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Partial eigenvalue assignment for stabilization of descriptor fractional discrete-time linear systems

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Abstract. In this article, a method by partial eigenvalue assignment for stabilization of descriptor fractional discrete-time linear system is presented. This system can be converted to standard descriptor system by definition of fractional-order derivative and considering a new state vector. Using forward and propositional state feedback we do not need to have a full rank open-loop matrix in this kind of systems. However, only a part of the open-loop spectrum which are not in stability region need to be reassigned while keeping all the other eigenvalues invariant. Using partial eigenvalue assignment, size of matrices are decreased while the stability is preserved. Finally, two methods of partial eigenvalue assignment are compared.

Keywords: Descriptor fractional discrete-time linear system, descriptor systems, eigenvalue assignment (EVA), partial eigenvalue assignment (PEVA).

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1 Introduction

Fractional calculus is a branch of mathematical analysis that studies the possibility of differentiation and integration of arbitrary real or complex orders of the differential operator. The idea of fractional calculus probably being associated with Leibniz and Hospital where half-order derivative was

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mentioned. It allows us to describe and model a real object more accurately than the classical integer methods. It has played an important role in physics [17], electrical engineering (electrical circuits theory and fractances) [20], control systems [18], robotics [14], chemical mixing [15], bioengineering [13], and so on.

Descriptor fractional systems describe a more complete class of dynamical models than the fractional state-space systems, which are not only theoretical interest but also have great importance in practice, like using of Kirchhoff's laws for the electrical circuits [8, 9]. In this article, stabilization of the descriptor fractional discrete-time linear system by partial eigenvalue assignment is presented. The descriptor fractional discrete-time linear system is converted to the standard descriptor model with unlimited delay in state by the fractional derivative definition whose control is impossible. Having decreasing sequence of coefficients of delays and defining a new state vector may help us to obtain a standard descriptor discrete-time linear system, but with large matrices. We may find several methods for stability of just positive standard and descriptor systems. Some of them were derived by the use of Drazin inverse [3] and Shuffle algorithm [6] in which some initial conditions like having full row rank matrices in every performed algorithm and finding index of Shuffle and Drazin are necessary but we do not need them in our method.

We compare some methods via forward state matrix, forward and propositional state matrix, partial eigenvalue assignment by forward and propositional state feedback, and partial eigenvalue assignment using orthogonality relations to stabilize the standard descriptor discrete-time linear systems. To gain forward and propositional state feedback matrices, two standard linear systems must exist. Assigning nonzero arbitrary eigenvalue to the first standard system and inverse of desired eigenvalue of standard descriptor system to the second one, desired eigenvalues are assigned to standard descriptor linear system. Using the forward and propositional state feedback matrices, one may not need to have a full rank open-loop matrix in standard descriptor systems. However, just a small number of open-loop system eigenvalues are not in the stability region or other desirable region. Therefore, if we use the method reassigns only those small number of undesired eigenvalues, keeping the remaining large number of desired eigenvalues invariant, we may obtain a new system with smaller sizes of matrices, input, and state vectors. So it is obvious that calculating of eigenvalue assignment is more accurate and obtaining state feedbacks is also easier. Likewise, we do not deal with some sufficient conditions like not having eigenvalues near zero and being distinct eigenvalues using orthogonality relations [19] by using partial eigenvalue assignment with similarity transformation.

This paper is organized as follow. Next section presents the conversion of the descriptor fractional discrete-time linear system to the standard descriptor discrete-time linear system but with unlimited delays whose control is impossible. Making control and stabilization of this system with delay is given in Section 3. In Section 4, existence and uniqueness theorem and two methods for partial eigenvalue assignment are discussed. Illustrative examples are presented in Section 5. Convergence of the state and input vectors to zero with their figures are also shown. At the final section, concluding remarks are given.

The following notations will be used: $\mathbb{R}$ – the set of real numbers, $\mathbb{C}$ – the set of complex numbers, $\mathbb{R}^{n\times m}$ – the set of $n \times m$ real matrices, $\mathbb{R}^m = \mathbb{R}^{m\times 1}$, A^t – the transpose matrix of A, and A^H– the Hermitian conjugate transpose matrix of A.

2 Statement of the problem

Consider the descriptor fractional discrete-time linear system described by

$$E\Delta^{\alpha}x_{k+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}, \tag{1}$$

where $\alpha \in \mathbb{R}^+$ is the fractional-order difference of state vector, $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are state and input vectors, the matrices $E \in \mathbb{R}^{n\times n}, A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$ are known constant matrices with $rank(E) < n, rank(B) = m$, $1 \leq m \leq n$, and x_0 is also a nonzero definite vector.

Definition 1. *([16]) The Grunwald-Letnikov fractional derivative of order α is defined as*

$${}_a\Delta_t^{\alpha} f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{i=0}^{[\frac{t-a}{h}]} (-1)^i \binom{\alpha}{i} f(t - ih),$$

which $[x]$ shows the integer part of x.

Definition 2. *([2, 7]) The fractional difference of the order $\alpha \in \mathbb{R}^+$ with zero initial point in discrete-time linear systems is defined by*

$$\Delta^{\alpha}x_k = \sum_{i=0}^{k} (-1)^i \binom{\alpha}{i} x_{k-i},$$

where

$$\binom{\alpha}{i} = \begin{cases} 1, & for \quad i = 0, \\ \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!}, & for \quad i = 1, 2, \ldots. \end{cases}$$

Theorem 1. *([12]) For $n \in \mathbb{N}$ and $0 < \alpha < 1$ we have*

$$D^{n+\alpha}x(t) = D^n D^\alpha x(t).$$

By this theorem, we can easily assume $0 < \alpha < 1$.

