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On nilpotent interval matrices

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Abstract. In this paper, we give a necessary and sufficient condition for the powers of an interval matrix to be nilpotent. We show an interval matrix $\boldsymbol{A}$ is nilpotent if and only if $\rho(\mathscr{B}) = 0$, where $\mathscr{B}$ is a point matrix, introduced by Mayer (Linear Algebra Appl. 58 (1984) 201-216), constructed by the $(*)$ property. We observed that the spectral radius, determinant, and trace of a nilpotent interval matrix equal zero but in general its converse is not true. Some properties of nonnegative nilpotent interval matrices are derived. We also show that an irreducible interval matrix $\boldsymbol{A}$ is nilpotent if and only if $|\boldsymbol{A}|$ is nilpotent.

Keywords: Interval matrix, nilpotent matrix, spectral radius.
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1 Introduction

First, we give a short survey of the interval operations being used in the sequel. We restrict ourselves to real intervals and real interval matrices, pointing out that all the given lemmas and theorems remain true if we choose rectangles or circular disks as complex intervals and corresponding complex interval matrices. A real interval is a set of the form $\mathbf{a} = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} | \ \underline{a} \le a \le \overline{a}\}$, where $\underline{a}$ and $\overline{a}$ are the lower and upper bounds (end-points) of real interval number $\mathbf{a}$, respectively. The set of real interval numbers is denoted by $\mathbb{IR}$. We say that $\mathbf{a}$ is degenerate if $\underline{a} = \overline{a}$. The width and absolute value of an interval $\mathbf{a} = [\underline{a}, \overline{a}]$ are defined by $d(\mathbf{a}) := \overline{a} - \underline{a}$ and $|\mathbf{a}| := \max\{|\underline{a}|, |\overline{a}|\}$, respectively.

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Let $\circledast \in \{+,-,.,/\}$ be one of the usual binary operations on the set of real numbers $\mathbb{R}$, then we define the same operations on $\mathbb{IR}$. Let $\mathbf{a},\mathbf{b} \in \mathbb{IR}$, then we have:

$$\mathbf{a} \circledast \mathbf{b} := \{a \circledast b : a \in \mathbf{a},\ b \in \mathbf{b}\}, \tag{1}$$

assuming that $0 \notin \mathbf{b}$ in the case of division [2,8]. A real $m\times n$ interval matrix is a $m \times n$ matrix $\boldsymbol{A}$ whose entries $\boldsymbol{a}_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]$, $1 \le i \le m$, $q \le j \le n$, are intervals. A real $m \times n$ interval matrix $\boldsymbol{A}$ is generally identified with the set of matrices A with

$$\boldsymbol{A} := \{A \in \mathbb{R}^{m\times n} |\ \underline{A} \le A \le \overline{A}\},$$

where $\underline{A} = (\underline{a}_{ij})$ and $\overline{A} = (\overline{a}_{ij})$. The set of all real $m \times n$ interval matrices are denoted by $\mathbb{IR}^{m\times n}$. Let $\boldsymbol{A}$ and $\boldsymbol{B} \in \mathbb{IR}^{n\times n}$, then the interval matrix operators $\circledast \in \{+,\ -\ ,\ .\}$ are defined as follows:

$$\boldsymbol{A} \circledast \boldsymbol{B} := \{A \circledast B : A \in \boldsymbol{A},\ B \in \boldsymbol{B}\}.$$

Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, then the powers $\boldsymbol{A}^k$ of $\boldsymbol{A}$ are defined as follows [6,7]:

$$\boldsymbol{A}^0 := I; \qquad \boldsymbol{A}^k := (\boldsymbol{A}^{k-1}).\boldsymbol{A}, \quad k = 1, 2, \ldots.$$

As noted by Mayer, the product of interval matrices is not associative in general [6]. Therefore, $\boldsymbol{A}^k := \boldsymbol{A}^{k-1}.\boldsymbol{A}$ may not equal to ${}^k\boldsymbol{A} = {}^{k-1}\boldsymbol{A}.\boldsymbol{A}$ $({}^0\boldsymbol{A} := I)$. For example, let $\boldsymbol{A} = \begin{pmatrix} 1 & [0,1] \\ 1 & -1 \end{pmatrix}$ then $\boldsymbol{A}^3 \ne {}^3\boldsymbol{A}$.

