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An economic group model for innovation diffusion of new product with delay of adoption for low income group

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Abstract. In this paper, an economic group delay model is established. Dynamical behavior and Basic influence number of the proposed system are studied. Asymptotic stability analysis is carried out for the steady-states. The critical value of the delay τ is determined. It is observed that for the interior steady-state remains stable if the adoption delay for the low-income group is less than the threshold value, i.e., $\tau < \tau_0^+$. If τ crosses its threshold, system perceives oscillating behavior, and Hopf bifurcation occurs. Moreover, sensitivity analysis is performed for the system parameter used in the interior steady-state. Finally, numerical simulations are conducted to support our analytical findings.

Keywords: Boundedness, positivity, delay, Hopf bifurcation, sensitivity analysis.
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1 Introduction

Innovation diffusion is an unending activity that relies upon a network and the dispersion. It means the practice that transmits between the members of the social arrangement through a particular channel across time [20]. The

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most recent four decades have a significant focus on the individual adoption aspects of the latest innovative product, including its diffusion into a social framework [20,22]. Individuals had accepted that innovation in the product is affecting our society and nation which results that innovation is an important aspect of financial progression. Several researchers have employed an innovation dissipation method as the product extension [1,4,5,8,18,25]. Researchers have organized several investigations with different angles for innovation diffusion by using several mathematical models [25]. One can follow the superb analysis of these innovation diffusion models with their applications. Mahajan and Peterson presented a popular diffusion model where the potential is changing with the time [19]. In an epidemic model as the contamination spread from an association of the infected individual to premature individual. Thus, in that way, in the innovation diffusion process, the non-adopted person can become the user of innovation by the direct interaction with the adopted person [27]. Various phenomenal models and their experimental analysis were exhibited [9,10,15–17]. Researchers studied further the short-term trends of e-commerce [7]. The global stability of steady-states is studied in the innovation diffusion model for two and three competing products in the market [29]. It was the first attempt to identify the homoclinic bifurcation toward infinity [2,3]. KK Models parameters are more significant than Bass models, and it is examined that multinational parameters are significant, and the KK Models through various countries or areas affected, innovation coefficient, imitation coefficient, and market potential comparatively greater than parameters of Bass Model [6]. In Chinese communication market, a few researchers have investigated the contention and diffusion for two innovative products [21, 31]. Sensitivity analysis of the parameters used in the utility function is analyzed, and their outcomes demonstrate during the initial period, and the later period advertisement strategy is better, while the intermediate stage, the communication from the mouth will be more helpful [30]. The researcher analyzed the competition of three simultaneous innovations in the market at the same time [11]. The ideas of [11] originate from the past work in biology, ecology, sociology and technology innovation [13, 14, 19, 28].

From the above literature survey, we noticed that gigantically remarkable work is completed at the direction of the product expansion. However, still, there are so many issues which have not been resolved yet. Here we will discuss one of the most crucial issues in the potential market, i.e., financial issues. In this world, each has a distinctive paying limit. A few people can purchase the new product right now on the opposite side few are the individuals who cannot purchase the product in a split second. It is expected that they can buy the product, later on, when they remain financially stable. It means adoption of the product depends on the financial status. Some product meant for the high-income group, and some meant for the low-income group. In the previous investigations, it is observed that every non-adopter can turn into the adopter irrespective of the financial status. It means that all the products are for everybody. That is not practically possible in our society. In our society, there are two major income groups, the high-income group, and the low-income group. Some products are within reach of the low-income group, and some are out of reach. The classification of the income groups was missed in the previous investigations that is the major research gap. That is a crucial issue in the present market scenario for the growth of a product in the market. Keeping in mind the above research gap, we proposed an economic group model with the delay of adoption for the low-income group regarding the acceptance of the new product in the market. The resulting sections of the present study are an introduction, the proposed mathematical system, positivity, boundedness, basic influence number, stability analysis of the steady-states, numerical simulations and conclusions.

2 Representation of proposed model

The presumptions of the proposed model are given below:

(i) In an appropriate region, total population is classified into three different groups as an adopter of the innovation $A(t)$, non-adopter from high-income group $N_1(t)$ and non-adopter from low-income group $N_2(t)$.