Using Definition 2, system (1) becomes

$$E\{x_{k+1} + \sum_{i=1}^{k+1}(-1)^i \binom{\alpha}{i} x_{k-i+1}\} = Ax_k + Bu_k. \tag{2}$$

System (2) is further simplified to

$$Ex_{k+1} = A_\alpha x_k + \sum_{i=1}^{k} c_i E x_{k-i} + Bu_k, \tag{3}$$

where

$$c_i = c_i(\alpha) = (-1)^i \binom{\alpha}{i+1}, \quad i = 1, 2, \ldots, k, \tag{4}$$

and $A_\alpha = A + \alpha E$.

Note that Eq. (3) describes a descriptor discrete-time linear system with unlimited delay in state. To make the control of this system possible we should change it to standard descriptor linear system. Although the converted standard descriptor linear systems may have large matrices, but stability of them is proved [1].

3 Stability of descriptor fractional discrete-time linear systems

The coefficients c_i in (4) strongly decrease for increasing i when $0 < \alpha < 1$. Assuming $c_i = 0$ for $i > h$, the system (3) is converted to a descriptor linear system with h delays [2, 7]

$$Ex_{k+1} = A_\alpha x_k + \sum_{i=1}^{h} c_i E x_{k-i} + Bu_k. \tag{5}$$

Now by defining the new state vector $X_k \in \mathbb{R}^{\bar{n}}$

$$X_k = \begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ \vdots \\ x_{k-h} \end{bmatrix},$$

which $\bar{n} = n(h+1)$ we may convert the time delay descriptor system (5) to a standard descriptor system

$$\bar{E}X_{k+1} = \bar{A}X_k + \bar{B}U_k, \tag{6}$$

where

$$\bar{A} = \begin{bmatrix} A_\alpha & c_1E & \cdots & c_{h-1}E & c_hE \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\bar{E} = \begin{bmatrix} E & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I \end{bmatrix}, \tag{7}$$

$U_k = u_k \in \mathbb{R}^m$ is the input vector, $\bar{E}, \bar{A} \in \mathbb{R}^{\bar{n}\times\bar{n}}, \bar{B} \in \mathbb{R}^{\bar{n}\times m}$, and $\bar{E}$ is singular because of singularity of E in system (1).

Definition 3. *([2, 7]) The descriptor fractional system (1) is called practically stable if and only if the time delay system (5) or equivalently the system (6) is asymptotically stable.*

3.1 Eigenvalue assignment with forward state feedback law

Consider system (6) by forward state feedback law

$$U_k = F'_f X_{k+1}. \tag{8}$$

The aim is to design the forward state feedback F'_f in (8) which produces a closed-loop system of (6) with the satisfactory response by assigning desirable eigenvalues $L = \{\lambda_1, \lambda_2, \ldots, \lambda_{\bar{n}}\}$, where $\lambda_i \in \mathbb{C}, \lambda_i \neq 0$, and are self-conjugate complex numbers for $i = 1, 2, \ldots, \bar{n}$.

Consider the following assumptions

$$I)\ \ rank[\bar{E}|\bar{B}] = \bar{n}, \quad II)\ \ rank[\bar{A}] = \bar{n}, \quad III)\ \ rank[\bar{B}] = m. \tag{9}$$

If assumption (I) holds, then there exists F'_f such that [1]

$$rank[\bar{E} - \bar{B}F'_f] = \bar{n}. \tag{10}$$

By substituting feedback (8) into the equation (6), one has

$$\bar{E}X_{k+1} = \bar{A}X_k + \bar{B}F'_f X_{k+1} \Rightarrow (\bar{E} - \bar{B}F'_f)X_{k+1} = \bar{A}X_k,$$

therefore

$$X_{k+1} = (\bar{E} - \bar{B}F'_f)^{-1}\bar{A}X_k, \tag{11}$$

is the standard linear system which is well-defined by (10).

Theorem 2. *([5]) The standard descriptor discrete-time linear system (11) is asymptotically stable if and only if eigenvalues of $(\bar{E} - \bar{B}F'_f)^{-1}\bar{A}$ lie in the unit disk.*

Theorem 3. *Define the matrices N', M' as*

$$N' = \bar{A}^{-1}\bar{E}, \quad M' = -\bar{A}^{-1}\bar{B}, \tag{12}$$

such that the pair of (M', N') be controllable. Let F'_f be state feedback matrix such that $\{\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_{\bar{n}}^{-1}\}$ is the set of eigenvalues of the closed-loop system

$$\begin{cases} z_{k+1} = N'z_k + M'w_k, \\ \quad w_k = F'_f z_k, \end{cases} \tag{13}$$

where arbitrarily assigned and $\lambda_i \in \mathbb{C}, \lambda_i \neq 0$, and are self-conjugate complex numbers for $i = 1, 2, \ldots, \bar{n}$. Then for this gained F'_f, the desired spectrum $L = \{\lambda_1, \lambda_2, \ldots, \lambda_{\bar{n}}\}$ is the set of eigenvalues of the controlled system (6) with forward feedback law (8) and the condition (10) also holds.

