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Interplay of resource distributions and diffusion strategies for spatially heterogeneous populations

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Abstract. In this paper, we consider a reaction-diffusion competition model describing the interactions between two species in a heterogeneous environment. Specifically, we study the impact of diffusion strategies on the outcome of competition between two populations while the species are distributed according to their respective carrying capacities. The two species differ in the diffusion strategies they employ as well as in their asymmetric growth intensities. In case of weak competition, both populations manage to coexist and there is an ideal free pair. If the resources are shared partially then one species emerge as the sole winner and the other goes extinct. The results have been verified and illustrated numerically.

Keywords: Resource distributions, adopted dynamics, competition, global analysis, asymptotic stability.
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1 Introduction

For a two species competition model, either both populations survive or one triumphs as the other goes extinct. In some cases both species leave the location under competition which yields neither coexistence nor extinction. In ecology or economy, this setting is important enough to warrant investigation of how one organism of a species changes the density over

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time to survive in competition (see, e.g., [1, 20, 2, 23, 4] and references therein). An illustrative case as such is in Fig. 1 presenting the significant dimensions of competition or cooperation for any two interacting populations in a habitat. To explore this idea, let us initially consider the classical

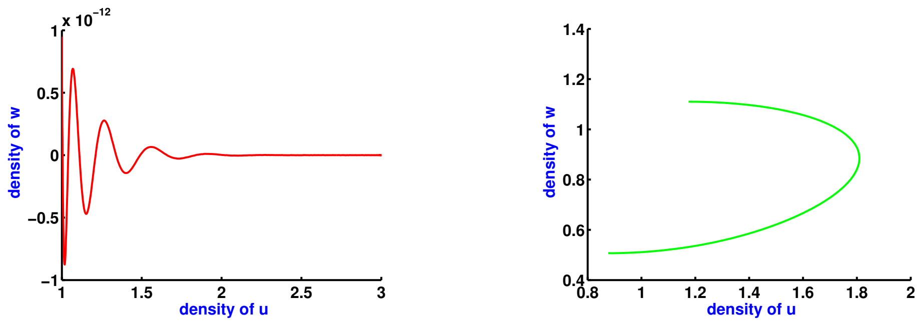


Figure 1: Extinction and coexistence patterns of two populations.

Lotka-Volterra two-species competition model with diffusion

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1 \Delta u + u(t,x)\left(r_1 - \alpha_1 u(t,x) - \beta_1 w(t,x)\right) \text{ in } \mathbb{R}_+ \times \Omega, \\ \dfrac{\partial w}{\partial t} = D_2 \Delta w + w(t,x)\left(r_2 - \alpha_2 u(t,x) - \beta_2 w(t,x)\right) \text{ in } \mathbb{R}_+ \times \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega, \\ u(0,x) = u_0(x) > 0,\ w(0,x) = w_0(x) > 0 \text{ in } \Omega, \end{cases} \tag{1}$$

where $\mathbb{R}_+ = (0, \infty)$, Δ is a usual Laplace operator and the coefficients $D_i,\ r_i,\ \alpha_i,\ \beta_i\,(i = 1, 2)$ are nonnegative. In addition, the constant r_i is the specific growth rate of the populations. The constants $\alpha_1,\ \beta_2$ are defined as the *intra-specific* competition rates whereas the constants $\alpha_2,\ \beta_1$ are known as the *inter-specific* competition rates. Unknown functions $u(t, x)$ and $w(t, x)$ denote the population densities of the two competing species.

The interaction terms represent logistic growth with competition. It is well known that the initial-boundary value problem (1) has a unique smooth nonnegative solution; see, e.g., [30] for systems with n equations and general semi-linearities. In mathematical modeling of ecology, the asymptotic behavior of the solutions of (1) has been central to understanding coexistence and spatial dissociation of two species, see [20, 2, 23, 24].

The long-time behavior depends on the values of the reaction coefficients $r_i,\ \alpha_i,\ \beta_i\,(i = 1, 2)$ in determining the asymptotic solutions of (1). For further study, these are classified into the following four classes by setting $r_{12} \equiv r_1/r_2$, $\alpha_{12} \equiv \alpha_1/\alpha_2$, $\beta_{12} \equiv \beta_1/\beta_2$ (see [20, 2, 24]):

1. If $r_{12} > \max\{\alpha_{12}, \beta_{12}\}$, the solution $(u, w) \to (r_1/\alpha_1, 0)$ uniformly on Ω as $t \to \infty$. Therefore, the species u dominates and the rest one, w becomes extinct eventually regardless of the initial values u_0, w_0.

2. Similarly, when $r_{12} < \min\{\alpha_{12}, \beta_{12}\}$, the species w dominates the species u and in the long run u is in extinction and the solution (u, w) converges to $(0, r_2/\beta_2)$ as $t \to \infty$.

3. The competition between species u and w is profusely weak as long as $r_{12} \in (\beta_{12}, \alpha_{12})$ and coexistence is expected then. In this case, the system (1) implies that the solution (u_s, w_s) is globally asymptotically stable. More precisely, $\lim_{t\to\infty}(u(t, x), w(t, x)) = (u_s, w_s)$ uniformly for $x \in \Omega$. This means that both species coexist.

4. Radical and rigorous story arises in the case of $r_{12} \in (\alpha_{12}, \beta_{12})$. The competition is strong, analysis is interesting and requires rather sophisticated methods. The steady states $(r_1/\alpha_1, 0)$ and $(0, r_2/\beta_2)$ are locally stable while the coexistence state (u_s, w_s) is unstable.

In the weak competition case, the positive stationary solution (u_s, w_s) is globally asymptotically stable independent of the diffusion coefficients. Materially, there exists a Lyapunov functional which allows for a long-time asymptotic analysis [21]. Thus, no non-constant stationary solution exists for any D_1 and D_2 and there is no pattern structure. Diffusion design in [5, 6] was based on the notion of the ideal free distribution to optimize the physical fitness. For the Volterra model with dispersion, for any number of interacting populations, the effect of uniform diffusion is to damp all spatial variations as shown in [25]. The effect of environment heterogeneity for Lotka-Volterra two species interactions was studied in [12]. General reaction rates and the stability of constant steady states was considered by [7]. In the strong competition case, the result follows from [15] that the problem (1) has no stable positive equilibrium solution if the domain is convex. For certain dumb-bell type domains, the system (1) has at least one stable coexistence solution [26]. According to the domains of attraction of $(r_1/\alpha_1, 0)$ and $(0, r_2/\beta_2)$, some attractive results were shown in [28]. One recent study of competition model has investigated the effects of diffusion rate and spatial heterogeneity on a two species Lotka-Volterra model under homogeneous Dirichlet boundary conditions [31]. The study derived that in weak competition and for certain range of diffusion speeds, there exists a unique coexistence solution which is globally asymptotically stable. However, a two species diffusive predator-prey model with bifurcation analysis was studied in [32]. In this paper, it is shown that if diffusion rate of the prey is treated as a bifurcation parameter, for some certain ranges of death rate of the predator, there exist multiply positive equilibria bifurcating from semi-trivial equilibrium of the model. Recently, in 2018, it has been established that the directed dispersal organism has evolutionary advantages to design its own habitat for a weak competition coupled species model [16] (also see the monographs [17, 18] for symmetric and non-symmetric directed dispersal models). In contrast to the competition model (1), we will consider the dynamics of a two-species adopted competition model (2) with logistic types growth. In this paper, we focus on evaluating the potential benefits of the species adopting their own resources according to (2) as compared to the well-established results from the random diffusion problem (1).