(ii) It is assumed that with the progression of time financial status of the peoples may be varied and their group might be changed. Non-adopters form the high group can migrate to the lower-income group and similarly non-adopters from the low-income group can migrate to the high-income group. At this stage, non-adopters from the low-income group can adopt the new product.

(iii) It is supposed that the non-adopter belongs to the high-income group can become the user of the innovation rapidly as compared to the non-adopter belongs to the low-income group.

(iv) τ is the delay of adoption for non-adopter belongs to the lower income group (N_2), to become the user of innovation.

(v) Let r_1 and r_2 are the recruitment rate of N_1 and N_2, respectively. Further, α_1 is the interaction rate between N_1 and A; δ_1 is the interaction rate between N_2 and A; β_1 is the migration rate of non-adopter N_2 to N_1; γ_1 is the migration rate of non-adopter N_1 to N_2. Finally, μ is the natural death rate of N_1, A and N_2.

The proposed system is of the form:

$$\frac{dN_1}{dt} = r_1 - \alpha_1 N_1 A + \beta_1 N_2 - \gamma_1 N_1 - \mu N_1, \tag{1}$$

$$\frac{dA}{dt} = \alpha_1 N_1 A + \delta_1 N_2(t-\tau)A - \mu A, \tag{2}$$

$$\frac{dN_2}{dt} = r_2 + \gamma_1 N_1 - \beta_1 N_2 - \delta_1 N_2(t-\tau)A - \mu N_2, \tag{3}$$

with initial values: $N_1(0) > 0,\ A(0) > 0,\ N_2(0) > 0$ for all $t \geq 0$.

3 Positivity and Boundedness of proposed model

For the positivity along with boundedness of the proposed model equations (1)-(3), we state and evaluate the mentioned below lemmas.

Lemma 1. *The solution for the proposed model equations (1)-(3), with initial values are non-negative, for all $t \geq 0$.*

Proof. Let N_1, A, N_2 be the solution of proposed model equations (1)-(3), with positive initial populations. For $t \in [0, \tau]$, Eq. (1) can be expressed as

$$\frac{dN_1}{dt} \geq -\alpha_1 N_1 A - \gamma_1 N_1 - \mu N_1.$$

It follows that

$$N_1(t) \geq \ N_1(0)\exp\left\{-\int_0^t (\alpha_1 A + \gamma_1 + \mu)\, dv\right\} > 0.$$

Eq. (2), as $t \in [0, \tau]$, may be recast as $\frac{dA}{dt} \geq -\mu A$, which exhibits that

$$A(t) \geq \ A(0)\exp\left\{-\int_0^t (\mu)\, dv\right\} > 0.$$

Eq. (3), as $t \in [0, \tau]$, may be recast as

$$\frac{dN_2}{dt} \geq -\beta_1 N_2 - \delta_1 N_2 A - \mu N_2,$$

which exhibit that

$$N_2(t) \geq N_2(0)\exp\left\{-\int_0^t (\delta_1 A + \beta_1 + \mu)\, dv\right\} > 0.$$

From the above discussion, it is established that $N_1(t) > 0, A(t) > 0$ and $N_2(t) > 0$ for all $t \geq 0$. Hence the population densities of the non-adopters from high-income group, non-adopters from low-income group and adopters always remain positive. □

Lemma 2. *The solution for the proposed model equations (1)-(3), is bounded uniformly in ζ, where $\zeta = \{(N_1, A, N_2) : 0 \leq N_1(t) + A(t) + N_2(t) \leq \frac{r}{\mu}\}$.*

Proof. Let $W(t) = N_1(t) + A(t) + N_2(t)$. By differenting $W(t)$ with respect to t, we have

$$\frac{dW}{dt} = r_1 + r_2 - \mu(N_1 + A + N_2) = r - \mu W,$$

where $r = r_1 + r_2$. It follows that, $0 \leq W(t) \leq \frac{r}{\mu}$ as $t \to \infty$. Hence, $W(t)$ is bounded. Therefore, the proposed system is bounded. □

4 Steady states, basic influence number and dynamical behavior

The proposed model equations (1)-(3), have two feasible steady-states:

[i] The adopter-free steady-state $E_0(\frac{r_2\beta_1 + r_1(\mu+\beta_1)}{\mu(\mu+\beta_1+\gamma_1)}, 0, \frac{r_1\gamma_1 + r_2(\mu+\gamma_1)}{\mu(\mu+\beta_1+\gamma_1)})$ always exists.