Proof. Considering that (M', N') is controlled, then one can find a state feedback matrix F'_f such that the controlled system (13) given by

$$z_{k+1} = (N' + M'F'_f)z_k, \tag{14}$$

has the eigenvalues $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_{\bar{n}}^{-1}$. Now by (12) note that

$$N' + M'F'_f = \bar{A}^{-1}(\bar{E} - \bar{B}F'_f), \tag{15}$$

so

$$(N' + M'F'_f)^{-1} = (\bar{E} - \bar{B}F'_f)^{-1}\bar{A}. \tag{16}$$

The closed-loop matrices of systems (13) and (6) via feedback law (8) are inverse of each other by (11), (14), (15), and (16). Therefore, (10) holds and the set of eigenvalues of the closed-loop system (6) with feedback law (8) is equal to $L = \{\lambda_1, \lambda_2, \ldots, \lambda_{\bar{n}}\}$. $\square$

Remark 1. *From Eq. (7), the matrices $\bar{E}$ and $\bar{A}$ in system (6) are singular because $rank(E) < n$ is the necessary condition in the fractional descriptor discrete-time linear system (1) and the matrix including last n columns and first n rows of $\bar{A}$, i.e. $[c_h E]$, is not full rank.*

The method based on using forward state feedback when $\bar{A}$ is singular, i.e., the condition (II) in (9) is not satisfied, does not work. This limitative condition is removed in Subsection 3.2.

3.2 Eigenvalue assignment with forward and propositional state feedback law

When we use the forward and propositional state feedback instead of the forward state feedback, we do not need the full rankness of matrix $\bar{A}$ in system (6).

Consider system (6) by forward and propositional state feedback law

$$U_k = F_f X_{k+1} + F_p X_k. \tag{17}$$

The aim is to design the forward and propositional state feedbacks F_f and F_p in (17) which produces a closed-loop system of (6) with the satisfactory response by assigning desirable eigenvalues $L = \{\lambda_1, \lambda_2, \ldots, \lambda_{\bar{n}}\}$ where $\lambda_i \in \mathbb{C}, \lambda_i \neq 0$, and are self-conjugate complex numbers for $i = 1, 2, \ldots, \bar{n}$.
To establish the proposed results, consider the following assumptions

$$I)\ \ rank[\bar{E}|\bar{B}] = \bar{n}, \quad II)\ \ rank[\bar{B}] = m.$$

If assumption (I) holds, then there exists F_f such that [1]

$$rank[\bar{E} - \bar{B}F_f] = \bar{n}. \tag{18}$$

Substituting feedback (17) into Eq. (6), one can write

$$\bar{E}X_{k+1} = \bar{A}X_k + \bar{B}F_f X_{k+1} + \bar{B}F_p X_k \Rightarrow (\bar{E} - \bar{B}F_f)X_{k+1} = (\bar{A} + \bar{B}F_p)X_k,$$

so

$$X_{k+1} = (\bar{E} - \bar{B}F_f)^{-1}(\bar{A} + \bar{B}F_p)X_k \tag{19}$$

is the standard linear system which is well-defined by (18).

Theorem 4. *([5]) The standard descriptor discrete-time linear system (19) is asymptotically stable if and only if eigenvalues of $(\bar{E} - \bar{B}F_f)^{-1}(\bar{A} + \bar{B}F_p)$ lie in the unit disk.*

First, the propositional feedback matrix F_p is obtained by assigning non-zero arbitrary eigenvalues to the closed-loop matrix of the system

$$\begin{cases} q_{k+1} = \bar{A}q_k + \bar{B}v_k, \\ v_k = F_p q_k. \end{cases} \tag{20}$$

Then, we obtain the forward state feedback matrix F_f by assigning $\{\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_{\bar{n}}^{-1}\}$to the closed-loop matrix of the system (22), where $\lambda_i \in \mathbb{C}, \lambda_i \neq 0$, are self-conjugate complex numbers for $i = 1, 2, \dots, \bar{n}$, and $L = \{\lambda_1, \lambda_2, \dots, \lambda_{\bar{n}}\}$ is the set of desired eigenvalues for the standard descriptor system (6) via state feedback (17).

Theorem 5. *Define the matrices N and M as*

$$N = (\bar{A} + \bar{B}F_p)^{-1}\bar{E}, \quad M = -(\bar{A} + \bar{B}F_p)^{-1}\bar{B} \tag{21}$$

such that the pair of (M, N) be controllable. Also let F_f be state feedback matrix such that $\{\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_{\bar{n}}^{-1}\}$ is the set of eigenvalues of the closed-loop system

$$\begin{cases} z_{k+1} = Nz_k + Mw_k, \\ w_k = F_f z_k, \end{cases} \tag{22}$$

where arbitrarily assigned and $\lambda_i \in \mathbb{C}$ are nonzero self-conjugate complex numbers for $i = 1, 2, \dots, \bar{n}$. Then for this gained F_f, the desired spectrum $L = \{\lambda_1, \lambda_2, \dots, \lambda_{\bar{n}}\}$ is the set of eigenvalues of the controlled system (6) with forward and propositional state feedback law (17) and the condition (18) also holds.

Proof. Considering that (M, N) is controlled, then one can find a state feedback matrix F_f such that the controlled system (22) given by

$$z_{k+1} = (N + MF_f)z_k \tag{23}$$

has eigenvalues equal to $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_{\bar{n}}^{-1}$. Now by (21) note that

$$N + MF_f = (\bar{A} + \bar{B}F_p)^{-1}(\bar{E} - \bar{B}F_f), \tag{24}$$

so

$$(N + MF_f)^{-1} = (\bar{E} - \bar{B}F_f)^{-1}(\bar{A} + \bar{B}F_p). \tag{25}$$

The closed-loop matrices of systems (23) and (6) via feedback law (17) are inverse of each other by (19), (23), (24), and (25). Therefore (18) holds and the set of eigenvalues of closed-loop system (6) with feedback law (17) is equal to $L = \{\lambda_1, \lambda_2, \dots, \lambda_{\bar{n}}\}$. $\square$

3.3 Eigenvalue assignment by similarity transformation in standard systems

In this subsection, we use the method based on similarity transformation to compute the forward state feedback matrix F'_f and also forward and propositional state feedback matrices F_p, F_f in standard systems (13), (20), and (22) in Subsections 3.1 and 3.2. Our assignment procedure is composed of two stages. First, we obtain a primary state feedback matrix Φ which assigns all the eigenvalues of closed-loop system to zero. Then, we produce a state feedback matrix F which assigns all the closed-loop system eigenvalues in desired region.

