([0,1]^n)$, put $\nu(X) = \lim_{\epsilon\to 0}\sup_{x\in X}\nu(x,\epsilon)$. It is known [1] that ν is a measure of noncompactness on $L^2([0,1]^n)$, in addition for X bounded subset of $L^2([0,1]^n)$ we have $\chi(X) \le \nu(X) \le 2\chi(X)$. So that the conditions (Hf) and (Hu) are checked with the following functions:

$$\begin{cases} h_1(t) = p(t) = \frac{1}{e^{-t}+1}, \\ h_2(t) = \widehat{q}(t) = 1. \end{cases}$$

Applying the result obtained in Theorem 3, we deduce that equation Eq. (1) has a mild solution.

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