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Weak Galerkin finite element method for an inhomogeneous Brusselator model with cross-diffusion

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Abstract. A new weak Galerkin finite element method is applied for time dependent Brusselator reaction-diffusion systems by using discrete weak gradient operators over discontinuous weak functions. In this work, we consider the lowest order weak Galerkin finite element space (P_0, P_0, RT_0). Discrete weak gradients are defined in Raviart-Thomas space. Thus we employ this approximate space on triangular mesh for solving unknown concentrations (u, v) in Brusselator reaction-diffusion systems. Based on a weak varitional form, semi-discrete and fully-discrete weak Galerkin finite element scheme are obtained. In addition, the paper presents some numerical results to illustrate the power of proposed method.

Keywords: Weak Galerkin finite element method, Reaction-diffusion system, Weak gradient.
AMS Subject Classification: 65M60, 35K57, 65M15.

1 Introduction

The Weak Galerkin finite element method (WGFEM) is a numerical approximation technique for partial differential equations (PDEs) which was developed for solving the second order elliptic problems based on local Raviart-Thomas (RT) or Brezzi-Douglas-Marini (BDM) elements [10]. This mehod is a combination of standard finite element method [1] and the idea

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of discontinuity by considering the value of function on the interior and interface of elements. Numerical implementation of the WGFEM were discussed in [7]. The main feature of this method is to approximate the differential operators (gradient, divergence, curl, etc.) by the aid of discrete weak derivatives. The WGFEM allows the use of totally discontinuous functions on arbitrary shape of polygons that makes the method highly flexible in practical computation. This approach is applied to PDEs such as, parabolic equation [2], Darcy equation [5], elliptic interface problems [8], Darcy-stokes equation [4], and etc. In this paper the WGFEM is applied to two-dimensional Brusselator reaction-diffusion systems. Brusselator reaction-diffusion systems arise in the study of chemical systems, biological systems, physical problems, and etc. In biological systems, when certain reaction is coupled with the process of diffusion, it is possible to obtain a stable spatial pattern [9]. We will focus on an inhomogeneous Brusselator model with cross-diffusion in the following form:

$$\begin{cases} \dfrac{\partial u(\mathbf{x},t)}{\partial t} - \Delta(D_{11}u + D_{12}v) = f(u,v), & (\mathbf{x},t) \in \Omega \times [0,\infty), \\ \dfrac{\partial v(\mathbf{x},t)}{\partial t} - \Delta(D_{21}u + D_{22}v) = g(u,v), & (\mathbf{x},t) \in \Omega \times [0,\infty), \\ \dfrac{\partial u(\mathbf{x},t)}{\partial \mathbf{n}} = \dfrac{\partial v(\mathbf{x},t)}{\partial \mathbf{n}} = 0, & \mathbf{x} \in \partial\Omega, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad v(\mathbf{x},0) = v_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{1}$$

where u and v represent concentrations of two reactants, $\Omega \subseteq \mathbb{R}^2$ is a bounded domain, Δ is the Laplace operator, $\mathbf{n}$ is the outward unit normal vector, D_{ij} is the cross-diffusion coefficient for $i \neq j$, $f(u,v) = u^2v - (B+1)u + A$, $g(u,v) = -u^2v + Bu$ and A, B are positive constants.

The outline of this paper is as follows. In Section 2 we introduce necessary definitions for the WGFEM. In Section 3 the weak Galerkin finite element scheme are proposed. In Section 4, we construct semi-discrete and fully-discrete weak Galerkin finite element scheme. In Section 5 we give some numerical results to verify the efficiency of proposed scheme. Finally, conclusions are presented in Section 6.

2 Weak gradient and discrete weak gradient

In this section, we will introduce the weak functions and the weak gradient operator which is the base of WGFEM. Let K be a polygonal domain with boundary ∂K and interior K^0. A weak function on the element K is a function $v = \{v_0, v_b\}$ such that $v_0 \in L^2(K)$ and $v_b \in H^{\frac{1}{2}}(\partial K)$. v_0 is the

value of v in the interior of element K and v_b is the value of v on the boundary of element K. The space of weak functions on the element K is represented by $W(K)$:

$$W(K) = \{v = \{v_0, v_b\} | v_0 \in L^2(K), v_b \in H^{\frac{1}{2}}(\partial K)\}.$$