[ii] The interior steady-state $E^*(N_1^*, A^*, N_2^*)$ exists, if H_1, H_2 and H_3 hold, where N_1^*, A^* and N_2^* are given by

$$\begin{cases} N_1^* = \frac{1}{2\mu\alpha_1(\alpha_1-\delta_1)}[\mu^2(\alpha_1 - \delta_1) - (r_1 + r_2)\alpha_1\delta_1 - \mu(\alpha_1\beta_1 + \gamma_1\delta_1) \\ \quad + \sqrt{4\mu^2\alpha_1(\alpha_1 - \delta_1)(\mu\beta_1 + r_1\delta_1) + (\mu(\mu + \gamma_1)\delta_1 + \alpha_1(\mu(-\mu + \beta_1) + (r_1 + r_2)\delta_1))^2}], \\ A^* = \frac{1}{2\mu\alpha_1\delta_1}[-\mu^2(\alpha_1 + \delta_1) + (r_1 + r_2)\alpha_1\delta_1 - \mu(\alpha_1\beta_1 + \gamma_1\delta_1) \\ \quad + \sqrt{4\mu^2\alpha_1(\alpha_1 - \delta_1)(\mu\beta_1 + r_1\delta_1) + (\mu(\mu + \gamma_1)\delta_1 + \alpha_1(\mu(-\mu + \beta_1) + (r_1 + r_2)\delta_1))^2}], \\ N_2^* = \frac{1}{2\mu\delta_1(-\alpha_1+\delta_1)}[\mu^2(-\alpha_1 + \delta_1) - (r_1 + r_2)\alpha_1\delta_1 - \mu(\alpha_1\beta_1 + \gamma_1\delta_1) \\ \quad + \sqrt{4\mu^2\alpha_1(\alpha_1 - \delta_1)(\mu\beta_1 + r_1\delta_1) + (\mu(\mu + \gamma_1)\delta_1 + \alpha_1(\mu(-\mu + \beta_1) + (r_1 + r_2)\delta_1))^2}]. \end{cases} \tag{4}$$

For the existence of the interior steady-state, H_1, H_2 and H_3 must hold. $H_1 : \alpha_1 > \delta_1$, i.e., if the the interaction rate between non-adopter from the high-income group and the adopter population is greater than the interaction rate between non-adopter from the low-income group and the adopter population.

$$H_2 : \frac{b_4 + b_6 - b_1 - b_2}{b_3} < R_A < \max\left\{\frac{b_1 + b_4 + b_6 - b_2}{b_3}, \frac{b_2 + b_4 + b_6 - b_5}{b_3}\right\},$$

i.e., if the basic influence number lies between the combination of these parameters. For the existence of the H_2 condition, H_3 must hold, i.e., $H_3 : b_4 + b_6 > b_1 + b_2$ and $b_2 + b_4 + b_6 > b_5$, where

$$\begin{aligned}
b_1 &= \mu^2(\alpha_1 - \delta_1), \quad b_2 = (r_1 + r_2)\alpha_1\delta_1, \quad b_3 = \frac{\mu^3(\mu + \beta_1 + \gamma_1)}{r_1 + r_2},\\
b_4 &= \frac{\mu^2(r_1\alpha_1 + r_2\delta_1)}{r_1 + r_2}, \quad b_5 = \mu^2(\alpha_1 + \delta_1),\\
b_6 &= \sqrt{4\alpha_1 b_1(\mu\beta_1 + r_1\delta_1) + (\mu(\mu + \gamma_1)\delta_1 + \alpha_1(\mu(-\mu + \beta_1) + b_2/\alpha_1))^2}.
\end{aligned}$$