Consider controllable standard system

$$\begin{cases} x_{k+1} = A_1 x_k + B_1 u_k, \\ u_k = F x_k, \end{cases} \tag{26}$$

and the state transformation

$$x_k = T\tilde{x}_k, \tag{27}$$

where T can be obtained by elementary similarity operations as described in [10, 11]. Substituting (27) into (26) yields

$$\tilde{x}_{k+1} = T^{-1}A_1T\tilde{x}_k + T^{-1}B_1u_k.$$

It is noted that the transformation matrix T is invertible. In this way, $\tilde{A}_1 = T^{-1}A_1T$ and $\tilde{B}_1 = T^{-1}B_1$ are in a compact canonical form known as vector companion form

$$\tilde{A}_1 = \begin{bmatrix} & G_0 & \\ I_{n-m} & , & 0_{n-m,m} \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} S_0 \\ 0_{n-m,m} \end{bmatrix}. \tag{28}$$

Here G_0 is a $m \times n$ matrix and S_0 is a $m \times m$ upper triangular matrix. Note that the Kronecker invariants of the pair (B_1, A_1) are regular if the difference between any of them is not greater than one. If Kronecker invariants of the pair of (B_1, A_1) are regular, then $\tilde{A}_1$ and $\tilde{B}_1$ are always in the above form [10]. In the case of irregular Kronecker invariants, some rows of I_{n-m} in $\tilde{A}_1$ are displaced [11].

The state feedback matrix which assigns all the eigenvalues to zero, for the transformed pair $(\tilde{B}_1, \tilde{A}_1)$ is then chosen as $\tilde{\Phi} = -S_0^{-1}G_0$,which results in the primary state feedback matrix for the pair (B_1, A_1) defined as $\Phi = \tilde{\Phi}T^{-1}$. The transformed closed-loop matrix

$$\tilde{\Gamma}_0 = \tilde{A}_1 + \tilde{B}_1\tilde{\Phi} = \begin{bmatrix} & 0_{m,n} & \\ I_{n-m} & , & 0_{n-m,m} \end{bmatrix}, \tag{29}$$

assumes a compact Jordan form with zero eigenvalues.

Theorem 6. *Let D be a block diagonal matrix in the form*

$$D = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_k \end{bmatrix},$$

where each D_j, $j = 1, 2, \cdots, k$ is either of the form

$$D_j = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}$$

(to designate the complex conjugate eigenvalues $\alpha_j + i\beta_j$) or in the case of real eigenvalues $D_j = \begin{bmatrix} d_j \end{bmatrix}$. If the diagonal matrix D with self-conjugate eigenvalue spectrum is added to the transformed closed-loop matrix, $\tilde{\Gamma}_0$, then the eigenvalues of the resulting matrix are the eigenvalues in the spectrum.

Proof. The primary compact Jordan form in the case of regular Kronecker invariants is in the form (29). The sum of $\tilde{\Gamma}_0$ with D has the form

$$\begin{aligned} \tilde{H} &= \tilde{\Gamma}_0 + D = \begin{bmatrix} & 0_{m,n} & \\ I_{n-m} & , & 0_{n-m,m} \end{bmatrix} + \begin{bmatrix} D_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_k \end{bmatrix} \\ &= \begin{bmatrix} D_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & D_l & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I_1 & 0 & \cdots & 0 & D_{l+1} & \cdots & 0 \\ \vdots & \ddots & \cdots & 0 & 0 & \ddots & \vdots \\ 0 & \cdots & I_r & 0 & 0 & \cdots & D_k \end{bmatrix}, \end{aligned} \tag{30}$$

where divided parts of matrix $\tilde{H}$ in (30), i.e.

$$\begin{bmatrix} D_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & D_l & 0 & \cdots & 0 \end{bmatrix}, \quad \begin{bmatrix} I_1 & 0 & \cdots & 0 & D_{l+1} & \cdots & 0 \\ \vdots & \ddots & \cdots & 0 & 0 & \ddots & \vdots \\ 0 & \cdots & I_r & 0 & 0 & \cdots & D_k \end{bmatrix},$$

include m and $n-m$ lines, respectively. And also I_s, $s = 1, 2, \cdots, r$ is the unit matrix of size 2 in case $n-m$ is even. In case $n-m$ is odd, only one I_s takes the form of unit matrix of size one.

By expanding $\det(\tilde{H} - \lambda I)$ along the first row, it is obvious that the eigenvalues of $\tilde{H}$ are the same as the eigenvalues of D. For the case of irregular Kronecker invariants [11], only some of the unit columns of I_{n-m} are displaced, since the unit elements are always below the main diagonal, the proof applies in the same manner. $\square$

Therefore the closed-loop system matrix (29) becomes (30). Simple elementary similarity operations can be used to obtain the matrix $\tilde{V}$ from $\tilde{H}$ such that

$$\tilde{V} = \begin{bmatrix} & G_\lambda & \\ I_{n-m} & , & 0_{n-m,m} \end{bmatrix}.$$

Thus $\tilde{F} = \tilde{\tilde{\Phi}} + S_0^{-1} G_\lambda = S_0^{-1}(-G_0 + G_\lambda)$, is the feedback matrix which assigns the eigenvalue spectrum to the closed-loop matrix $\tilde{\Gamma} = \tilde{A}_1 + \tilde{B}_1 \tilde{F}$, and F may then be obtained by $F = \tilde{F} T^{-1}$.