The rest of the paper is organized as follows. The mathematical model is described in Section 2 and some auxiliary results are presented that are utilized in the rest of the paper. Also the statements of the main results of the paper are articulated in this section. Competitive analysis due to the competition coefficients for coexistence solutions are investigated in Section 3. The contents of Section 4 are analysis of the semi-trivial equilibria in non-homogeneous environment. In this section, we consider different types of interactions between distribution functions to prove our main results of the paper. For weak competition, it is shown that there is a unique coexistence solution. Some applications are presented in Section 5 to justify the theoretical results. Finally, Section 6 presents summary and discussion of the results.

2 Adopted dispersal model and main results

In the present study, the problem deals with the competition for the same basic resources between two species $u(t,x)$ and $w(t,x)$ pertaining to the diffusion strategies in their adopted dynamics. This corresponds to the following system of differential equations with homogeneous Neumann boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1 \Delta \left(\dfrac{u(t,x)}{K_1(x)} \right) + u(t,x) \left(K(x) - u - \mu w \right) \text{ in } \mathbb{R}_+ \times \Omega, \\ \dfrac{\partial w}{\partial t} = D_2 \Delta \left(\dfrac{w(t,x)}{K_2(x)} \right) + w(t,x) \left(K(x) - \nu u - w \right) \text{ in } \mathbb{R}_+ \times \Omega, \\ \dfrac{\partial (u/K_1)}{\partial n} = \dfrac{\partial (w/K_2)}{\partial n} = 0 \text{ on } \partial\Omega, \\ u(0,x) = u_0(x),\ w(0,x) = w_0(x) \text{ in } \Omega. \end{cases} \tag{2}$$

The habitat Ω is a bounded region in $\mathbb{R}^n$ with $\partial\Omega \in C^{2+\beta}$, $\beta > 0$.

We consider that all parameters $D_i, K_i(x)\,(i = 1,2)$, μ, ν and $K(x)$ are positive. The constants D_i, $i = 1,2$ represent the migration rates, and are known as the diffusion coefficients. The function $K(x)$ denotes the resource distribution (carrying capacity) of the environment and the function is spatially distributed only. We assume that the carrying capacity $K(x)$ is in the class of $C^{1+\beta}(\overline{\Omega})$, $\beta > 0$ for any $x \in \overline{\Omega}$ and $l_1 \times l_2$ is a bounded subset of $\mathbb{R}^2$. The set $l_1 \times l_2$ corresponds to the range of the solutions to (2) and is determined by the corresponding upper and lower solutions. It is important to note that, we consider the common carrying capacity and the specific growth rates are proportional in (2), which makes the reaction patterns different from the primary system (1). The following result is well-established for monotone dynamical system (2) (see the monographs [1, 13, 29]).

Theorem 1. *[14, 27, 11] If the system (2) has no coexistence equilibrium then one of the semi-trivial equilibrium is unstable while the other is globally asymptotically stable [14]; if both semi-trivial equilibria are unstable, the problem (2) has at least one stable coexistence equilibrium [27, 11].*

For the sake of comprehension and clarity, we state our key results at this point. For simplicity, it is also noted that throughout the paper, we consider *const* instead of *constant* while the rational functions are defined.

Theorem 2. *Assume that $\dfrac{K_1(x)}{K(x)} \equiv const$. If $\mu \in (0,1)$ and $\nu > 1$, then the equilibrium $(K(x), 0)$ of (2) is globally asymptotically stable. That is, for any nonnegative nontrivial initial values $u_0, w_0 \in C(\overline{\Omega})$, the solution (u,w) of (2) satisfies $(u,w) \to (K(x),0) \quad$ as $\quad t \to \infty$ uniformly in $x \in \overline{\Omega}$.*

Theorem 3. *Suppose that $K_1 \equiv K_2$, $\dfrac{K_i(x)}{K(x)} \equiv const$, $(i = 1,2)$, and $D_1 = D_2 = D$, then the system (2) has a stable positive coexistence equilibrium, which is $\left(\frac{(1-\mu)K(x)}{1-\mu\nu}, \frac{(1-\nu)K(x)}{1-\mu\nu}\right)$ as long as $\mu, \nu \in (0,1)$.*

Theorem 4. *Let $\dfrac{K_1(x)}{K(x)} \not\equiv const$ and $\dfrac{K_2(x)}{K(x)} \equiv const$. If $\mu, \nu \in (0,1)$, then the system (2) has at least one coexistence equilibrium (u_s, w_s), which is stable for any nonnegative and nontrivial initial conditions.*

The proofs of Theorems 2-4 are formulated through a series of steps in the Section 4.

3 Classical approach to equilibrium analysis

Analysis of the problem designated in (2) is undertaken at this stage.

Substituting $h(t,x) = u(t,x)/K_1(x)$ and $m(t,x) = w(t,x)/K_2(x)$, the problem (2) becomes

$$\begin{cases} K_1(x)\dfrac{\partial h}{\partial t} = D_1\Delta h + K_1 h\left(K(x) - K_1 h - \mu K_2 m\right) \text{ in } \mathbb{R}_+ \times \Omega, \\ K_2(x)\dfrac{\partial m}{\partial t} = D_2\Delta m + K_2 m\left(K(x) - \nu K_1 h - K_2 m\right) \text{ in } \mathbb{R}_+ \times \Omega, \\ \dfrac{\partial h}{\partial n} = \dfrac{\partial m}{\partial n} = 0 \text{ on } \partial\Omega, \\ h(0,x) = u_0(x)/K_1(x),\ m(0,x) = w_0(x)/K_2(x) \text{ in } \Omega. \end{cases} \tag{3}$$

In a bid to provide exposition, we will consider the system (3) for numerical simulations so as to corroborate the theoretical results. In the following, we will state the results on stability of the semi-trivial equilibria for the system (2).

The functions $u(t,x)$ and $w(t,x)$ are the solutions of (2) and exist for all $t > 0$. The system (2) has at most four nonnegative equilibria; the trivial equilibrium $(0,0)$, two semi-trivial equilibria $(u^*(x),0), (0,w^*(x))$ and the coexistence equilibrium $(u_s(x), w_s(x))$. If $(u(x), w(x))$ is any stationary coexistence solution of (2), consider the following eigenvalue problem to analyze the linear stability of the system,

$$\begin{cases} D_1\Delta\left(\dfrac{\phi(x)}{K_1(x)}\right) + \phi(x)(f + uf_u) + \psi(x)\cdot uf_w = \sigma\phi(x) \text{ in } \Omega, \\ D_2\Delta\left(\dfrac{\psi(x)}{K_2(x)}\right) + \psi(x)(g + wg_w) + \phi(x)\cdot wg_u = \sigma\psi(x) \text{ in } \Omega, \\ \dfrac{\partial(\phi/K_1)}{\partial n} = \dfrac{\partial(\psi/K_2)}{\partial n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{4}$$

where f_u and g_w are the derivatives of f and g with respect to u and w, respectively, and

$$f(x,u,w,K) = \left(K(x) - u(x) - \mu w(x)\right),$$

$$g(x,u,w,K) = \left(K(x) - \nu u(x) - w(x)\right).$$

The function $\phi(x)$ and $\psi(x)$ are two eigenfunctions and σ is the corresponding eigenvalue.

Next, let us proceed to test the stability of trivial equilibrium solution.