Basic influence number is the noteworthy edge parameter that characterizes mathematical problems concerning the adoption of the new innovative product. Basic influence number is represented by R_A and identified as the required part of consequent adoption created by an adopted person which is correlated with basic reproduction number in epidemic models [24]. The advantage of this matrix is to state either the innovative products will spread to population or not. Assume that f and v are vectors to signify the new users by the personal impact of user population on the non-user population and enduring transfer terms, sequentially from model equations of user sections [12, 24]. Therefore when $\tau = 0$:

$$\frac{dA}{dt} = f - v,$$

where $f = (\alpha_1 N_1 A + \delta_1 N_2 A)$ and $v = (\mu A)$. Now by calculating the Jacobian of f and v, at the adopter-free steady-state E_0, we get

$$F = J(f) = \left(\ \alpha_1 N_1 + \delta_1 N_2\ \right) = \frac{(r_1 + r_2)(\beta_1\alpha_1 + \gamma_1\delta_1) + \mu(r_1\alpha_1 + r_2\delta_1)}{\mu(\mu + \beta_1 + \gamma_1)},$$

and $V = J(v) = \left(\ \mu\ \right)$, $\quad V^{-1} = \left(\ 1/\mu\ \right)$, where F is positive, and the matrix V is non-singular. Therefore, the next generation matrix $K = FV^{-1}$ is positive. The basic influence number is given by

$$R_A = \rho(FV^{-1}) = \frac{(r_1 + r_2)(\beta_1\alpha_1 + \gamma_1\delta_1) + \mu(r_1\alpha_1 + r_2\delta_1)}{\mu^2(\mu + \beta_1 + \gamma_1)}, \qquad (5)$$

which is the largest eigenvalue of FV^{-1}. Here, we will study the stability behavior of the system in terms of R_A.

Now, the local behavior of distinct steady-states of the proposed model (1)-(3), is as follows:

Theorem 1. *If $R_A < 1$ holds, then adopter-free steady-state*

$$E_0\left(\frac{r_2\beta_1 + r_1(\mu+\beta_1)}{\mu(\mu+\beta_1+\gamma_1)}, 0, \frac{r_1\gamma_1 + r_2(\mu+\gamma_1)}{\mu(\mu+\beta_1+\gamma_1)}\right), \tag{6}$$

is asymptotically stable for all τ.

Proof. The variational matrix corresponding to model equation(1)-(3), is given by

$$J = \begin{pmatrix} -\mu - A\alpha_1 - \gamma_1 & -N_1\alpha_1 & \beta_1 \\ \alpha_1 A & -\mu + N_1\alpha_1 + N_2\delta_1 & A\delta_1 e^{-\lambda\tau} \\ \gamma_1 & -N_2\delta_1 & -\mu - \beta_1 - A\delta_1 e^{-\lambda\tau} \end{pmatrix}.$$

Now, the characteristic equation for the adopter-free steady-state (6) is given by $(\lambda+\mu)(\lambda+\mu+\beta_1+\gamma_1)(\lambda+\mu-N_1\alpha_1-N_2\delta_1) = 0$. From this characteristic equation we get $\lambda = -\mu, -(\mu+\beta_1+\gamma_1), N_1\alpha_1 + N_2\delta_1 - \mu$. For locally asymptotically stable, all the values of λ should be negative. From the above discussion first two values of λ are negative and third value is conditionally negative. The third value of the λ is negative if $N_1\alpha_1 + N_2\delta_1 < \mu$, i.e.,

$$\frac{(r_1+r_2)(\beta_1\alpha_1 + \gamma_1\delta_1) + \mu(r_1\alpha_1 + r_2\delta_1)}{\mu^2(\mu+\beta+\gamma_1)} < 1.$$

It shows that $R_A < 1$. Hence, it is established that adopter-free steady-state is asymptotically stable if the basic influence number is less than one. □

Now, the second-degree transcendental polynomials equation with delay as follows

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau} = 0, \tag{7}$$

has been studied [23,26] wherein, the following results have been discussed:

[B1] $p + s > 0$;

[B2] $q + r > 0$;

[B3] either $s^2 - p^2 + 2r < 0$ and $r^2 - q^2 > 0$ or $(s^2 - p^2 + 2r)^2 < 4(r^2 - q^2)$;

[B4] either $r^2-q^2<0$ or $s^2-p^2+2r>0$ and $(s^2-p^2+2r)^2=4(r^2-q^2)$;

[B5] either $r^2-q^2>0$, $s^2-p^2+2r>0$ and $(s^2-p^2+2r)^2>4(r^2-q^2)$.