4 Partial eigenvalue assignment

In this section, we propose the existence and uniqueness theorem and an algorithm to find the state feedback matrices in standard systems. The aim of partial eigenvalue assignment is reassigning undesired eigenvalues of open-loop spectrums in new system with smaller sizes of matrices such that other eigenvalues unchanged. Therefore, the stability in partial eigenvalue assignment for the standard descriptor system is kept by reassigning eigenvalues in the unit disk and unchanging the remaining eigenvalues in the standard system (22). Also we present some sufficient conditions in another algorithm using orthogonality relations which are not necessary for partial eigenvalue assignment algorithm based on similarity transformation.

4.1 Existence and uniqueness

Theorem 7. *([4]) (Eigenvector criterion of controllability). The standard system (26) or, equivalently, the matrix pair (B_1, A_1) is controllable with respect to the eigenvalue λ of A_1 if $y^H B_1 \neq 0$ for all $y \neq 0$ such that $y^H A_1 = \lambda y^H$.*

Definition 4. The standard system (26) or the matrix pair (B_1, A_1) is partially controllable with respect to the subset $\{\lambda_1, \ldots, \lambda_p\}$ of the spectrum of A_1 if it is controllable with respect to each of the eigenvalues $\lambda_j, j = 1, \ldots, p$.

Definition 5. *The standard system (26) or the matrix pair* (B_1, A_1) *is completely controllable if it is controllable with respect to every eigenvalue of* A_1.

Theorem 8. *([4]) (Existence and uniqueness for eigenvalue assignment problem). The eigenvalue assignment problem for the pair* (B_1, A_1) *is solvable for any arbitrary set* $S = \{\mu_1, \ldots, \mu_p\}$ *if and only if* (B_1, A_1) *is completely controllable. The solution is unique if and only if the system is a single-input system (that is, if* B_1 *is a vector). In the multi-input case, there are infinitely many solutions, whenever a solution exists.*

Theorem 9. *([4]) (Existence and uniqueness for partial eigenvalue assignment problem). Let* $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_p; \lambda_{p+1}, \cdots, \lambda_n\}$ *be the diagonal matrix containing the eigenvalues* $\lambda_1, \ldots, \lambda_n$ *of* $A_1 \in \mathbb{C}^{n\times n}$. *Assume that the sets* $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ *and* $\{\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_n\}$ *are disjoint. Let the eigenvalues* $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ *to be changed to* $\{\mu_1, \mu_2, \ldots, \mu_p\}$ *and the remaining eigenvalues stay invariant. Then the partial eigenvalue assignment problem for the pair* (B_1, A_1) *is solvable for any choice of the closed-loop eigenvalues* $\{\mu_1, \mu_2, \cdots, \mu_p\}$ *if and only if the pair* (B_1, A_1) *is partially controllable with respect to the set* $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. *The solution is unique if and only if the system is a completely controllable single-input system. In the multi-input case, and in the single-input case when the system is not completely controllable, there are infinitely many solutions, whenever a solution exists.*

4.2 Partial eigenvalue assignment algorithm using orthogonality relations

There exists an algorithm for partial eigenvalue assignment using orthogonality relations as follows [19].

Inputs:
(I) $\{M_k, M_{k-1}, \cdots, M_0\}$ are $n \times n$ real non-symmetric constant matrices.
(II) b is an n-vector and $D = diag(\mu_1, \ldots, \mu_p)$ closed under complex conjugation.
Output:
The feedback vectors $\{f_i\}_{i=1}^{k}$ such that the spectrum of modified matrix polynomial

$$P(\lambda) = M_k\lambda^k + (M_{k-1} - bf_1^T)\lambda^{k-1} + (M_0 - bf_k^T),$$

is $\{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \ldots, \lambda_{kn}\}$, where $\{\lambda_{p+1}, \ldots, \lambda_{kn}\}$ are the last $kn - p$ eigenvalues of matrix polynomial $P(\lambda) = \lambda^k M_k + \lambda^{k-1}M_{k-1} + \cdots + \lambda M_1 +$

M_0.

Assumption:

(I) M_k is a nonsingular matrix.

(II) The sets $\{\mu_1, \dots, \mu_p\}$ and $\{\lambda_1, \dots, \lambda_p\}$ are distinct and closed under complex conjugation, where $\{\lambda_1, \dots, \lambda_{kn}\}$ are the eigenvalues of matrix polynomial $P(\lambda) = \lambda^k M_k + \lambda^{k-1} M_{k-1} + \cdots + \lambda M_1 + M_0$.

(III) $\Lambda_1 = diag(\lambda_1, \dots, \lambda_p)$.

Step 1. Obtain the first p eigenvalues $\{\lambda_1, \dots, \lambda_p\}$ of matrix polynomial $P(\lambda) = \lambda^k M_k + \lambda^{k-1} M_{k-1} + \cdots + \lambda M_1 + M_0$ that need to be reassigned and the corresponding left eigenvectors $Y_1 = (y_1, y_2, \dots, y_p)$.

Step 2. Compute the explicit expression for β

$$\beta_j = \frac{1}{b^T \bar{y}_j} \frac{\mu_j - \lambda_j}{\lambda_j} \prod_{i=1, i\neq j}^{p} \frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j}, \quad j = 1, \dots, p.$$

Step 3. Form

$$f_i = \sum_{j=1}^{i} [M_{k-i+j}^T \bar{Y}_1 \Lambda_1^j] \beta^T, \ f_k = -M_0^T \bar{Y}_1 \beta^T, \ i = 1, \dots, k-1, \ \beta^T \in \mathbb{C}^p.$$

By Step 2, it is clear that sufficient conditions for the existence of β, and consequently for a solution to the partial pole assignment problem are:

(1) No $\lambda_j, j = 1, \dots, p$ vanishes,

(2) The $\{\lambda_i\}_{i=1}^p$ are distinct,

(3) The vector b must be not orthogonal to $\bar{y}_j, j = 1, \dots, p$.