For a $m \times n$ interval matrix $\boldsymbol{A} = (\mathbf{a}_{ij})$, we define the real matrices

$$d(\boldsymbol{A}) := (d(\mathbf{a}_{ij})), \quad |\boldsymbol{A}| := (|\mathbf{a}_{ij}|).$$

Here we state some important properties of the width and absolute value of interval matrices [6]. Let $\boldsymbol{A},\boldsymbol{B} \in \mathbb{IR}^{n\times n}$ and m and k be two positive integer numbers, then we have

$$d(\boldsymbol{A}) \ge O, \quad |\boldsymbol{A}| \ge O, \tag{2}$$

$$d(\boldsymbol{A}) = O \quad \Leftrightarrow \quad \boldsymbol{A} \textit{ is a point matrix}, \tag{3}$$

$$|\boldsymbol{A}| = O \quad \Leftrightarrow \quad \boldsymbol{A} = O, \tag{4}$$

$$d(\boldsymbol{A})|\boldsymbol{B}| \le d(\boldsymbol{A}\boldsymbol{B}) \le d(\boldsymbol{A})|\boldsymbol{B}| + |\boldsymbol{A}|d(\boldsymbol{B}), \tag{5}$$

$$|\boldsymbol{A}\boldsymbol{B}| \le |\boldsymbol{A}|\ |\boldsymbol{B}|, \tag{6}$$

$$d(\boldsymbol{A})|\boldsymbol{A}|^{k-1}|\boldsymbol{B}|^m \le d(\boldsymbol{A}^k\boldsymbol{B}^m), \tag{7}$$

$$d(\boldsymbol{A}^m)|\boldsymbol{A}|^k \le d(\boldsymbol{A}^{m+k}), \tag{8}$$

$$|\boldsymbol{A}^k| \le |\boldsymbol{A}|^k, \tag{9}$$

where (2)-(6) are found in [1] and (7)-(9) are consequences of (5) and (6).

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ belong to $\mathbb{IR}^{m\times n}$. We define the usual partial ordering $\leq$ by $\boldsymbol{A} \leq \boldsymbol{B}$ if and only if $\mathbf{a}_{ij} \leq \mathbf{b}_{ij}$, $i = 1, \dots, m$, $j = 1, \dots, n$. An interval matrix $\boldsymbol{A}$ is nonnegative if $\boldsymbol{A} \geq O$. We denote the set of all $n \times n$ nonnegative interval matrices by $\mathbb{IR}_{+}^{n\times n}$.

Definition 1. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{IR}^{n\times n}$. We say that $\boldsymbol{A}$ is permutationally similar to $\boldsymbol{B}$ if there exists a $n \times n$ permutation matrix P such that $\boldsymbol{B} = P^T \boldsymbol{A} P$.

Definition 2. [3] A matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is said to be reducible if $\boldsymbol{A}$ be permutationally similar to a block upper triangular. Otherwise, $\boldsymbol{A}$ is an irreducible matrix.

We note that $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is reducible if and only if $|\boldsymbol{A}|$ is reducible.

Definition 3. Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. The spectral radius of $\boldsymbol{A}$ is defined by

$$\rho(\boldsymbol{A}) := \max\{\rho(A) \, : A \in \boldsymbol{A}\}, \tag{10}$$

where $\rho(A) := \max\{|\lambda| \, : \, \lambda \in \mathbb{C}, \, Ax = \lambda x \text{ for some } x \in \mathbb{C}^n \backslash \{0\}\}$.

The interval matrix $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$ under the real function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is defined by

$$f(\boldsymbol{A}) := \{f(A) : A \in \boldsymbol{A}\}. \tag{11}$$

For instance, $\det(\boldsymbol{A})$ and $trace(\boldsymbol{A})$ give the range of determinant and trace of $\boldsymbol{A}$, respectively.

2 Main Results

The convergence of powers of an interval matrix was studied by Mayer [6,7]. Recently, the limit behavior of max product powers of a nonnegative interval matrix was presented in [4,5,13]. Based on the results of [7], we present some properties of a nilpotent interval matrix.