Lemma 1. *[19, 3] The trivial equilibrium $(0,0)$ of the system (2) is unstable. Moreover, it is a repelling equilibrium.*

Proof. Let us consider the eigenvalue problem (4) around the origin with corresponding boundary condition in (4)

$$D_1\Delta\left(\frac{\phi(x)}{K_1(x)}\right) + \phi(x)K(x) = \sigma\phi(x) \text{ in } \Omega,\ \frac{\partial(\phi/K_1)}{\partial n} = 0 \text{ on } \partial\Omega. \tag{5}$$

Now it suffices to show that the principal eigenvalue is positive. According to the variational characterization of the eigenvalues [8], the principal eigenvalue of (5) is given by

$$\sigma_1 = \sup_{\phi\neq 0, \phi\in W^{1,2}} \frac{-D_1 \int\limits_\Omega |\nabla(\phi/K_1)|^2 dx + \int\limits_\Omega K(x)\frac{\phi^2}{K_1} dx}{\int\limits_\Omega (\phi^2/K_1) dx}.$$

For eigenfunction $\phi(x) = K_1(x)$, we have

$$\sigma_1 \geq \frac{\int\limits_\Omega K(x)K_1(x)dx}{\int\limits_\Omega K_1(x)dx} \geq \frac{\min\{\inf\limits_{x\in\Omega} K(x)\}\int\limits_\Omega K_1(x)dx}{\int\limits_\Omega K_1(x)dx} = \min\{\inf_{x\in\Omega} K(x)\} > 0,$$

and the trivial equilibrium $(0,0)$ of (2) is unstable.

Next step is to prove that the trivial equilibrium $(0,0)$ of (2) is a repeller. Let

$$\delta = \min\left\{\inf_{x\in\Omega}\frac{K}{4}, \inf_{x\in\Omega}\frac{K}{4\mu}, \inf_{x\in\Omega}\frac{K}{4\nu}\right\} > 0,$$

$u_0(x) \geq 0$ and $w_0(x) \geq 0$ be such that $u_0(x) < \delta$, $w_0(x) < \delta$ for $(u_0(x), w_0(x)) \neq (0,0)$. Add the first two equations of (2) and integrate over Ω using Neumann boundary conditions [13, 8], we get

$$\frac{d}{dt}\int\limits_\Omega (u+w)dx = \int\limits_\Omega u\left(K - u - \mu w\right)dx + \int\limits_\Omega w\left(K - w - \nu u\right)dx. \tag{6}$$

Note that $\gamma = \min\{\inf\limits_{x\in\Omega} K(x)\} > 0$ on condition that $u \leq \delta$ and $w \leq \delta$ there holds

$$\frac{d}{dt}\int\limits_\Omega (u+w)\,dx \geq \gamma\int\limits_\Omega u\left(1 - \frac{1}{2}\right)\,dx + \gamma\int\limits_\Omega w\left(1 - \frac{1}{2}\right)\,dx = \frac{\gamma}{2}\int\limits_\Omega (u+w)\,dx.$$

Therefore using Gronwall's lemma [24, 29], we get

$$\int\limits_\Omega (u(t,x) + w(t,x))\,dx \geq e^{\gamma t/2}\int\limits_\Omega (u_0(x) + w_0(x))\,dx.$$

Since $\int\limits_\Omega (u_0(x)+w_0(x))dx > 0$, then the integral $\int\limits_\Omega (u(t,x)+w(t,x))dx$ grows up exponentially with time as far as $u \leq \delta$ and $w \leq \delta$. So, we can say that there exists $t_0 > 0$ such that $u(t_0,x) > \delta$ and $w(t_0,x) > \delta$ for some $x \in \Omega$ and the equilibrium point $(0,0)$ is a repeller. $\square$

3.1 Preliminaries

The function $u(x)$ is the solution of the following elliptic boundary value problem and it corresponds to a state when only species u survives in (2):

$$D_1\Delta\left(\frac{u(x)}{K_1(x)}\right)+u(x)\left(K(x)-u(x)\right)=0 \text{ in } \Omega,\ \frac{\partial(u/K_1)}{\partial n}=0 \text{ on } \partial\Omega. \quad (7)$$

If one of $u,\ w$ is identically equal to zero, we obtain semi-trivial equilibrium solutions, where the non-trivial part solves a single equation. For single-species u of (2), the proof of the following lemma can be found in [19].

Lemma 2. *[19] Suppose that* $\dfrac{K_1(x)}{K(x)} \not\equiv$ *const and let* $u(t,x)$ *be a solution to*

$$\frac{\partial u}{\partial t}=D_1\Delta\left(\frac{u(t,x)}{K_1(x)}\right)+u\left(K(x)-u\right)\ \textit{in}\ \Omega,\ \frac{\partial(u/K_1)}{\partial n}=0\ \textit{on}\ \partial\Omega. \quad (8)$$

Then there is a unique stationary solution $u^*(x)$ *of (8). Also, with non-trivial nonnegative initial condition any solution* $u(t,x)>0$*, and* $u(t,x)\to u^*(x)$ *as* $t\to\infty$*. Moreover*

$$\int_\Omega \left(K(x)-u^*(x)\right)K(x)dx=\int_\Omega \left(u^*(x)-K(x)\right)^2 dx>0. \quad (9)$$

For single species w, if $\dfrac{K_2(x)}{K(x)}\not\equiv$ const, then there exists a unique positive stationary solution $w^*(x)$ in (2) such that

$$\int_\Omega \left(K(x)-w^*(x)\right)K(x)dx=\int_\Omega \left(w^*(x)-K(x)\right)^2 dx>0. \quad (10)$$

We need the following definition concerning the elliptic eigenvalue problem of (2).

Definition 1. *[8, 22] Given a positive constant* D *and a function* $p\in L^\infty(\Omega)$*, we define the* n^{th} *eigenvalue* $\sigma_n(D,p)$ *with counting multiplicities*

$$D\Delta\varphi+p\varphi=-\sigma\varphi,\ x\in\Omega,\ \frac{\partial\varphi}{\partial n}=0,\ x\in\partial\Omega, \quad (11)$$

where $\sigma_1(D,p)$ *is defined as the first eigenvalue of (11) and* φ *is the eigenfunction.*

Proposition 1. *[8] If* $p_1\le p_2$ *within* Ω*, then* $\sigma_1(D,p_1)\ge\sigma_1(D,p_2)$ *and the equality holds only when* $p_1=p_2$ *in* Ω*. In addition, if* p *is nonconstant, then* $\sigma_1(D_1,p)<\sigma_1(D_2,p)$ *as long as* $D_1<D_2$*.*

The proof of the Proposition 1 is available in [8], pp. 95.

3.2 Stability analysis of coexistence solutions

The asymptotic behavior of the solutions $(u(t,x), w(t,x))$ for the system (2) with nonnegative and nontrivial initial functions $(u(0,x), w(0,x))$ can be categorized into four classes: (i) $0 < \mu < 1$, $\nu > 1$; (ii) $0 < \nu < 1$, $\mu > 1$; (iii) $0 < \mu$, $\nu < 1$; and (iv) μ, $\nu > 1$. In this section, our discussion is limited to the first two classes, exploring the coexistence of both populations.

The next result is concerned with the coexistence of both populations in case of competition coefficients $0 < \mu < 1$ and $\nu > 1$.