Lemma 3. *[i] If [B1]-[B3] hold, entire roots of Eq.* (7) *possess negative real parts for all* $\tau \geq 0$.

[ii] If [B1], [B2] and [B4] hold, also $\tau=\tau_j^+$, *then Eq.* (7) *possess a pair of purely imaginary roots* $\pm iw_+$. *When* $\tau=\tau_j^+$ *then entire roots of* (7) *except* $\pm iw_+$ *possess negative real parts.*

[iii] If [B1], [B2] and [B5] hold, and $\tau=\tau_j^+$ *(*$\tau=\tau_j^-$ *respectively) then Eq.* (7) *possess a pair of purely imaginary roots* $\pm iw_+$ *(*$\pm iw_-$*, respectively). Moreover* $\tau=\tau_j^+$ *(*τ_j^-*, respectively),then entire roots of Eq.* (7) *except* $\pm iw_+$ *(*$\pm iw_-$*, respectively) possess negative real parts.*

Theorem 2. *For the proposed model (1)-(3), we have:*

(i) The interior steady-state $E^*(N_1^*, A^*, N_2^*)$ *is always locally asymptotically stable for all* $\tau=0$.

(ii) If H_4 *and* H_5 *hold, then interior steady-state* $E^*(N_1^*, A^*, N_2^*)$ *is locally asymptotically stable for all* $\tau\in[0,\tau_0^+)$ *and unstable when* $\tau \geq \tau_0^+$.

Proof. The characteristic equation for steady-state $E^*(N_1^*, A^*, N_2^*)$ can be formulated as: $(\lambda+\mu)F(\lambda)=0$. From this equation $\lambda=-\mu$ and

$$\begin{aligned} F(\lambda) \;=\; & \lambda^2+(\mu+A^*\alpha_1+\beta_1+\gamma_1)\lambda+(A^*\mu\alpha_1+A^*\alpha_1\beta_1-A^*N_2^*\alpha_1\delta_1) \\ & +(A^*\delta_1\lambda+A^{*2}\alpha_1\delta_1+A^*\gamma_1\delta_1+A^*N_2^*\delta_1^2)e^{-\lambda\tau}. \qquad (8) \end{aligned}$$

From Eq. (8), $F(\lambda)$ can be represented as

$$\lambda^2+p\lambda+r+(s\lambda+q)e^{-\lambda\tau}=0, \qquad (9)$$

where $p=\mu+A^*\alpha_1+\beta_1+\gamma_1$, $r=A^*\mu\alpha_1+A^*\alpha_1\beta_1-A^*N_2^*\alpha_1\delta_1$, $s=A^*\delta_1$, $q=A^{*2}\alpha_1\delta_1+A^*\gamma_1\delta_1+A^*N_2^*\delta_1^2$.

Case I: If there is no delay of adoption to adopt the product for the non-adopters belong to the low-income group, i.e., $\tau=0$, the transcendental equation (9) becomes

$$\lambda^2+(p+s)\lambda+(q+r)=0. \qquad (10)$$

Let $C_1=p+s=\mu+A^*\alpha_1+\beta_1+\gamma_1+A^*\delta_1>0$, $C_2=q+r=A^*[\delta_1(A^*\alpha_1+\gamma_1+N_2^*\delta_1)+\alpha_1(\alpha_1N_1^*+\beta_1)]>0$, which shows $C_1>0, C_2>0$. With the help of Routh-Hurwitz method, entire roots of the equation (10), have

negative real parts. Therefor the interior steady-state E^* is always locally asymptotically stable.
Case II: If there is delay of adoption to adopt the product for the non-adopters belong to the low-income group, i.e., $\tau > 0$.