First, we recall some properties of a point matrix. A square matrix $A \in \mathbb{R}^{n\times n}$ is said to be nilpotent if $A^k = 0$ for some integer number k. The characteristic polynomial of a nilpotent matrix is x^n, so its determinant, trace and eigenvalues are always zero. Every strictly upper/lower triangular matrix is nilpotent. $A \in \mathbb{R}^{n\times n}$ is nilpotent if and only if A is similar to a triangular matrix with zero diagonal entries [9].

Definition 4. For a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ we define a directed graph (or simply a digraph) $D(A)$ with vertices $\{1, \dots, n\}$ such that $a_{ij} \neq 0$ if and only if there is an edge (i, j) from node i to node j. A path in the digraph $D(A)$ (of length k) is a sequence of edges $(i_1, i_2), \dots, (i_{k-1}, i_k)$ such that $a_{i_1 i_2} \dots a_{i_{k-1} i_k} \neq 0$. If $i_1 = i_k$, then the path is called a cycle of length k or k-cycle. An edge (i,i) or 1-cycle is called a loop. A digraph is called acyclic if it does not contain cycles of any length.

For nonnegative nilpotent matrices we have the next result.

Theorem 1. *[4, 9] Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. Then the following statements are equivalent:*
(1) A is nilpotent.
(2) $A^n = 0$.
(3) The directed graph associated with A contains no cycle.
(4) There exists a permutation matrix P such that $P^T AP$ is a strictly triangular matrix.
(5) $\rho(A) = 0$.

The directed graph corresponding to a nonnegative nilpotent matrix contains no cycles so an irreducible nonnegative matrix A is non-nilpotent. Now, we define a nilpotent interval matrix.

Definition 5. [9] We say that $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is a nilpotent matrix if and only if $\boldsymbol{A}^k = O$, for some positive integer k. The smallest positive integer that $\boldsymbol{A}^k = O$ is called the index of nilpotency.

As noted by Mayer [6], the right and left powers of an interval matrix may be different. We say that $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is a left nilpotent matrix if and only if ${}^k\boldsymbol{A} = O$, for some positive integer k. The following example shows ${}^k\boldsymbol{A} = O$ does not guarantee $\boldsymbol{A}^k = O$ and vice-versa.

Example 1. [6] Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & [0, \frac{1}{2}] & 0 \end{pmatrix},$$

then

$$\boldsymbol{A}^k = 2^{k-4} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ [-1,1] & [-1,1] & 0 \end{pmatrix}, \quad k \geq 3.$$

It shows that $\boldsymbol{A}$ is not nilpotent. However, ${}^3\boldsymbol{A} = O$.

Since $({}^k\boldsymbol{A})^T = (\boldsymbol{A}^T)^k$, then ${}^k\boldsymbol{A} = O$ if and only if $(\boldsymbol{A}^T)^k = O$.

Theorem 2. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be a nilpotent matrix. Then, all $A \in \boldsymbol{A}$ is nilpotent.*

Proof. Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be a nilpotent matrix whose index is k. Since for all $A \in \boldsymbol{A}$, $O = A^k \in \boldsymbol{A}^k$, then each $A \in \boldsymbol{A}$ is also nilpotent and so $\rho(\boldsymbol{A}) = trace(\boldsymbol{A}) = 0$. □

The converse of Theorem 2 is not generally true. We illustrate that the matrix $\boldsymbol{A}$ in Example 1 is not nilpotent, although $\boldsymbol{A}$ contains only real matrices A having a spectral radius $\rho(A) = 0$ and according to (10) and (11), $\rho(\boldsymbol{A}) = \det(\boldsymbol{A}) = trace(\boldsymbol{A}) = 0$.

Let $\boldsymbol{A}$ and $\boldsymbol{B} \in \mathbb{IR}^{n\times n}$ be two nilpotent matrices that commute, i.e. $\boldsymbol{A}.\boldsymbol{B} = \boldsymbol{B}.\boldsymbol{A}$, then $\boldsymbol{A}.\boldsymbol{B}$ and $\boldsymbol{A}+\boldsymbol{B}$ are nilpotent. Next, we present the conditions for interval nilpotent matrices. To do this we use some point matrices associated with the interval matrix.