Lemma 3. *Suppose that* $\dfrac{K_1(x)}{K(x)} \equiv const$. *If* $\mu \in (0,1)$ *and* $\nu > 1$, *there is no coexistence state* $(u_s(x), w_s(x))$ *of (2).*

Proof. Let us assume the contrary, i.e., that there exists a strictly positive solution $(u_s(x), w_s(x))$ of (2). Then an equilibrium $(u_s(x), w_s(x))$ must satisfy

$$\begin{cases} D_1 \Delta \left(\dfrac{u_s(x)}{K_1(x)}\right) + u_s(x)\left(K(x) - u_s(x) - \mu w_s(x)\right) = 0 \text{ in } \Omega, \\ D_2 \Delta \left(\dfrac{w_s(x)}{K_2(x)}\right) + w_s(x)\left(K(x) - \nu u_s(x) - w_s(x)\right) = 0 \text{ in } \Omega, \\ \dfrac{\partial(u_s/K_1)}{\partial n} = \dfrac{\partial(w_s/K_2)}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{12}$$

After multiplying the second equation in (12) by μ, adding them and integrating over Ω using boundary conditions in (12), we obtain

$$0 = \int_\Omega \left[u_s(x)\left(K(x) - u_s - \mu w_s\right) + \mu w_s(x)\left(K(x) - \nu u_s - w_s\right)\right] dx. \tag{13}$$

If $0 < \mu < 1$ and $\nu > 1$, then $(K(x) - u_s - \mu w_s) > (K(x) - \nu u_s - w_s)$ and it follows that

$$\int_\Omega \left(u_s(x) + \mu w_s(x)\right)\left(K(x) - u_s(x) - \mu w_s(x)\right) dx > 0. \tag{14}$$

Integrating the equality

$$(u_s + \mu w_s)(K - u_s - \mu w_s) = (K - u_s - \mu w_s)(u_s + \mu w_s - K) + (K - u_s - \mu w_s) K,$$

over Ω using (14) leads to

$$\int_\Omega K(x)\left(K(x) - u_s - \mu w_s\right) dx > \int_\Omega \left(K(x) - u_s - \mu w_s\right)^2 dx, \tag{15}$$

and (15) is valid for any positive $(u_s(x),\, w_s(x))$ and $\mu \in (0,1)$ which exclude the possibility of $u_s(x) + \mu w_s(x) \equiv K(x)$. Thus the left integral is positive only when $u_s(x) + \mu w_s(x) \not\equiv K(x)$ and we define the eigenvalue problem

$$D_1 \Delta \left(\frac{\phi}{K_1} \right) + \phi \left(K - u_s - \mu w_s \right) = \sigma \phi(x) \text{ in } \Omega, \frac{\partial (\phi/K_1)}{\partial n} = 0 \text{ on } \partial\Omega. \quad (16)$$

According to the variational characterization of eigenvalues described in [8], its principal eigenvalue is given by

$$\sigma_1 = \sup_{\phi \neq 0, \phi \in W^{1,2}} \frac{-D_1 \int\limits_{\Omega} |\nabla(\phi/K_1)|^2 \, dx + \int\limits_{\Omega} \frac{\phi^2}{K_1} \left(K - u_s - \mu w_s \right) \, dx}{\int\limits_{\Omega} (\phi^2/K_1) \, dx}. \quad (17)$$

For eigenfunction $\phi(x) = K(x)$ and using (15), we have

$$\sigma_1 \geq \frac{\int\limits_{\Omega} \left(K(x) - u_s(x) - \mu w_s(x) \right) K(x) \, dx}{\int\limits_{\Omega} K(x) \, dx} > 0, \quad \text{since} \quad K_1(x)/K(x) \equiv const.$$

Since 0 is always a principal eigenvalue with a positive principal eigenfunction of (16), this contradicts $\sigma_1 > 0$. □

Remark 1. *Suppose that* $\frac{K_2(x)}{K(x)} \equiv const$. *If* $\nu \in (0,1)$ *and* $\mu > 1$ *then there is no coexistence state* $(u_s(x), w_s(x))$ *of (2).*

If $\mu = \nu = 1$, $K_1(x) \equiv K(x)$ and $\frac{K_2(x)}{K(x)} \not\equiv const$ then there is no coexistence, and this result is reported in [19].

Lemma 4. *Suppose that* $K_1 \equiv K_2$, $\frac{K_i(x)}{K(x)} \equiv const$, $(i = 1,2)$, *and* $D_1 = D_2 = D$, *then for each* $\mu, \nu \in (0,1)$, *the system (2) has a unique positive coexistence equilibrium, and that is given by* $\left(\frac{(1-\mu)K}{1-\mu\nu}, \frac{(1-\nu)K}{1-\mu\nu} \right)$.

Proof. For stationary solutions $(u_s(x), w_s(x))$, the system (2) can be written as

$$\begin{cases} D\Delta \left(\dfrac{u_s(x)}{K_1(x)} \right) + u_s \left(K(x) - u_s - \mu w_s \right) = 0 \text{ in } \Omega, \\ D\Delta \left(\dfrac{w_s(x)}{K_2(x)} \right) + w_s \left(K(x) - \nu u_s - w_s \right) = 0 \text{ in } \Omega, \\ \dfrac{\partial (u_s/K_1)}{\partial n} = \dfrac{\partial (w_s/K_2)}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \quad (18)$$

By direct substitution it is easy to check that $\left(\frac{(1-\mu)K}{1-\mu\nu}, \frac{(1-\nu)K}{1-\mu\nu}\right)$ is a coexistence stationary solution of (18). To show the uniqueness, suppose that (u_s, w_s) is any coexistence equilibrium of (18) except $\left(\frac{(1-\mu)K}{1-\mu\nu}, \frac{(1-\nu)K}{1-\mu\nu}\right)$.

Claim: $(1-\nu)u_s(x) = (1-\mu)w_s(x)$.

Assume to the contrary that $(1-\nu)u_s(x) \not\equiv (1-\mu)w_s(x)$ and let $v_s(x) = (1-\nu)u_s(x) - (1-\mu)w_s(x) \not\equiv 0$. Dividing the last equation by K_1 and for equality of diffusion coefficients, assume that $K_1 = K_2$. Since $\frac{K_1}{K} \equiv \frac{K_2}{K} \equiv$ *const*, after simplification we have

$$\frac{v_s(x)}{K_1} = (1-\nu)\frac{u_s(x)}{K_1} - (1-\mu)\frac{w_s(x)}{K_2} \not\equiv 0.$$

Multiplying the first equation of (18) by $(1-\nu)$ and second equation by $(1-\mu)$, we obtain

$$\begin{cases} D(1-\nu)\Delta\left(\dfrac{u_s(x)}{K_1(x)}\right) + (1-\nu)u_s\left(K(x) - u_s - \mu w_s\right) = 0 \text{ in } \Omega, \\ D(1-\mu)\Delta\left(\dfrac{w_s(x)}{K_2(x)}\right) + (1-\mu)w_s\left(K(x) - \nu u_s - w_s\right) = 0 \text{ in } \Omega, \\ \dfrac{\partial(u_s/K_1)}{\partial n} = \dfrac{\partial(w_s/K_2)}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{19}$$

Subtracting the first two equations in (19), v_s satisfies

$$D\Delta\left(\frac{v_s(x)}{K_1(x)}\right) + v_s(x)\left(K(x) - u_s - w_s\right) = 0 \text{ in } \Omega, \frac{\partial(v_s/K_1)}{\partial n} = 0 \text{ on } \partial\Omega$$

and therefore, $\sigma_n\left(D, K(x) - u_s(x) - w_s(x)\right) = 0$ for some $n \geq 1$.