A discrete weak function is a weak function $v = \{v_0, v_b\}$ such that $v_0 \in P^m(K^0)$, $(m \geq 0)$ and $v_b \in P^n(\partial K)$, $(n \geq 0)$, where $P^m(K^0)$, $P^n(\partial K)$ are polynomials with degree no more than m and n in interior of element K and on boundary of element K, respectively. The space of discrete weak function is denoted by:

$$W(K, m, n) = \{v = \{v_0, v_b\} | v_0 \in P^m(K^0), v_b \in P^n(\partial K)\}.$$

For any weak function $v \in W(K)$, its weak gradient $\nabla_w v$ is defined as a linear functional on $H(div, K)$ which is given by:

$$(\nabla_w v, w) = -\int_{K^0} v_0(\nabla . w) + \int_{\partial K} v_b(w.\mathbf{n}), \qquad \forall w \in H(div, K),$$

where $H(div, K) = \{\mathbf{v} : \mathbf{v} \in (L^2(K))^2, \nabla.\mathbf{v} \in L^2(K)\}$ and $\mathbf{n}$ is the outward normal direction to ∂K. A discrete weak gradient of $v = \{v_0, v_b\}$ on each element K is given by the following equation:

$$(\nabla_{w,d} v, w) = -\int_{K^0} v_0(\nabla . w) + \int_{\partial K} v_b(w.\mathbf{n}), \qquad \forall w \in [P_k(K)]^2 .$$

Here, $[P_k(K)]^2$ is the space of vector valued polynomials of degree k [3].

3 The weak Galerkin finite element method

Let $K \in \mathcal{K}_h$ be a triangular partition of domain with shape regular property. For each $K \in \mathcal{K}_h$ the finite element space is defined by patching $W(K, m, n)$ over all triangles $K \in \mathcal{K}_h$. The weak Galerkin finite element space is

$$V_h(m, n) = \{v = \{v_0, v_b\}, v|_K \in W(K, m, n), \forall K \in \mathcal{K}_h\},$$

the subspace of $V_h(m, n)$ with vanishing boundary value on $\partial\Omega$ is

$$V_h^0(m, n) = \{v = \{v_0, v_b\} \in V_h(m, n); v_b|_{\partial K \cap \partial\Omega} = 0, \quad \forall K \in \mathcal{K}_h\}.$$

By employing weak Galerkin finite element method, the variational form for system (1) is to find $(u_h, v_h) \in V_h$ such that for all test functions $w = \{w_0, w_b\} \in V_h$:

$$\begin{cases} \int \frac{\partial u(\mathbf{x},t)}{\partial t} w dx + D_{11} \int \nabla_w u_h . \nabla_w w dx + D_{12} \int \nabla_w v_h . \nabla_w w dx \\ \qquad = \int f(u_h, v_h) w dx, \\ \int \frac{\partial v(\mathbf{x},t)}{\partial t} w dx + D_{21} \int \nabla_w u_h . \nabla_w w dx + D_{22} \int \nabla_w v_h . \nabla_w w dx \\ \qquad = \int g(u_h, v_h) w dx, \\ u_h(\mathbf{x}, 0) = Q_h u_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \\ v_h(\mathbf{x}, 0) = Q_h v_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \end{cases}$$

where $Q_h u = \{Q_0 u, Q_b u\}$ is the L^2 projection operator onto $P^m(K^0) \times P^n(\partial K)$. In other words, $Q_0 u$ is the L^2 projection of u on $P^m(K^0)$ and $Q_b u$ is the L^2 projection of u in $P^n(\partial K)$ [3].

4 Weak Galerkin finite element discretization

We define the semi-discrete weak Galerkin finite element scheme for (1) as find $(u_h, v_h) = \{u_0(.,t), u_b(.,t), v_0(.,t), v_b(.,t)\} \in V_h(m,n)$ such that for all $w = \{w_0, w_b\} \in V_h$:

$$\begin{cases} (u_{h,t}, w) + a(u_h, w) = (f(u_h, v_h), w), \\ (v_{h,t}, w) + a(v_h, w) = (g(u_h, v_h), w), \\ u_h(\mathbf{x}, 0) = Q_h u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ v_h(\mathbf{x}, 0) = Q_h v_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{cases} \tag{2}$$