Theorem 3. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. If $\rho(|\boldsymbol{A}|) = 0$, then $\boldsymbol{A}$ is nilpotent.*

Proof. By Theorem 1, $|\boldsymbol{A}|$ is a nonnegative nilpotent point matrix. Let $|\boldsymbol{A}|^k = 0$, for some integer number k, then $|\boldsymbol{A}^k| \le |\boldsymbol{A}|^k = 0$. This completes the proof. □

Note that the converse of Theorem 3 is not generally true. Let

$$\boldsymbol{A} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

$\boldsymbol{A}$ is a nilpotent matrix, while $2 = \rho(|\boldsymbol{A}|) \ne 0$ results in $|\boldsymbol{A}|$ is non-nilpotent.

Corollary 1. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be a zero diagonal triangular matrix. Then $\boldsymbol{A}$ is nilpotent.*

Proof. By assumption $|\boldsymbol{A}|$ is a zero diagonal triangular matrix and it is nilpotent. Using Theorem 1 and Theorem 3, $\boldsymbol{A}$ is a nilpotent matrix. □

Next, we present necessary and sufficient conditions for the sequence $\{\boldsymbol{A}^k\}$ of the powers of an interval matrix to converge to the null matrix. First, we recall some results.

Theorem 4 (see [10])**.** *If A, $B \in \mathbb{R}^{n\times n}$, then*

(a) $|A| \le B$ *if and only if* $\rho(A) \le \rho(B)$,

(b) $\rho(A) \le \rho(|A|)$,

(c) *If* $0 \le A \le B$, *then* $\rho(A) \le \rho(B)$.

Definition 6. [6] We say that the jth column of $\boldsymbol{A}$ has property $(*)$ if and only if there exists a power of $\boldsymbol{A}$ such as $\boldsymbol{A}^m$ containing at least one interval not degenerated to a point interval in the same jth column.

The directed graph $G(\boldsymbol{A})$ of the $n \times n$ interval matrix $\boldsymbol{A} = (\mathbf{a}_{ij})$ is the directed graph of the real matrix $|\boldsymbol{A}|$. $G(\boldsymbol{A})$ consists of n nodes $V_1, \dots, V_n$, where two equal or distinct nodes V_i and V_j, are connected if and only if $|\mathbf{a}_{ij}| \neq 0$. Note that $i = j$ is allowed. The jth column of the interval matrix $\boldsymbol{A}$ has property $(*)$ if and only if there exists a directed path in $G(\boldsymbol{A})$ which ends in V_j contains two neighboring nodes V_k and V_l such that $d(a_{kl}) > 0$. Since in the graph of an irreducible matrix, any two nodes V_i and V_j are connected by a directed path, then each column of an irreducible interval matrix $\boldsymbol{A}$ has property $(*)$ if $d(\boldsymbol{A}) \neq O$ [6].

Theorem 5. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be irreducible and $d(\boldsymbol{A}) \neq 0$. Then, $\boldsymbol{A}$ is not a nilpotent matrix.*

Proof. Since $\boldsymbol{A}$ is irreducible any two nodes V_i and V_j are connected by a directed path in the corresponding directed graph $G(\boldsymbol{A})$. Then, V_i and V_j are connected in $G(\boldsymbol{A}^k)$ for every integer number $k > 0$ and $\boldsymbol{A}^k \neq O$. Therefore, $\boldsymbol{A}$ is not nilpotent. □

Here, we present some results for nilpotent nonnegative interval matrices. Let $\boldsymbol{a} = [\underline{a}, \overline{a}]$ and $\boldsymbol{b} = [\underline{b}, \overline{b}]$ belong to $\mathbb{IR}_+$. According to (1), $\boldsymbol{a}.\boldsymbol{b} = [\underline{a}\ \underline{b}, \overline{a}\ \overline{b}]$ [8]. Therefore, we have the next result.