By the method of Subsection 4.3, we do not deal with the sufficient conditions (1) until (3).

4.3 Partial eigenvalue assignment algorithm using similarity transformation

The following algorithm presents a partial eigenvalue assignment method on the standard system (26).

Inputs:

(a) The $n \times n$ matrix A_1.

(b) The $n \times m$ control matrix B_1.

(c) The set $\{\mu_1, \mu_2, \dots, \mu_p\}$, closed under complex conjugation.

(d) The self-conjugate subset $\{\lambda_1, \dots, \lambda_p\}$ of the spectrum $\{\lambda_1, \dots, \lambda_n\}$ of the matrix A_1 and the associated right eigenvector set $\{y_1, \dots, y_p\}$.

Outputs:

The real feedback matrix F such that the spectrum of the closed-loop matrix $A_1 + B_1 F$ is $\{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \ldots, \lambda_n\}$.
Assumptions:
(a) The matrix pair (B_1, A_1) is partially controllable with respect to the eigenvalues $\{\lambda_1, \ldots, \lambda_p\}$.
(b) The sets $\{\lambda_1, \ldots, \lambda_p\}, \{\lambda_{p+1}, \ldots, \lambda_n\}$, and $\{\mu_1, \ldots, \mu_p\}$ are disjoint.
Step 1. Form $\Lambda_1 = diag(\lambda_1, \ldots, \lambda_p), \quad Y_1 = (y_1, \ldots, y_p)$. **Step 2.** Find feedback K such that $eig(\Lambda_1 + Y_1^H B_1 K) = \{\mu_1, \ldots, \mu_p\}$ by the method in Subsection 3.3.

Step 3. Form $F = KY_1^H$. Now we have $eig(A_1 + B_1 F) = \{\mu_1, \cdots, \mu_p; \lambda_{p+1}, \cdots, \lambda_n\}$.

In the next section, examples are presented in order to compare the numerical results obtained by our methods.

5 Numerical Results

Consider the following examples.

Example 1. Consider the system (1) with $\alpha = 0.5, h = 2$, and matrices A, B, and E as

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -1 & -2 & -4 \\ 3 & 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & -3 \\ 0 & -2 & 6 \end{bmatrix},$$

where $rank(E) = 2 < 3$. The matrices $\bar{A}, \bar{B}$, and $\bar{E}$ are obtained as

$$\bar{A} = \begin{bmatrix} 2.5 & 3 & 0.5 & 0.12 & 0 & 0.37 & -0.06 & 0 & -0.18 \\ -0.5 & -1 & -5.5 & 0.12 & 0.25 & -0.37 & -0.06 & -0.12 & 0.18 \\ 3 & 0 & -2 & 0 & -0.25 & 0.75 & 0 & 0.12 & -0.37 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalue assignment via forward state feedback is not applicable. The pair (M, N) cannot be defined because of singularity of the matrix $\bar{A}$, $rank(\bar{A}) = 8 < 9$.

Now consider the standard systems (20) and (22) by propositional and forward state feedbacks F_p and F_f, respectively. Only obtaining the forward feedback matrix F_f is displayed by the propositional state feedback matrix F_p as

$$F_p = \begin{bmatrix} 0.13 & 0.23 & -0.42 & 0.82 & 0.31 & 1.17 & -0.09 & 0.01 & -0.29 \\ 0 & -0.03 & 0.07 & -0.12 & -0.04 & -0.18 & 0.01 & 0 & 0.04 \end{bmatrix},$$

which all eigenvalues are assigned to 0.1.

The pair of (M, N) is

$$N = 10^8 \times \begin{bmatrix} 0 & 0 & 0 & 10^{-8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^{-8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10^{-8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10^{-8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-8} \\ 0.09 & 0.78 & -2.05 & 0.01 & -0.03 & 0.12 & 0 & 0 & -0.01 \\ -0.1 & -0.87 & 2.31 & -0.01 & 0.03 & -0.14 & 0 & 0 & 0.01 \\ -0.03 & -0.29 & 0.77 & 0 & 0.01 & -0.04 & 0 & 0 & 0 \end{bmatrix},$$

$$M = 10^7 \times \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.98 & 6.79 \\ -1.1 & -7.64 \\ -0.36 & -2.54 \end{bmatrix}.$$

Case (a). Consider the method in Subsection 4.2. Because $eig(N) = \{64.61 \pm 85.43i, -82.3, 15.8 \pm 11.42i, 12.94, -2.19, 0.72, 0\}$, we may reassign $p = 2$ eigenvalues like $\{10, 10\}$ instead of $\{0, 0.72\}$ while leaving the other eigenvalues unchanged. The first sufficient conditions, i.e., $\{\lambda_1, \lambda_2\}$ should not vanish, is not satisfied for the solution of the partial eigenvalue assignment using orthogonality relations. Therefore, this method cannot be used in this example.