However, using Proposition 1,

$$\sigma_n\left(D, K - u_s - w_s\right) \geq \sigma_1\left(D, K - u_s - w_s\right) > \sigma_1\left(D, K - u_s - \mu w_s\right) = 0,$$

is a contradiction, where the last equality is satisfied by the first equation of (18).

Therefore, $(1-\nu)u_s(x) = (1-\mu)w_s(x)$ which implies $w_s(x) = \dfrac{(1-\nu)u_s(x)}{(1-\mu)}$ and by substituting $w_s(x)$ in the first equation of (18), we obtain

$$D\Delta\left(\frac{u_s(x)}{K_1(x)}\right) + u_s\left(1 - \frac{(K-\mu\nu)u_s}{(1-\mu)}\right) = 0 \text{ in } \Omega,\ \frac{\partial(u_s/K_1)}{\partial n} = 0 \text{ on } \partial\Omega.$$

Hence by uniqueness $u_s(x) = \dfrac{(1-\mu)K}{(1-\mu\nu)}$ and $w_s(x) = \dfrac{(1-\nu)K}{(1-\mu\nu)}$. □

4 Global analysis of semi-trivial equilibria

In this section, the stability properties of the equilibria $(u^*(x), 0)$ and $(0, w^*(x))$ are examined for the system (2) for different cases of μ and ν. Besides, when $K_i(x)/K(x) \equiv const$, $(i = 1, 2)$, and $\mu, \nu \in (0, 1)$ then the stability of $(K(x), 0)$ and $(0, K(x))$ are analyzed as part of our study. In this case, the coexistence solution $(u_s(x), w_s(x)) \equiv \left(\frac{(1-\mu)K}{1-\mu\nu}, \frac{(1-\nu)K}{1-\mu\nu}\right)$ of (2) is the intersection of the lines $u(x) + \mu w(x) = K(x)$ and $\nu u(x) + w(x) = K(x)$ whenever it exists. Also it is well established that if a stationary solution of (2) is linearly stable then it is asymptotically stable [29].

4.1 Effects of higher consumption rate

First we note that in random movement of the species [20, 2, 24] if $0 < \mu < 1$, $\nu > 1$, then the solution $(u(t, x), w(t, x))$ converges to $(u^*(x), 0)$ as $t \to \infty$ for any positive $(u_0(x), w_0(x))$. Similarly, the solution $(u(t, x), w(t, x)) \to (0, w^*(x))$ as $t \to \infty$ for $\nu < 1 < \mu$ regardless of the initial conditions. For the adopted dynamical system (2), we will show some applications by considering various relations between $K_i(x)\,(i = 1, 2)$ and $K(x)$ as well as for different values of μ, ν. We illustrated that the results of random diffusion are not always true for the modified directed dynamics model, coexistence is also possible.

The following lemma shows the instability of the semi-trivial equilibrium $(0, w^*(x))$ (as well as possible $(0, K(x))$) under certain conditions.

Lemma 5. *Assume that* $\dfrac{K_1(x)}{K(x)} \equiv const$ *and* $\nu > 1$, *then the semi-trivial equilibrium* $(0, w^*(x))$ *of (2) is unstable if* $\mu \in (0, 1)$.

Proof. Note that if $\frac{K_2(x)}{K(x)} \not\equiv const$ then $(0, w^*(x))$ is an equilibrium of (2). The associated eigenvalue problem of (4) around $(0, w^*(x))$ is

$$D_1 \Delta \left(\frac{\phi}{K_1}\right) + \phi\,(K - \mu w^*) = \sigma\phi(x) \text{ in } \Omega,\ \frac{\partial(\phi/K_1)}{\partial n} = 0 \text{ on } \partial\Omega. \tag{20}$$

The variational characterization [8] leads to the principal eigenvalue of (20)

$$\sigma_1 = \sup_{\phi\neq 0, \phi\in W^{1,2}} \frac{-D_1 \int\limits_\Omega |\nabla(\phi/K_1)|^2\,dx + \int\limits_\Omega \dfrac{\phi^2}{K_1}\,(K(x) - \mu w^*(x))\,dx}{\int\limits_\Omega (\phi^2/K_1)\,dx}.$$

Choosing the eigenfunction $\phi(x) = K(x)$ and for $0 < \mu < 1$, we obtain

$$\sigma_1 \geq \frac{\int\limits_\Omega (K(x) - \mu w^*(x))\, K(x)\, dx}{\int\limits_\Omega K(x)\, dx} > \frac{\int\limits_\Omega (K(x) - w^*(x))\, K(x)\, dx}{\int\limits_\Omega K(x)\, dx}. \tag{21}$$

since $\frac{K_1(x)}{K(x)} \equiv const$. Therefore, σ_1 is positive using (10) and the equilibrium $(0, w^*(x))$ is unstable. In particular, $w^*(x) \equiv K(x)$ if $\frac{K_2(x)}{K(x)} \equiv const$ and the equilibrium $(0, K(x))$ is also unstable. Here, in fact, the principal eigenvalue (21) becomes

$$\sigma_1 \geq \frac{\int\limits_\Omega (K(x) - \mu w^*(x))\, K(x)\, dx}{\int\limits_\Omega K(x)\, dx} = \frac{\int\limits_\Omega (1 - \mu)\, K^2(x)\, dx}{\int\limits_\Omega K(x)\, dx}.$$

Thus σ_1 is positive when $\mu \in (0, 1)$ and $K(x) > 0$, which concludes the proof. $\square$

Now we are ready to give the proof of the Theorem 2.

Proof of Theorem 2:

Proof. It can be checked that the system (2) satisfies all the conditions of a strong monotone dynamical system. The system (2) has no coexistence solution by Lemma 3 and trivial equilibrium $(0, 0)$ is unstable by Lemma 1. Moreover, $(0, w^*(x))$ is unstable according to Lemma 5 while $0 < \mu < 1$ and $\nu > 1$. Hence the equilibrium $(K(x), 0)$ is globally asymptotically stable as a consequence of Theorem 1. $\square$

Remark 2. *Assume that* $\dfrac{K_2(x)}{K(x)} \equiv const$ *and* $\mu > 1$, *then two equilibria* $(u^*(x), 0)$ *and* $(K(x), 0)$ *of (2) are unstable according to* $\dfrac{K_1(x)}{K(x)} \not\equiv const$ *and* $\dfrac{K_1(x)}{K(x)} \equiv const$, *respectively if* $\nu \in (0, 1)$. *Finally the semi-trivial equilibrium* $(0, K(x))$ *of (2) is globally asymptotically stable for any nonnegative nontrivial* $u_0, w_0 \in C(\overline{\Omega})$ *if* $\nu \in (0, 1)$.

If $K_1(x)$ and $K(x)$ are arbitrary whereas $K_2(x)$ and $K(x)$ are proportional then we have following two results prescribed in Propositions 2 and 3.

Proposition 2. *Assume that* $\dfrac{K_1(x)}{K(x)} \not\equiv const$, $\dfrac{K_2(x)}{K(x)} \equiv const$, *then the semi-trivial equilibrium* $(0, K(x))$ *of (2) is unstable if* $\mu \in (0, 1)$.

The proof is similar to Lemma 5 and thus is omitted.

Proposition 3. *Let* $\frac{K_1(x)}{K(x)} \not\equiv const$ *and* $\frac{K_2(x)}{K(x)} \equiv const$*. If* $\mu, \nu \in (0,1)$*, then the semi-trivial steady state* $(0, K(x))$ *of (2) is unstable.*

The proof of proposition 3 is fairly straightforward and thus is omitted.