Theorem 6. *[4] Let $\boldsymbol{A} = [\underline{A}, \overline{A}] \in \mathbb{IR}_+^{n\times n}$. Then $\boldsymbol{A}^k = [\underline{A}^k, \overline{A}^k]$, for all $k \geq 1$.*

Proof. We prove by induction. It is trivial if $k = 1$. Let for nonnegative integer number m, $\boldsymbol{A}^m = [\underline{A}^m, \overline{A}^m]$. Assume that $\boldsymbol{A}^{m+1} = (\boldsymbol{a}_{ij}^{(m+1)})$, then

$$\boldsymbol{a}_{ij}^{(m+1)} = \sum_{l=1}^{n} \boldsymbol{a}_{il}^{(m)} \boldsymbol{a}_{lj} = \sum_{l=1}^{n} [\underline{\boldsymbol{a}}_{il}^{m}, \overline{\boldsymbol{a}}_{il}^{m}][\underline{\boldsymbol{a}}_{lj}, \overline{\boldsymbol{a}}_{lj}] = [\underline{\boldsymbol{a}}_{ij}^{m+1}, \overline{\boldsymbol{a}}_{ij}^{m+1}],$$

which completes the proof. □

Here, we present the necessary and sufficient conditions for the powers of a nonnegative interval matrix to be nilpotent.

Theorem 7. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be a nonnegative matrix. Then, the following statements are equivalent:*
(1) $\boldsymbol{A}$ is nilpotent,
(2) $\boldsymbol{A}^n = 0$,
(3) The directed graph associated with $\boldsymbol{A}$ contains no cycle.
(4) There exists a permutation matrix P such that $P^T\boldsymbol{A}P$ is a strictly triangular matrix.

(5) $|\boldsymbol{A}|$ *is nilpotent.*
(6) All $A \in \boldsymbol{A}$ *is nilpotent.*
(7) $\rho(\boldsymbol{A}) = 0$.

Proof. Since, $|\boldsymbol{A}| = \overline{A}$, using Theorems 1, 2 and 3, the cases (1), (5), (6), (7) are equivalent. Furthermore, (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) are trivial.

(1) $\Rightarrow$ (2) Let $\boldsymbol{A} = (\boldsymbol{a}_{ij}) = ([\underline{a}_{ij}, \overline{a}_{ij}])$ and k be a positive integer such that $\boldsymbol{A}^k = O$. Suppose, by contradiction, that $\boldsymbol{A}^n \neq O$, then there are $1 \leq j_1 \leq j_2 \ldots \leq j_{n+1} \leq n$ such that $\overline{a}_{j_1,j_2}.\overline{a}_{j_2,j_3} \ldots \overline{a}_{j_n,j_{n+1}} \neq 0$. Therefore, there exist $1 \leq t \leq r \leq n$ so that $j_t = j_r$ and then $\overline{a}_{j_t,j_{t+1}} \ldots \overline{a}_{j_{r-1},j_r} \neq 0$. Thus, $(\boldsymbol{A}^{r-t})_{j_r,j_r} \neq 0$ implies witch $\boldsymbol{A}^{(r-t)k_{j_r,j_r}} \neq O$ and contradicts to $\boldsymbol{A}^k = O$.

(1) $\Rightarrow$ (3) If $\boldsymbol{A}$ is nilpotent, then $\boldsymbol{A}^n = O$. This means that there are no paths of any kind of length n and then the directed graph has no cycle. If a graph contains no cycles, under an ordering of the vertices, the resulting adjacency matrix is triangular with 0s on the diagonal and it is therefore nilpotent. Furthermore, any other adjacency matrix of the same graph will be similar to this one and will therefore $\boldsymbol{A}$ also be nilpotent.