By using Lemma 3, the conditions [B1], [B2] and [B4] hold, if H_4 and H_5 exist, where

$$H_4 : \alpha_1 < \frac{\delta_1(\mu + \gamma_1)}{\beta_1 + N_1^*(\gamma_1 + \delta_1)},$$

i.e., if the rate of interaction between adopter and non-adopter from the high-income group is less than the ratio between the rest of the combination of the parameters and

$$H_5 : \delta_1^2 > \frac{1}{A^{*2}}[(\mu + A^*\alpha_1)^2 + (\beta_1 + \gamma_1)^2 + 2\gamma_1(\mu + A^*\alpha_1 + \beta_1) + 2A^*\alpha_1^2 N_1^*],$$

i.e., if the square of the rate of interaction between the adopter and non-adopter from the low-income group is greater than the ratio between the rest of the combination of the parameters. Hence, the proposed model equations (1)-(3), posses a pair of purely imaginary roots.

By substituting $\lambda = iw$ in Eq. (9), we have

$$(iw)^2 + p(iw) + r + (iws + q)e^{-iw\tau} = 0. \tag{11}$$

Comparing real and imaginary parts from Eq. (11), we have

$$-w^2 + r + sw\sin w\tau + q\cos w\tau = 0, \tag{12}$$

$$pw + sw\cos w\tau_1 - q\sin w\tau = 0. \tag{13}$$

On simplifying Eqs. (12) and (13), we get

$$\sin w\tau = \frac{sw^3 + (pq - rs)w}{s^2w^2 + q^2}, \tag{14}$$

$$\cos w\tau = \frac{(q - ps)w^2 - qr}{s^2w^2 + q^2}, \tag{15}$$

and

$$w^4 + (p^2 - 2r - s^2)w^2 + (r^2 - q^2) = 0. \tag{16}$$

Let $F(w) = w^4 + (p^2 - 2r - s^2)w^2 + (r^2 - q^2) = 0$. With Descarte's rule concerning the sign, there is at least one positive root of $F(w) = 0$. Suppose w_0 is the non-negative root of $F(w) = 0$. From Eq. (15), we have

$$\tau_k^+ = \frac{1}{w_0}\left[\cos^{-1}\left(\frac{(q - ps){w_0}^2 - qr}{s^2{w_0}^2 + q^2}\right) + 2k\pi\right],$$

where $k = 0, 1, 2, \ldots$.

As for the occurrence concerning Hopf bifurcation at τ_0^+, it is mandatory that the transversely situation

$$Re\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_0^+} \neq 0,$$

should exist. Therefore taking the derivative of λ with respect to τ in Eq. (9), we get

$$\frac{d\lambda}{d\tau} = \frac{\lambda(s\lambda + q)e^{-\lambda\tau}}{2\lambda + p + se^{-\lambda\tau} - (s\lambda + q)\tau e^{-\lambda\tau}}.$$

Put $\lambda = iw_0$ and $\tau = \tau_0^+$, we have

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{qG - sw_0 H}{w_0(q^2 + s^2 w_0^2)}, \tag{17}$$

where $G = p \sin w_0\tau_0 + 2w_0 \cos w_0\tau_0$ and $H = s + p \cos w_0\tau_0 - 2w_0 \sin w_0\tau_0$.

On analyzing Eq. (17), we have

$$Re\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_0^+} \neq 0,$$

if $qG \neq sw_0 H$. □

Further, with the help of Lyapunov direct method, we have studied the global stability for the proposed system (1)-(3), and constructed the positive definite function:

$$V(N_1, A, N_2) = \frac{\alpha_a}{2}(N_1 - N_1^*)^2 + \frac{\alpha_b}{2}(A - A^*)^2 + \frac{\alpha_c}{2}(N_2 - N_2^*)^2,$$

where α_a, α_b and α_c are positive constants. These constants are so chosen that the condition $V'(N_1, A, N_2)$ become negative definite. In such a situation, the local stability of the equilibrium $E^* = (N_1^*, A^*, N_2^*)$ become globally stable in a particular region of concern.