where a bilinear form $a(u, w)$ and $a(v, w)$ are defined as:

$$\begin{aligned} a(u,w) = {} & D_{11} \sum_{K \in \mathcal{K}_h} (\nabla_w u, \nabla_w w) + D_{12} \sum_{K \in \mathcal{K}_h} (\nabla_w v, \nabla_w w) \\ & + \sum_{K \in \mathcal{K}_h} h^{-1} \prec u_0 - u_b, w_0 - w_b \succ_{\partial K} \\ & + \sum_{K \in \mathcal{K}_h} h^{-1} \prec v_0 - v_b, w_0 - w_b \succ_{\partial K}, \end{aligned}$$

$$
\begin{aligned}
a(v,w) = {} & D_{21}\sum_{K\in\mathcal{K}_h}(\nabla_w v,\nabla_w w) + D_{22}\sum_{K\in\mathcal{K}_h}(\nabla_w u,\nabla_w w)\\
& + \sum_{K\in\mathcal{K}_h} h^{-1} \prec v_0 - v_b, w_0 - w_b \succ_{\partial K}\\
& + \sum_{K\in\mathcal{K}_h} h^{-1} \prec u_0 - u_b, w_0 - w_b \succ_{\partial K}.
\end{aligned}
$$

Two last terms are stabilization terms. Let τ denote the time step size, and $t_n = n\tau (n = 0, 1, \ldots)$, at time $t = t_n$ adopting the backward Euler difference quotient

$$
\bar{\partial}_t u_h^n = \frac{u_h^n - u_h^{n-1}}{\tau} \quad \text{and} \quad \bar{\partial}_t v_h^n = \frac{v_h^n - v_h^{n-1}}{\tau},
$$

to approximate $u_{h,t}$ and $v_{h,t}$ in semi-discrete scheme (2), then the fully-discrete scheme for (2) is to find $(u_h^n, v_h^n) = \{u_0^n, u_b^n, v_0^n, v_b^n\} \in V_h(m,n)$ such that for all $w = \{w_0, w_b\} \in V_h$:

$$
\begin{cases}
(\bar{\partial}_t u_h^n, w) + a(u_h, w) = (f(u_h^{n-1}, v_h^{n-1}), w),\\
(\bar{\partial}_t v_h^n, w) + a(v_h, w) = (g(u_h^{n-1}, v_h^{n-1}), w),\\
u_h^0(\mathbf{x}, 0) = Q_h u_0(\mathbf{x}), \quad \mathbf{x}\in\Omega,\\
v_h^0(\mathbf{x}, 0) = Q_h v_0(\mathbf{x}), \quad \mathbf{x}\in\Omega.
\end{cases} \tag{3}
$$

5 Numerical experiments

In this section, we examine the performance of WGFEM on two dimensional Brusselator reaction-diffusion systems. We construct triangular mesh $\mathcal{K}_h$ on square domain Ω and employ weak Galerkin finite element space $WG(P_0, P_0, RT_0)$ for each element K.

Example 1. As numerical validation of the spatial accuracy of our method we consider model (1) defined as [6]

$$
\begin{cases}
\dfrac{\partial u(\mathbf{x},t)}{\partial t} - \Delta(D_{11}u + D_{12}v) = D_{12}v, & (\mathbf{x},t)\in\Omega\times[0,\infty),\\
\dfrac{\partial v(\mathbf{x},t)}{\partial t} - \Delta(D_{21}u + D_{22}v) = 4D_{21}u, & (\mathbf{x},t)\in\Omega\times[0,\infty),\\
\dfrac{\partial u(\mathbf{x},t)}{\partial \mathbf{n}} = \dfrac{\partial v(\mathbf{x},t)}{\partial \mathbf{n}} = 0, & \mathbf{x}\in\partial\Omega,\\
u(\mathbf{x},0) = \cos(2x) + \cos(2y), & \mathbf{x}\in\Omega,\\
v(\mathbf{x},0) = \cos(x) + \cos(y), & \mathbf{x}\in\Omega,
\end{cases} \tag{4}
$$

Table 1: The errors for WGFEM applied to Brusselator model Example 1 at time $t = 0.5$.