(1) $\Rightarrow$ (4) Let $\boldsymbol{A} = (\boldsymbol{a}_{ij})$ be a nilpotent matrix, then $\boldsymbol{A}^n = 0$. We prove, by induction on dimension n, $\boldsymbol{A}$ is permutationally similar to a strictly lower triangular interval matrix. For $n = 1$ it is trivial. Let the assertion is true for all $m < n$ where $n > 1$. We show that there exists $1 \leq i \leq n$ such that $\boldsymbol{a}_{ij} = 0$ for all $j = 1, \ldots, n$. Suppose to the contrary that there is the index j such that $\boldsymbol{a}_{ij} \neq 0$. Then, for $1 \leq j_1 \leq n$ there exists $1 \leq j_2 \leq n$ such that $\boldsymbol{a}_{j_1 j_2} > 0$. Corresponding to j_2, there exists $1 \leq j_3 \leq n$ such that $\boldsymbol{a}_{j_2 j_3} > 0$. By repeatedly applying the process, we obtain a sequence $j_1 \ldots j_{n+1}$ such that $\boldsymbol{a}_{j_k j_{k+1}} > 0$, for $k = 1, \ldots, n$. Therefore, there exists $1 \leq t \leq r \leq n$ such that $j_t = j_r$ and $(\boldsymbol{A}^{r-t})_{j_t j_t} > 0$. Then, $(\boldsymbol{A}^{(r-t)n})_{j_t j_t} > 0$ which is a contradiction. Now, let P_1 be a permutation matrix such that

$$P_1 \boldsymbol{A} P_1^T = \begin{pmatrix} 0 & 0 \\ v & \boldsymbol{A}_{n-1} \end{pmatrix}.$$

By assumption, for nonnegative nilpotent $(n-1) \times (n-1)$ matrix $\boldsymbol{A}_{n-1}$, there exists a permutation matrix P_2 such that $P_2 \boldsymbol{A}_{n-1} P_2^T$ is a strictly triangular matrix.

$$P \boldsymbol{A} P^T = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \boldsymbol{A}_{n-1} P_2^T \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & P_2 \end{pmatrix} P_1,$$

is a strictly lower triangular matrix [4]. □

We are now ready to present a necessary and sufficient condition for $\boldsymbol{A}$ to be nilpotent.

Definition 7. Let $\boldsymbol{A} = (\mathbf{a}_{ij}) \in \mathbb{IR}^{n\times n}$ and

$$M_j(\boldsymbol{A}) := \{\boldsymbol{A}^m : \text{the jth column of } \boldsymbol{A}^m \text{ contains at least one interval not degenerate to a point interval}\}.$$

We say that $\boldsymbol{A}$ is majorized by a point matrix $\mathscr{B} = (b_{ij})$ if

$$b_{ij} := \begin{cases} |\mathbf{a}_{ij}| & if \quad M_j(\boldsymbol{A}) \neq \emptyset, \\ a_{ij} & if \quad M_j(\boldsymbol{A}) = \emptyset. \end{cases}$$

For a nonempty $M_j(\boldsymbol{A})$, the jth column of $\boldsymbol{A}$ has property $(*)$.

From the above definition, we can conclude that

$$\boldsymbol{A} \leq \mathscr{B} \leq |\boldsymbol{A}|,$$

and $\boldsymbol{A} = \mathscr{B}$ if and only if $d(\boldsymbol{A}) = 0$. Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval matrix which is majorized by $\mathscr{B}$, then the inequality

$$\rho(\boldsymbol{A}) \leq \rho(\mathscr{B}),$$

holds (see [12]). We note that $\rho(\boldsymbol{A}) = \rho(\mathscr{B})$ does not hold in general (see Example 2 in [12]).

Lemma 1. *[7] Let $\boldsymbol{A} = (\boldsymbol{a}_{ij}) \in \mathbb{IR}^{n\times n}$ with $\boldsymbol{A}^k = (a_{ij}^{(k)})$ and $\mathscr{B}^k = (b_{ij}^{(k)})$. Therefore $a_{ij}^{(k)} = b_{ij}^{(k)}$, $i = 1, \ldots, n$, for all columns j which do not have property $(*)$.*

Lemma 2. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be nilpotent and $\boldsymbol{A}^k = O$, for some integer number k. If the jth column of $\boldsymbol{A}$ has property $(*)$, then the jth row of the matrix $|\boldsymbol{A}|^k$ equals to 0^T.*

Proof. According to the assumption there exist the integer numbers m and i so that $d(a_{ij}^{(m)}) > 0$. Then we have

$$\begin{aligned} 0 = d(a_{il}^{(m+k)}) &= (d(\boldsymbol{A}^{m+k}))_{il} \geq (d(\boldsymbol{A}^m)|\boldsymbol{A}|^k)_{il} \geq (d(\boldsymbol{A}^m))_{ij}(|\boldsymbol{A}|^k)_{jl} \\ &= d(a_{ij}^{(m)})(|\boldsymbol{A}|^k)_{jl} \geq 0, \quad l = 1, \ldots, n. \end{aligned}$$

Therefore, $(|\boldsymbol{A}|^k)_{jl} = 0$, for $l = 1, \ldots, n$. This completes the proof. □

According to Theorem 3 and Lemma 2, we have the next result.