5 Sensitivity analysis

In this section, sensitivity analysis of state variables for the proposed system (1)-(3), concerning with context of the model parameters at the interior steady-state has been accomplished. The corresponding forward sensitive indices of the state variable at the interior steady-state are shown in Table

Table 1: The sensitivity indices $\gamma_{y_k}^{x_m} = \frac{\partial x_m}{\partial y_k} \times \frac{y_k}{x_m}$ of the state variables for the proposed system (1)−(3), to the parameters y_k for the parameter values $r_1 = 0.09, \alpha_1 = 0.1, \beta_1 = 0.06, r_2 = 0.085, \gamma_1 = 0.012, \delta_1 = 0.09, \mu = 0.08$.

Parameter (y_k)	$\gamma_{y_k}^{N_1^*}$	$\gamma_{y_k}^{A^*}$	$\gamma_{y_k}^{N_2^*}$
r_1	0.37	1.12	-0.35
α_1	-0.66	0.64	-0.32
β_1	0.13	-0.005	-0.130
r_2	-0.39	1.20	0.37
γ_1	-0.13	-0.004	1.23
δ_1	-0.36	0.68	-0.66
μ	1.03	-3.65	0.96

1, by using the parameter values $r_1 = 0.09$, $\alpha_1 = 0.1$, $\beta_1 = 0.06$, $r_2 = 0.085$, $\gamma_1 = 0.012$, $\delta_1 = 0.09$, $\mu = 0.08$. It is established that μ is the extremely sensitive parameter to N_1^* and A^* as compared to the rest of the parameters. Hence, the significant change in N_1^* and A^* is observed with a small variation in the parameter μ. Further, γ_1 is the most sensitive parameter as contrasted to the other parameters for N_2^*.

6 Numerical simulations

In this section, we exhibit the numerical simulations to enhance our comprehension of the model. To support the analytic outcomes of the model (1)-(3), numerical simulations are achieved by implementing in MATLAB. It gives an impression of fulfillment to the systematic ends. The parameter values for the Sets (1)-(4) are shown in the following Table 2, to describe the Figures (1)-(6).

7 Conclusions

In this paper, an economic group model with the delay of adoption for the low-income group is proposed. Positivity, Boundedness and Basic influence number are analyzed for the proposed system. Asymptotic stability behavior is examined for the adopter-free and interior steady-states. The adopter-free steady-state E_0 is stable if the basic influence number is less than one, i.e., $R_A < 1$. It signifies that there is no adoption of the product in the market by the people only non-adopter population will exist in this

Table 2: Parameter values for different sets.

Parameters	Set-1	Set-2	Set-3	Set-4
r_1	0.1	0.1	0.09	0.09
α_1	0.015	0.018	0.1	0.1
β_1	0.012	0.013	0.06	0.06
r_2	0.07	0.08	0.085	0.085
γ_1	0.01	0.015	0.012	0.012
δ_1	0.012	0.025	0.09	0.09
μ	0.06	0.05	0.08	0.08
τ	–	0	28	31.2
τ_0^+	–	–	30.97	30.97

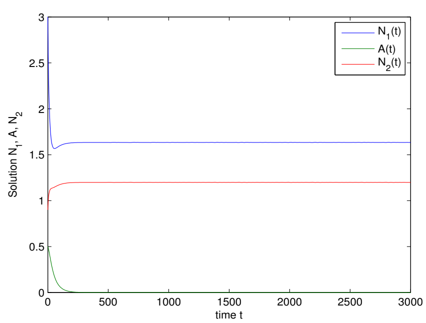

Figure 1: The adopter-free steady-state $E_0(1.63, 0, 1.20)$ is stable for the parameter values mentioned in the Set-1.

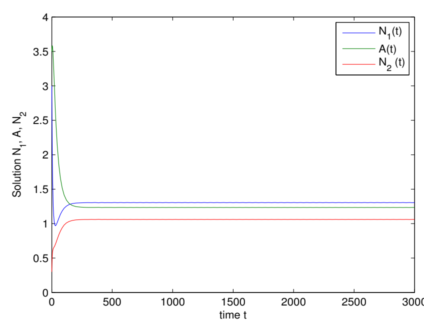

Figure 2: The interior steady-state $E_0(1.30, 1.23, 1.06)$ at $\tau = 0$ is stable for the parameter values mentioned in the Set-2.