If $\frac{K_i(x)}{K(x)} \not\equiv const$, $(i = 1,2)$, then both semi-trivial equilibria $(u^*(x), 0)$ and $(0, w^*(x))$ are stable whenever $\mu \in (0,1), \nu > 1$ and the coexistence solution is unstable. If $(u^*(x), 0)$ is an equilibrium of (2), then the corresponding eigenvalue problem of (2) around $(u^*(x), 0)$ is

$$D_2 \Delta \left(\frac{\psi}{K_2}\right) + \psi \left(K - \nu u^*\right) = \sigma \psi(x) \text{ in } \Omega, \ \frac{\partial(\psi/K_2)}{\partial n} = 0 \text{ on } \partial\Omega, \tag{22}$$

and we will apply it in the subsequent analysis.

Remark 3. *Assume that* $\frac{K_2(x)}{K(x)} \not\equiv const$*,* $\frac{K_1(x)}{K(x)} \equiv const$ *and* $\mu > 0$*, then the semi-trivial equilibrium* $(K(x), 0)$ *of (2) is unstable if* $\nu \in (0,1)$*.*

4.2 Case of weak competition

In a competitive system, one of the fundamental problems in ecology is to determine which species will survive. In this subsection, we will consider the problem (2) under weak competition to determine whether both species persist or one goes extinct.

The next result shows that one equilibrium point is instable if the competition coefficients, μ, $\nu \in (0,1)$ and the resource functions are proportional.

Proposition 4. *Assume that* $\frac{K_i(x)}{K(x)} \equiv const$*,* $(i = 1,2)$ *and* $\nu \in (0,1)$*, then the semi-trivial equilibrium* $(0, K(x))$ *of (2) is unstable if* $\mu \in (0,1)$*.*

Proof. Consider the eigenvalue problem (4) about $(0, K(x))$ using the boundary condition in (4) and we obtain

$$D_1 \Delta \left(\frac{\phi}{K_1}\right) + \phi \left(K - \mu K\right) = \sigma \phi(x) \text{ in } \Omega, \ \frac{\partial(\phi/K_1)}{\partial n} = 0 \text{ on } \partial\Omega. \tag{23}$$

According to the variational characterization of the eigenvalues [8], the principal eigenvalue of (23) is given by

$$\sigma_1 = \sup_{\phi \neq 0, \phi \in W^{1,2}} \frac{-D_1 \int\limits_\Omega |\nabla(\phi/K_1)|^2 \, dx + \int\limits_\Omega \frac{\phi^2}{K_1} \left(K(x) - \mu K(x)\right) \, dx}{\int\limits_\Omega (\phi^2/K_1) \, dx}.$$

The circumstance $\frac{K_i(x)}{K(x)} \equiv const$, $(i = 1, 2)$ implies that $\frac{K_2(x)}{K_1(x)} \equiv const$. Choosing the eigenfunction $\phi(x) = K_2(x)$ such that

$$\sigma_1 \geq \frac{\int\limits_\Omega (K(x) - \mu K(x))\, K_2(x)\, dx}{\int\limits_\Omega K_2(x)\, dx} > 0,$$

where $0 < \mu < 1$, $K(x) > 0$, $K_2(x) > 0$. This concludes the proof. $\square$

Proposition 5. *Assume that* $\frac{K_i(x)}{K(x)} \equiv const$, $(i = 1, 2)$ *and* $\mu \in (0, 1)$, *then the semi-trivial equilibrium* $(K(x), 0)$ *of (2) is unstable if* $\nu \in (0, 1)$.

Analogous to Proposition 4, it is easy to show that the equilibrium $(K(x), 0)$ is unstable.

Now, it is time to give the proof of the Theorem 3.

Proof of Theorem 3:

Proof. If $\mu, \nu \in (0, 1)$ then two semi-trivial equilibria $(0, K(x))$ and $(K(x), 0)$ are unstable according to Propositions 4 and 5, respectively. Also the trivial equilibrium is a repeller by Lemma 1. Since (2) is a strongly monotone dynamical system, therefore, as a consequence of Theorem 1 and Lemma 4 the unique coexistence solution $\left(\frac{(1-\mu)K}{1-\mu\nu}, \frac{(1-\nu)K}{1-\mu\nu}\right)$ is stable as far as $\mu, \nu \in (0, 1)$. $\square$

Remark 4. *If* $K_1 \not\equiv K_2$ *but* $\frac{K_i(x)}{K(x)} \equiv const$, $(i = 1, 2)$, *then the Theorem 3 is valid continually for* $0 < \mu, \nu < 1$.

Proposition 6. *Let* $\frac{K_1(x)}{K(x)} \not\equiv const$ *and* $\frac{K_2(x)}{K(x)} \equiv const$. *If* $\mu, \nu \in (0, 1)$, *then the semi-trivial steady state* $(u^*(x), 0)$ *of (2) is unstable.*

Proof. Notice that $\frac{K_1(x)}{K(x)} \not\equiv const$ implies that $(u^*(x), 0)$ is a semi-trivial equilibrium of (2). Consider the principal eigenvalue of (22) about $(u^*(x), 0)$ and we have

$$\sigma_1 = \sup_{\psi \neq 0, \psi \in W^{1,2}} \frac{-D_2 \int\limits_\Omega |\nabla(\psi/K_2)|^2\, dx + \int\limits_\Omega \frac{\psi^2}{K_2} (K(x) - \nu u^*(x))\, dx}{\int\limits_\Omega (\psi^2/K_2)\, dx}.$$

Since $\frac{K_2(x)}{K(x)} \equiv const$, then for $\psi(x) = K(x)$, we obtain

$$\sigma_1 \geq \frac{\int\limits_{\Omega} (K - \nu u^*)\, K(x)\, dx}{\int\limits_{\Omega} K(x)\, dx} > \frac{\int\limits_{\Omega} (K - u^*)\, K(x)\, dx}{\int\limits_{\Omega} K(x)\, dx}, \quad \text{since } 0 < \nu < 1,$$

and σ_1 is positive using (9) in Lemma 2. □

At this phase, we are in stable setting to establish the last Theorem 4 when both species are cooperating with each other.

Proof of Theorem 4:

Proof. Two equilibria $(0, K(x))$ and $(u^*(x), 0)$ of (2) are unstable as a consequence of the Propositions 3 and 6, respectively, due to $\frac{K_1(x)}{K(x)} \not\equiv const$, $\frac{K_2(x)}{K(x)} \equiv const$ and for $\mu, \nu \in (0, 1)$. Therefore, for a monotone dynamical system (2), the system has at least one stable coexistence solution according to the monotone properties of Theorem 1. □

5 Examples and applications

In this section, we will use the following mathematical notations to define the average solutions of u, w and K:

- Average solution of u is u_{ave},
- Average solution of w is w_{ave}, and
- Average solution of K is K_{ave}.

For numerical simulations, first, let us consider the case when the competition coefficients $\mu \in (0, 1)$, $\nu > 1$. The following example shows that in the case of space-dependent resource distribution, only the population u survives, coexistence solution being unstable.