Corollary 2. *Assume that $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ has at least one nondegenerate interval in each column. Then $\boldsymbol{A}$ is nilpotent if and only if $\rho(|\boldsymbol{A}|) = 0$.*

Lemma 3. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and there exist a nonempty index subset $I \subset \{1, 2, \ldots, n\}$ such that $|I| = k < n$ and*

$$j \in I \quad if \quad M_j(\boldsymbol{A}) = \varnothing, \qquad j \in I^c \quad if \quad M_j(\boldsymbol{A}) \neq \varnothing.$$

Then, there exists a permutation matrix P such that

$$P^{-1}\boldsymbol{A}\,P = \begin{pmatrix} A_1 & \boldsymbol{A}_2 \\ O & \boldsymbol{A}_3 \end{pmatrix},$$

where, A_1 is a $k \times k$ point matrix and each column located in the last $n-k$ columns has property $(*)$.

Proof. See Lemma 2 in [11]. □

Theorem 8. *Let $\boldsymbol{A} = (\boldsymbol{a}_{ij}) \in \mathbb{IR}^{n\times n}$ be an interval matrix which is majorized by $\mathscr{B}$. Then, $\boldsymbol{A}$ is nilpotent, if and only if $\rho(\mathscr{B}) = 0$.*

Proof. If all columns of $\boldsymbol{A}$ do not have property $(*)$, then $\boldsymbol{A} = \mathscr{B}$ which results $\boldsymbol{A}$ is nilpotent if and only if $\rho(\mathscr{B}) = 0$. If each column of $\boldsymbol{A}$ has property $(*)$, then $\mathscr{B} = |\boldsymbol{A}|$ and according to Corollary 2, $\boldsymbol{A}$ is nilpotent if and only if $\rho(\mathscr{B}) = 0$. Now, let there exists a subset $I \subset \{1, \ldots, n\}$ with $|I| = k < n$ and a permutation matrix P such that

$$P^{-1}\boldsymbol{A}\,P = \begin{pmatrix} A_1 & \boldsymbol{A}_2 \\ O & \boldsymbol{A}_3 \end{pmatrix}.$$

where, A_1 is a $k \times k$ point matrix and each column located in the last $n-k$ columns has property $(*)$. Then, $P^{-1}\,\boldsymbol{A}\,P$ is majored by $P^{-1}\,\mathscr{B}\,P$, where

$$P^{-1}\,\mathscr{B}\,P = \begin{pmatrix} \mathscr{B}_1 & \mathscr{B}_2 \\ O & \mathscr{B}_3 \end{pmatrix} = \begin{pmatrix} A_1 & |\boldsymbol{A}_2| \\ O & |\boldsymbol{A}_3| \end{pmatrix}.$$

Let $\boldsymbol{A}^k = O$, then $P^{-1}\boldsymbol{A}^kP = O$. Since, the last $n-k$ columns of $P^{-1}\boldsymbol{A}P$ have property $(*)$, by Lemma 2, the last $n-k$ rows of $|P^{-1}\boldsymbol{A}P|^k$ equals to 0^T. Therefore, $|\boldsymbol{A}_3|^k = O$ and since $A_1^k = O$, thus $\mathscr{B}$ is a nilpotent matrix. Now, let $\rho(\mathscr{B}) = 0$, thus there exists an integer number k so that $\mathscr{B}^k = 0$. Then, $A_1^k = \mathscr{B}_1^k = O$ and $\boldsymbol{A}_3^k \leq |\boldsymbol{A}_3|^k = \mathscr{B}_3{}^k = O$. This, complete the proof. □

3 Conclusion

We introduced the nilpotent interval matrices. A necessary and sufficient condition for the powers of an interval matrix to be nilpotent was presented. Some properties of nonnegative nilpotent interval matrices were established.

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