Iteration	100	200	300	400	500
Error	$1.36e-3$	$6.87e-4$	$4.60e-4$	$3.45e-4$	$2.76e-4$

on the square domain $\Omega = (0, 2\pi)^2$, with parameters $D_{11} = D_{22} = 1, D_{12} = 1.5, D_{21} = 0.5$. The exact solutions are

$$\begin{aligned} u(x, y, t) &= \exp(-4t)(\cos(2x) + \cos(2y)), \\ v(x, y, t) &= \exp(-t)(\cos(x) + \cos(y)). \end{aligned}$$

We evolve system (4) until final time $t_f = 0.5$, and the contour plot of

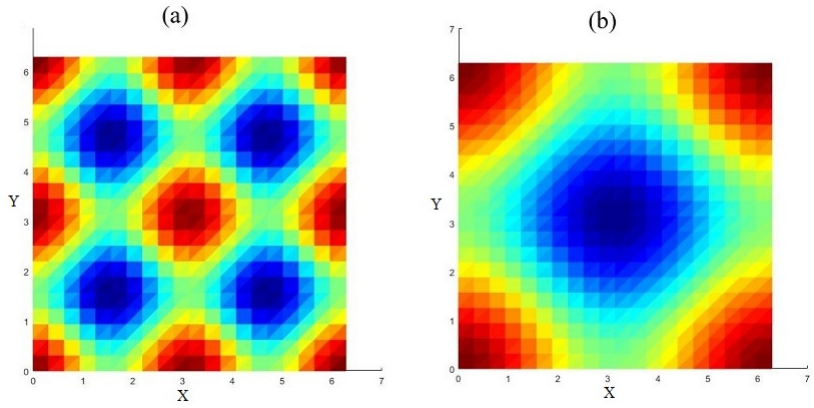

Figure 1: Contour plots of numerical concentration (a) $u(x, y, t)$, (b) $v(x, y, t)$ at final time $t_f = 0.5$.

approximate solutions are plotted in Fig. 1. Table 1 displays the errors obtained by this method.

Example 2. Consider the Brusselator system in the following form [9]:

$$\begin{cases} \dfrac{\partial u(\mathbf{x},t)}{\partial t} - \Delta(D_{11}u + D_{12}v) = u^2v - (B+1)u + A, & (\mathbf{x},t) \in \Omega \times [0,\infty), \\ \dfrac{\partial v(\mathbf{x},t)}{\partial t} - \Delta(D_{21}u + D_{22}v) = -u^2v + Bu, & (\mathbf{x},t) \in \Omega \times [0,\infty), \\ \dfrac{\partial u(\mathbf{x},t)}{\partial \mathbf{n}} = \dfrac{\partial v(\mathbf{x},t)}{\partial \mathbf{n}} = 0, & \mathbf{x} \in \partial\Omega, \\ u(\mathbf{x},0) = 2 + 0.25y, & \mathbf{x} \in \Omega, \\ v(\mathbf{x},0) = 1 + 0.8x, & \mathbf{x} \in \Omega, \end{cases}$$

Table 2: The errors for WGFEM applied to Brusselator model Example 2 at time $t = 3$.

Iteration	100	200	300	400	500
Error	$2.52e-5$	$1.23e-5$	$8.17e-6$	$6.10e-6$	$4.55e-6$

on the square domain $\Omega = (0,1)^2$, with parameters $D_{11} = D_{22} = 0.002, D_{12} = D_{21} = 0$ and $A = 2$, $B = 1$.