circumstance. For an interior steady-state, E^* is always stable if there is no delay in adoption to adopt the product, i.e., $\tau = 0$ for the non-adopters belong to the low-income group. It demonstrates that the adoption remains constant in the market when there is no delay in adoption. Further, if there is the delay of adoption to adopt the product for the non-adopters belong to the low-income group, i.e., $\tau > 0$, then the interior steady-state E^* is stable for a specific threshold parameter $\tau \in [0, \tau_0^+)$. When τ crosses the threshold parameter, i.e., $\tau > \tau_0^+$ interior steady-state is the unstable and the oscillatory character occurs, i.e., Hopf bifurcation. It establishes that adoption is fluctuating in the market when the delay of the adoption is more than the threshold parameter. Sensitivity analysis for the interior steady-

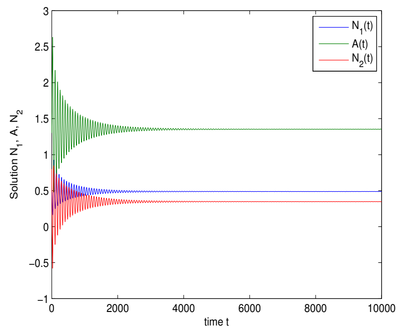

Figure 3: The interior steady-state with delay $E_0(0.49, 1.35, 0.35)$ is stable, at $\tau < \tau_0^+$ for the parameter values mentioned in the Set-3.

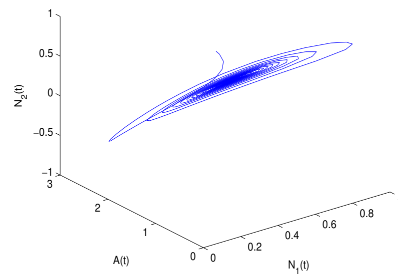

Figure 4: The interior steady-state with delay $E_0(0.49, 1.35, 0.35)$ is stable, at $\tau < \tau_0^+$ for the parameter values mentioned in the Set-3.

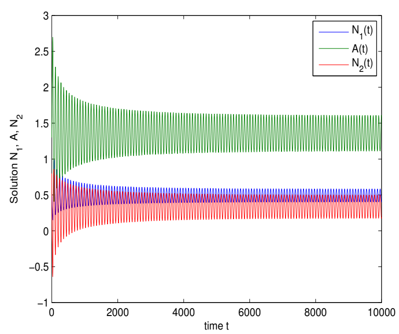

Figure 5: The interior steady-state with delay $E_0(0.49, 1.35, 0.35)$ is unstable, and Hopf bifurcation occurs at $\tau > \tau_0^+$ for the parameter values mentioned in the Set-4.

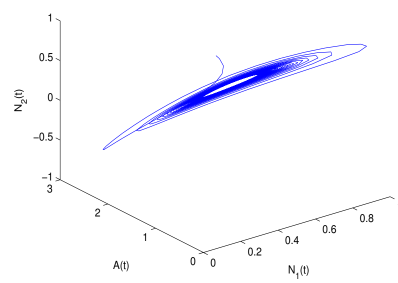

Figure 6: The interior steady-state with delay $E_0(0.49, 1.35, 0.35)$ is unstable, and Hopf bifurcation occurs at $\tau > \tau_0^+$ for the parameter values mentioned in the Set-4.

state is also studied. It is observed that the natural death rate is the most sensitive parameter with the positive effect in N_1^* and with a negative in the A^*. Furthermore, the migration rate of non-adopter from high-income group to low income-group is the most delicate parameter in N_2^*. Finally, to justify the analytical investigation, we perform the numerical simulations with the different set of parameters.

From the marketing viewpoint, our analysis has the following key features:
(i) Identifies the combination of parameters for which the growth of the product remains stable in the market.
(ii) Explore the state when the adoption is fluctuating (i.e., Hopf bifurcation).
(iii) Identifies the controlling parameters to increase the growth of the product in the market.
Keeping in view the economic status of the population in the region an organization or specialist can improve their strategy before launching the product in the market. Hence, the adoption will expand in the competitive market. As a result, the product will stay for a long time.

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