Example 1. Consider the problem (2), where $\Omega = (0, 1)$, $\frac{K_1(x)}{K(x)} \equiv const$, $\frac{K_2(x)}{K(x)} \not\equiv const$, where $K(x) = 2 + \cos(\pi x)$, $K_1(x) = 0.3K(x)$, $K_2(x) = 1.5 + \cos(\pi x)$, and $D_1 = D_2 = 1$. If $\mu \in (0, 1)$, $\nu > 1$, then according to Theorem 2, the semi-trivial equilibrium $(K(x), 0)$ is globally asymptotically stable as $t \to \infty$. This result guarantees the survival of species u, similar results were established for random diffusion models in [20, 2, 24].

The numerical simulations show that in this situation, the average solution $(u_{ave}, w_{ave}) \to (K_{ave}, 0)$, see Fig. 2. However, when ν is large enough compared to μ, for example $\nu = 1.1$ and $\mu = 0.75$, then the convergence of u_{ave} tend to K_{ave} is faster than the other values. The figure represents that there is no coexistence, consistent with the Lemma 3 and that the equilibrium $(0, w^*(x))$ is unstable in accord with Lemma 5.

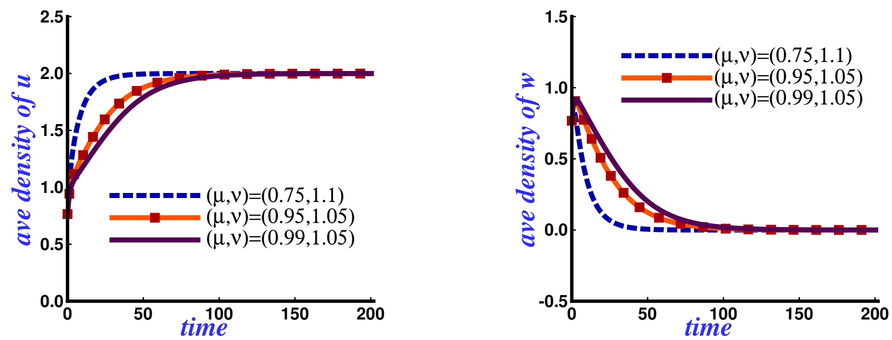


Figure 2: Average solutions of (2) for different functional values and for various (μ, ν).

We next consider various illustrations of the problem (2) in the following cases: (a) $\dfrac{K_1(x)}{K(x)} \not\equiv const$, $\dfrac{K_2(x)}{K(x)} \equiv const$ and (b) $\dfrac{K_i(x)}{K(x)} \not\equiv const$, $(i = 1, 2)$ by showing the effects of different competition coefficients, where $\mu \in (0, 1)$, $\nu > 1$.

Example 2. In Fig. 3, we have taken into account $\dfrac{K_1(x)}{K(x)} \not\equiv const$, $\dfrac{K_2(x)}{K(x)} \equiv const$ for $K(x) = 2+\cos(\pi x)$, $K_1(x) = 1.1+\cos(\pi x)$ and $K_2(x) = 0.7K(x)$. The parameters represent the effects applicable to the other species. Numerical simulations depicted in Fig. 3 indicates that the first population, u survives while the species, w goes extinct and coexistence is also possible. According to Proposition 2, the steady state $(0, K(x))$ is unstable and all positive solutions converge to the semi-trivial equilibrium $(u^*(x), 0)$ when μ is small enough compared to ν.

Next consider the state $\dfrac{K_i(x)}{K(x)} \not\equiv const$, $(i = 1, 2)$ in the logistic growth model (2). Different types of scenarios are seen in Fig. 4. When $\mu \approx 0.99 < 1$ and $\nu \approx 1.05 > 1$, then the semi-trivial steady state $(0, w^*(x))$ is globally asymptotically stable as $t \to \infty$. In all other cases in Fig. 4, the effects of species w on species u is less than the effects of species u on its own

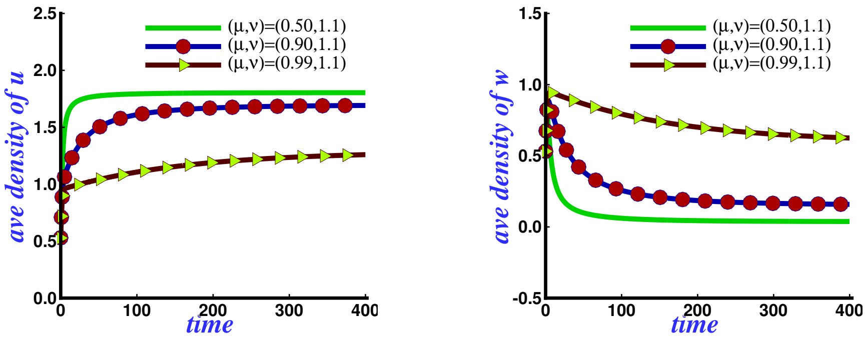


Figure 3: Average solutions of (2) for different set of functional values while (μ, ν) varies.

members and only the population u sustains in competition. Numerical results give evidence that there is no coexistence in these events.

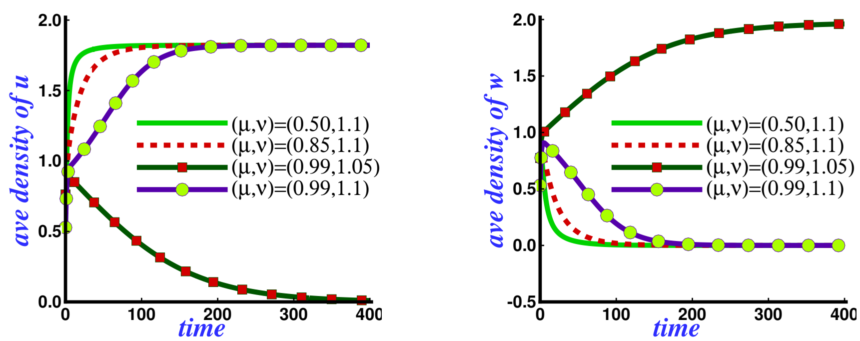


Figure 4: Average solutions of (2) for several parametric values of (μ, ν).

The next example investigated the small deviations between two functions $K(x)$ and $K_1(x)$ when either μ or ν is greater than one but close to 1 (approaches 1 from above) or equal to 1.

Example 3. Let us consider $\dfrac{K_1(x)}{K(x)} \not\equiv const$ and $|K_1(x) - K(x)| = \varepsilon$ on the domain $\Omega = (0, 1)$. Here $\varepsilon > 0$ is sufficiently small whereas $\dfrac{K_2(x)}{K(x)} \not\equiv const$ and choosing the values of competition coefficients $\mu = \nu = 1$ or using $\mu \downarrow 1,\ \nu \downarrow 1$, here (μ, ν) is eventually drop down to 1. In this example, we consider $K(x) = 2 + \cos(\pi x)$, $K_1(x) = 1.97 + 0.98\cos(\pi x)$, $K_2(x) = 1.9 + \cos(\pi x)$.

Fig. 5 illustrates average solutions in a time interval of length 400. In symmetric competition, the first population u is approximately choosing the carrying capacity dependent diffusion strategy because of small deviations between $K(x)$ and $K_1(x)$. It is well established that the semi trivial equilibrium $(K(x), 0)$ is globally asymptotically stable when $t \to \infty$ whenever $\mu = \nu = 1$, $K_1(x) = K(x)$ and $\frac{K_2(x)}{K(x)} \not\equiv const$ [19]. The left diagram in Fig. 5 illustrates that for $\mu = \nu = 1$, $\frac{K_2(x)}{K(x)} \not\equiv const$ and $K(x), K_1(x)$ chosen, there exists $\varepsilon > 0$ such that both populations coexist.