Case (b). Consider the method in Subsection 4.3. Similar to case (a), because $eig(N) = \{64.61 \pm 85.43i, -82.3, 15.8 \pm 11.42i, 12.94, -2.19, 0.72, 0\}$, we reassign $\{10, 10\}$ instead of $\{0, 0.72\}$ while leaving the other eigenvalues unchanged. New pair $(Y_1^H M, \Lambda_1)$ and forward state feedback F_f are

$$Y_1^H M = \begin{bmatrix} 0 & 0.22 \\ -0.04 & -0.13 \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} 0.72 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F_f = \begin{bmatrix} 28.8 & -79.9 & 308.5 & -10.6 & 31.7 & -123.4 & 0.2 & -4.1 & 13.2 \\ 4.7 & 14.6 & -30.1 & 1.4 & -4.7 & 18.2 & 0 & 0.8 & -2.7 \end{bmatrix}.$$

The eigenvalues of the closed-loop matrix of the standard system (22) and the standard descriptor system (6) via the feedback law (17) are $\{64.61 \pm 85.54i, -82.3, 15.8 \pm 11.42i, 12.94, -2.2, 10, 10\}$ and $\{-0.45, 0.04 \pm 0.03i, -0.012, 0.005 \pm 0.007i, 0.077, 0.1, 0.1\}$, respectively. The Figures 1 and 2 show that the state and input variables $x_i(t), i = 1, 2, 3$ and $u_i, i = 1, 2$ converge to zero by considering

$$X_0 = \begin{bmatrix} -0.1 & 0.1 & -0.1 & 0.1 & 0.1 & -0.1 & 0.1 & 0.1 & -0.1 \end{bmatrix}.$$

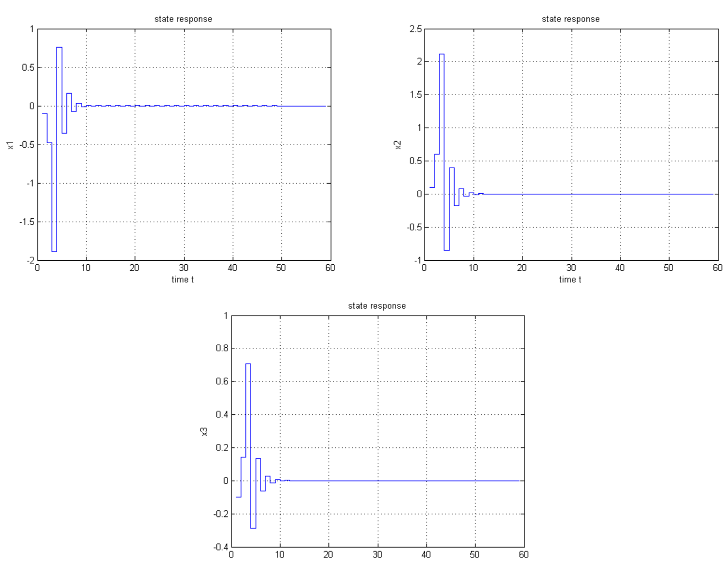

Figure 1: State vector in Example 1.

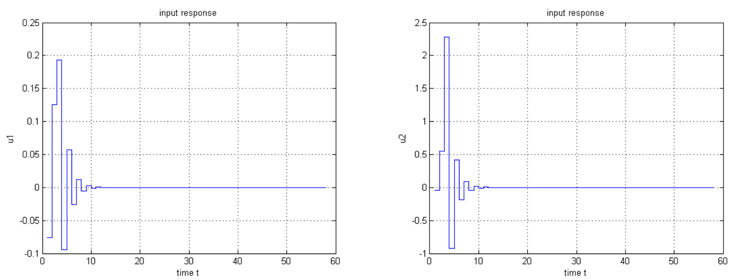

Figure 2: Input vector in Example 1.

Example 2. Consider system (1) with $\alpha = 0.3, h = 2$, and matrices A, B, and E as

$$A = \begin{bmatrix} 3 & -3 & -4 \\ 3 & 1 & -1 \\ 4 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} -4 & 1 & 5 \\ 8 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix},$$

where $rank(E) = 2 < 3$.

The matrices $\bar{A}, \bar{B}$, and $\bar{E}$ are obtained as

$$\bar{A} = \begin{bmatrix} 1.8 & -2.7 & -2.5 & -0.42 & 0.1 & 0.52 & 0.23 & -0.05 & -0.29 \\ 5.4 & 1.6 & -0.1 & 0.84 & 0.21 & 0.31 & -0.47 & -0.11 & -0.17 \\ 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} -4 & 1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalue assignment via forward state feedback is not applicable. The pair (M, N) may not be defined because of singularity of the matrix $\bar{A}$, $rank(\bar{A}) = 8 < 9$.

Now consider the standard systems (20) and (22) by propositional and forward state feedbacks F_p and F_f, respectively. Only obtaining the forward feedback matrix F_f is displayed by the propositional state feedback matrix F_p as

$$F_p = \begin{bmatrix} -3.07 & 0.57 & -0.5 & 0.68 & 0.21 & 0.4 & -0.13 & 0 & 0.06 \\ -0.46 & 0.6 & 1.53 & -0.69 & -0.29 & -0.65 & 0.14 & 0.04 & 0.07 \end{bmatrix},$$

which all eigenvalues are assigned to 0.1.

The pair of (M, N) is

$$N = 10^7 \times \begin{bmatrix} 0 & 0 & 0 & 10^{-7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10^{-7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10^{-7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10^{-7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-7} \\ 0.01 & 0.05 & 0.18 & -0.01 & 0 & -0.01 & 0 & 0 & 0 \\ -0.1 & -0.37 & -1.18 & 0.07 & 0.03 & 0.09 & 0 & 0 & -0.01 \\ 0.03 & 0.11 & 0.34 & -0.02 & -0.01 & -0.02 & 0 & 0 & 0 \end{bmatrix},$$

$$M = 10^6 \times \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -0.75 & -0.78 \\ 4.95 & 5.16 \\ -1.46 & -1.52 \end{bmatrix}.$$

Case (a). Consider the method in Subsection 4.2. Because $eig(N) = \{34.6 \pm 40.14i, -33.53, 24.81 \pm 18i, 4.77, -0.04 \pm 0.92i, 0\}$, we may reassign $p = 3$ eigenvalues like $\{10 \pm 10i, 10\}$ instead of $\{-0.04 \pm 0.92i, 0\}$ while leaving the other eigenvalues unchanged. The first sufficient conditions, i.e. $\{\lambda_1, \lambda_2, \lambda_3\}$ should not vanish, is not satisfied for the solution of the partial eigenvalue assignment using orthogonality relations. Therefore, this method cannot be used in this example.