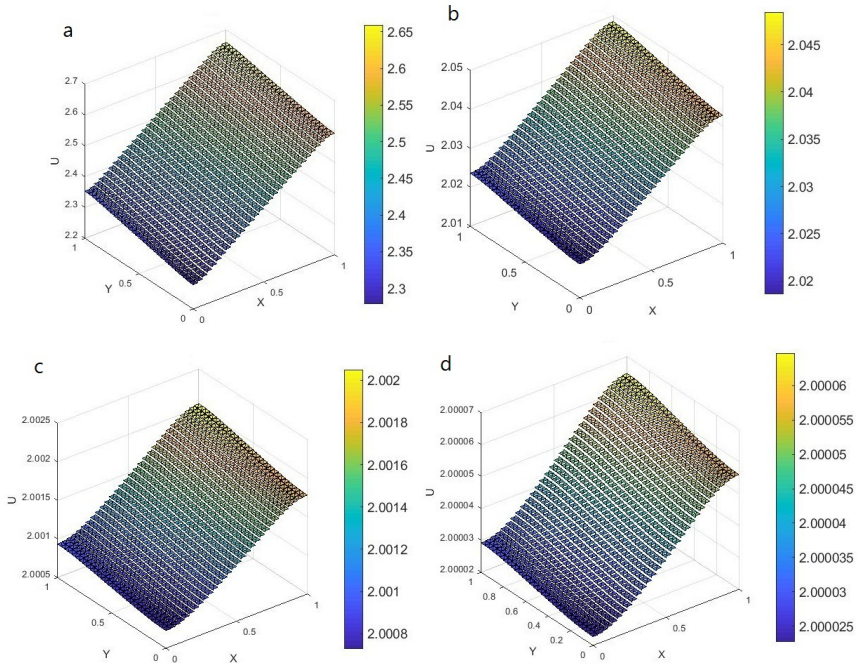

Figure 2: Plots of concentrations $u(x, y, t)$ at different times (a) $t = 1$, (b) $t = 3$, (c) $t = 5$, and (d) $t = 7$.

Fig. 2 and Fig. 3 depict concentration profiles of $u(x, y, t)$ and $v(x, y, t)$ at different times. Table 2 and Table 3 display the errors obtained by this method in two different times.

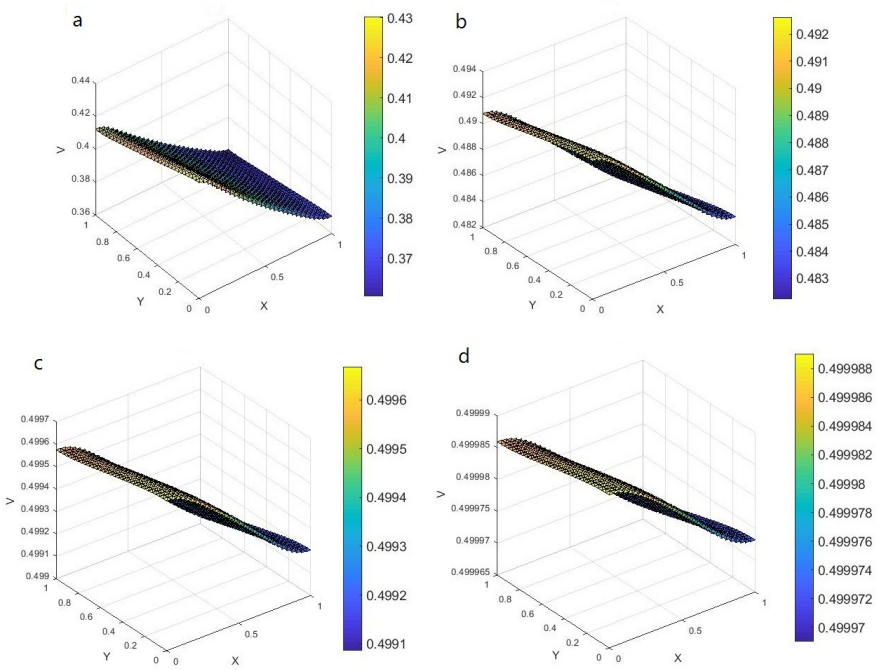

Figure 3: Plots of concentrations $v(x, y, t)$ at different times (a) $t = 1$, (b) $t = 3$, (c) $t = 5$, and (d) $t = 7$.

6 Conclusion

In this paper, the weak Galerkin finite element method (WGFEM) were used for the numerical solution of two dimensional Brusselator reaction-diffusion systems. The spatial dimension has been discretized using the weak Galerkin finite element method with selecting $WG(P_0, P_0, RT_0)$ space, that produced a semi-discrete weak Galerkin finite element scheme. A fully-discrete weak Galerkin finite element scheme is obtained by applying time difference approximation. Finally, the numerical results are shown the effect of cross-diffusion coefficients in formation of patterns and convergence to

Table 3: The errors for WGFEM applied to Brusselator model Example 2 at time $t = 5$.

Iteration	100	200	300	400	500
Error	$2.14e-6$	$1.00e-6$	$6.52e-7$	$4.83e-7$	$3.84e-7$

steady state point.

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