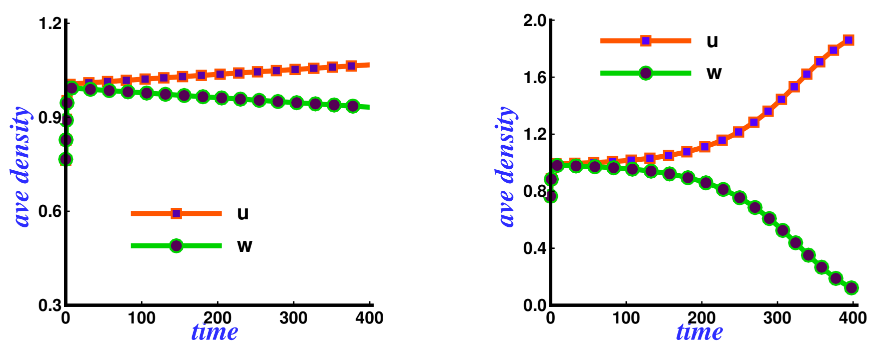
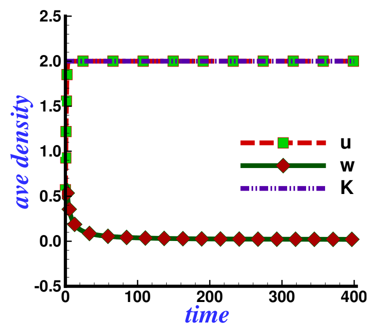


Figure 5: Average solutions of (2) for (left) $\mu = \nu = 1$ and (right) $\mu = \nu = 1.03$.

In Example 4, we show the limiting case of exact solution for proportional functions by choosing various $\mu, \nu \in (0, 1)$.

Example 4. If $\dfrac{K_i(x)}{K(x)} \equiv const$, $(i = 1, 2)$, then according to Theorem 3, there is a exact stable coexistence solution, and that is $(K_1, K_2) \equiv \left(\frac{(1-\mu)K}{1-\mu\nu}, \frac{(1-\nu)K}{1-\mu\nu}\right)$ for $\mu, \nu \in (0, 1)$. From Fig. 6 (left), it can be observed that Theorem 3 is no longer valid when $\mu \to 0$ and $\nu \to 1$. In Fig. 6 (left), it is apparent that when $t \to \infty$, the average density of u tends to the average density of K whereas $w_{ave} \to 0$. Moreover, if $\mu \to 0$ and $\nu \to 1$, then it can be easily checked that the exact solution $\left(\frac{(1-\mu)K}{1-\mu\nu}, \frac{(1-\nu)K}{1-\mu\nu}\right) \to (K, 0)$. The continuity of similar scenario is also observable in Fig. 6 (right) for $\mu \gtrapprox 0$ as well as for $\mu \lessapprox 1$ wherein $\nu = 0.5$ and $\dfrac{K_i(x)}{K(x)} \not\equiv const$, $(i = 1, 2)$. For fixed $\nu = 0.5$, the variational values of $\mu \in (0, 1)$ except close to one, Fig. 6 (right) shows that there is always a stable coexistence solution. The average density of species u is decreasing and converges to zero while w_{ave} tends to K_{ave} at time $t = 400$.



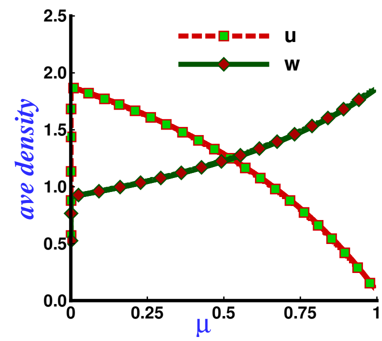


Figure 6: Average solutions of (2) for (left) $\mu = 0.001,\ \nu = 0.99$ and (right) $\nu = 0.5,\ \mu \in (0, 1)$ at $t = 400$.

The following example illustrates the accuracy of numerical solutions and explore how the average population densities depend on the relations between carrying capacities, $K(x),\ K_1(x)$ and $K_2(x)$ whenever $\mu, \nu \in (0, 1)$.

Example 5. Here we consider $\dfrac{K_i(x)}{K(x)} \equiv const,\ (i = 1, 2)$ in Fig. 7 (left) and see that there is an excellent agreement between numerical runs and exact solution $\left(\frac{(1-\mu)K}{1-\mu\nu}, \frac{(1-\nu)K}{1-\mu\nu}\right)$ for $\mu, \nu \in (0, 1)$.

In Fig. 7 (middle and right), we have $\dfrac{K_i(x)}{K(x)} \not\equiv const,\ (i = 1, 2)$, where $K(x) = 2 + \cos(\pi x),\ K_1(x) = 1.25 + \cos(\pi x),\ K_2(x) = 1.5 + \cos(\pi x)$ and different $\mu, \nu \in (0, 1)$. All possible types of stability are displayed in the two diagrams of Fig. 7 (right) depending on the numerical values of the competition coefficients. When μ is very close to 1 and $\nu = 0.5$, then the average rates of u is in extinction while the average rates of w is globally asymptotically stable as $t \to \infty$. Similarly, for $(\mu, \nu) = (0.05, 0.99)$, the population $u \to u^*$ and the second population w is very close to 0 (see similar behavior of μ, ν in Fig. 6). In all other cases of Fig. 7 (middle and right), the coexistence solution (u_s, w_s) is stable.

6 Conclusion

In this paper, we have studied a two species Lotka type competition model where two species are diffused according to their own carrying capacity and the environment has an individual resource function. We established the following results:

1. If the first species is adopted with the carrying capacity and the

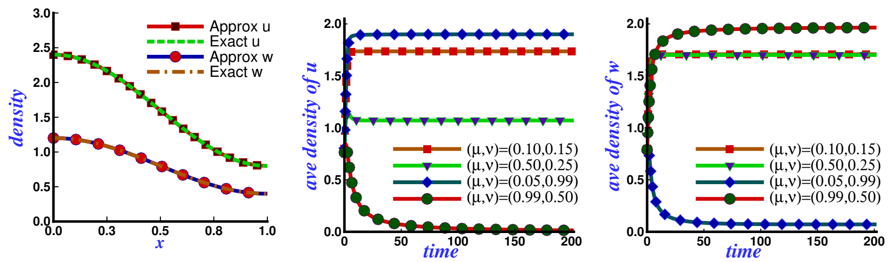


Figure 7: Solutions and average solutions of (2) for arbitrary functional values as long as $\mu, \nu \in (0, 1)$.

resources are shared only partially between the two organisms ($\mu < 1 < \nu$), then the second species goes extinct (see Fig. 2).

2. When both species share the resources with each other in a habitat then a unique coexistence solution exist provided that $\mu,\ \nu \in (0, 1)$ and $K_1 \equiv K_2$ with $K_i,\ (i = 1, 2)$ and K are rational, (see Fig. 7, left). In this case both populations are distributed ideally, maximize their fitness and make an ideal free pair [9, 10].

3. It is also proven that at least one stable coexistence solution exists when K_1 and K are arbitrary and $\mu,\ \nu \in (0, 1)$.

4. Even for weak competition, the coexistence solution is not guaranteed as shown in Fig. 7 (middle and right) as long as the selection of all functions is random.

We consider the case when one species dominates the other and also cooperates with each other. It appears that for strong competition, $\mu,\ \nu > 1$, the expected outcome is that two semi-trivial steady states are locally asymptotically stable but the proof and study of this issue is left for future study.

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