Case (b). Consider the method in Subsection 4.3. Similar to case (a), because $eig(N) = \{34.6 \pm 40.14i, -33.53, 24.81 \pm 18i, 4.77, -0.04 \pm 0.92i, 0\}$, we reassign $\{10 \pm 10i, 10\}$ instead of $\{-0.04 \pm 0.92i, 0\}$ while leaving the other eigenvalues unchanged. New pair $(Y_1^H M, \Lambda_1)$ and forward state feedback

F_f are obtained by

$$Y_1^H M = \begin{bmatrix} -0.25 - 0.38i & -0.59 - 0.5i \\ -0.25 + 0.38i & -0.59 + 0.5i \\ -0.33 & -0.67 \end{bmatrix},$$

$$\Lambda_1 = \begin{bmatrix} -0.04 - 0.92i & 0 & 0 \\ 0 & -0.04 + 0.92i & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$F_f = 10^2 \times \begin{bmatrix} -0.53 & 0.31 & 1.32 & -0.23 & -0.13 & -0.32 & 0.07 & 0.04 & 0.11 \\ -0.02 & -0.13 & -0.42 & 0.1 & 0.05 & 0.13 & -0.02 & -0.01 & -0.02 \end{bmatrix}.$$

The eigenvalues of the closed-loop matrix of the standard system (22) and the standard descriptor system (6) via feedback law (17) are $\{34.6 \pm 40.14i, -33.53, 24.81 \pm 18i, 4.77, 10 \pm 10i, 10\}$ and $\{0.2, -0.03, 0.02 \pm 0.01i, 0.01 \pm 0.01i, 0.05 \pm 0.05i, 0.1\}$, respectively. The Figures 3 and 4 show that the state and input variables $x_i(t), i = 1, 2, 3$ and $u_i, i = 1, 2$ converge to zero by considering

$$X_0 = \begin{bmatrix} 0.02 & 0.02 & 0.02 & -0.02 & -0.02 & -0.02 & 0.02 & 0.02 & 0.02 \end{bmatrix}.$$

6 Concluding remarks

Stabilization and control of descriptor fractional discrete-time linear systems are presented. Assigning desired eigenvalues in unit disk to the converted standard descriptor discrete-time linear system is done by eigenvalue assignment with forward state feedback. This method needs the sufficient condition (II) in (9) for all examples where we do not have it by using eigenvalue assignment with forward and propositional state feedback but with large matrices. To decrease dimensions of matrices, input, and state vectors, partial eigenvalue assignment may be used. Although the partial eigenvalue assignment algorithm using orthogonality relations is not doable for reassigning indistinct and vanished eigenvalues. However, in partial eigenvalue assignment via similarity transformation, undesired indistinct and even zero eigenvalues can be reassigned, while leaving the rest of the spectrum invariant. Also, the eigenvalues of closed-loop matrix in last method lie in desired region and state and input vectors $x_i(t), i = 1, \ldots, n$ and $u_i(t), i = 1, \ldots, m$ converge to zero. The results presented in this article are also applicable in stabilization of discrete-time descriptor, fractional, delayed, and two-dimensional systems. The subject of minimum norm of feedback matrices is remarkable, too. An extension of these considerations for continuous-time descriptor fractional linear systems is still an open problem.

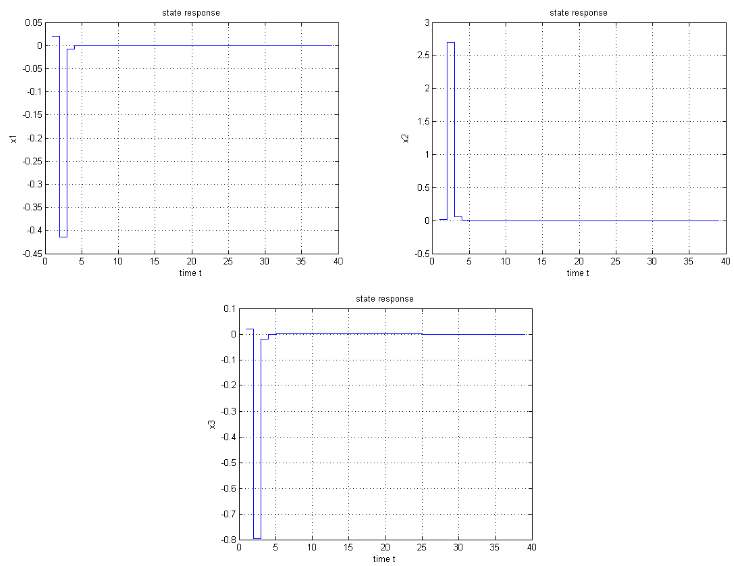

Figure 3: State vector in Example 2.

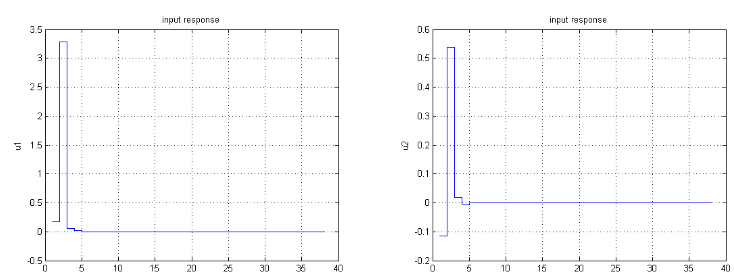

Figure 4: Input vector in Example 2.

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