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Rationalized Haar wavelet bases to approximate the solution of the first Painlevé equations

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Abstract. In this article, using the properties of the rationalized Haar (RH) wavelets and the matrix operator, a method is presented for calculating the numerical approximation of the first Painlevé equations solution. Also, an upper bound of the error is given and by applying the Banach fixed point theorem the convergence analysis of the method is stated. Furthermore, an algorithm to solve the first Painlevé equation is proposed. Finally, the reported results are compared with some other methods to show the effectiveness of the proposed approach.

Keywords: Wave equation, first Painlevé equation, Volterra integral equation, RH wavelet
AMS Subject Classification: 34A34, 65L05.

1 Introduction and preliminaries

One of the best group of equations that discovered about 100 years ago is the six Painlevé equations ($P_I - P_{VI}$) applied in theoretical physics, modeling of the electric field in a semiconductor, quantum spin models, scattering theory, self-similar solutions to nonlinear dispersive wave PDEs,

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string theory, and random matrix theory; see [3, 6, 10, 11, 14, 16]. In this paper, we study the first Painlevé equation as follows:

$$\frac{\partial^2 w}{\partial x^2} = 6w^2(x) + x, \tag{1}$$

with the initial conditions $w(0) = 0$ and $w'(0) = 1$.

There exist some transformations to the Painlevé equation (see [11, 12] and the references therein). The following expresses some cases in which the first Painlevé equation (P_I) is equal to Korteweg-de Vries (KdV) and Cylindrical Korteweg-de Vries (CKdV) equations; see [18]. Several approaches are used to compute the numerical solution of this equation such as the variational iteration method (VIM) [5], homotopy perturbation method (HPM) [13], collocation method (CM) [7], Adomian's decomposition method (ADM), modified Adomian's decomposition method (MADM), modified variational iteration method (MVIM), modified homotopy perturbation method (MHPM), and homotopy analysis method (HAM) [2].

Motivated by the former discussion, in this paper, we are going to solve the first Painlevé equation by the rationalized Haar (RH) wavelet. We transform the equation to an integral equation and then solve the integral equation using the RH wavelet. Recently, the solution of many problems have been approximated by the RH wavelet (see [1,4,8,9] and the references therein). Now, consider the first Painlevé equation (1). By integrating two times from the equation with respect to x and using the initial conditions, we obtain

$$w(x) = x + \frac{1}{6}x^3 + 6\int_0^x \int_0^x w^2(x)dxdx,$$

and by using the initial conditions, we obtain

$$\int_0^x \int_0^x w^2(x)dxdx = \int_0^x (x-t)w^2(t)dt,$$

implies

$$w(x) = x + \frac{1}{6}x^3 + 6\int_0^x (x-t)w^2(t)dt. \tag{2}$$

If

$$Z(x,t,w(t)) = 6(x-t)w^2(t), \quad f(x) = x + \frac{1}{6}x^3,$$

then Eq. (2) yields

$$w(x) = f(x) + \int_0^x Z(x,t,w(t))dt, \tag{3}$$

the unknown function $w : [0,1] \to \mathbb{R}$ to be determined. Also, $f : [0,1] \to \mathbb{R}$ and $Z : [0,1]^2 \times \mathbb{R} \to \mathbb{R}$ are a known continuous functions.

2 Properties of the RH functions

Definition 2.1. The Haar wavelet family are defined on subintervals of $[0, 1)$, as follows:

$$h_i(x) = \begin{cases} 1, & t \in [\alpha_1, \alpha_2), \\ -1, & t \in [\alpha_2, \alpha_3), \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $i = 2, 3, \ldots, 2^{\lambda+1}$ and λ is a maximal level of the resolution and

$$\alpha_1 = \frac{k}{m}, \qquad \alpha_2 = \frac{k+0.5}{m}, \qquad m = 2^r, \quad r = 0, 1, \ldots, \lambda,$$

$$\alpha_3 = \frac{k+1}{m}, \quad k = 0, 1, \ldots, 2^r - 1.$$

Also, k is a translation parameter.

The relationship between i, m and k is $i = m + k + 1$. By deleting the irrational numbers and introducing the integral powers of two, Lynch and Reis in [15] introduced the RH transformation, that maintains all of the properties of the original Haar function. The orthogonal set of Haar functions is a group of square waves with a magnitude of $+2^{r/2}$, $-2^{r/2}$ and 0, but the RH function are composed of only three values of $+1$, -1, and 0, so we have the following.

Definition 2.2. *In the real number line $\mathbb{R}$ the RH wavelet function is defined as*

$$\mathrm{RH}(x) = \begin{cases} 1, & 0 < x \leq \frac{1}{2}, \\ -1, & \frac{1}{2} < x < 1, \\ 0, & \text{elsewhere}. \end{cases}$$

In general, for $n = 2^r + k$, RH wavelet is defined as

$$h_n(x) = RH(2^r x - k), \qquad r, k \in \mathbb{N} \cup \{0\}, \qquad k = 0, 1, \ldots, 2^r - 1.$$

Furthermore, the integral of $h_n(x)$ is given by

$$\int_0^1 h_n(x)dx = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Since the sequence $\{h_n\}_{n=0}^{\infty}$ is a complete orthogonal system in $L^2[0, 1]$, we have

$$\langle h_n(t), h_i(t) \rangle = \int_0^1 h_n(t) h_i(t) dt = \begin{cases} 1/m, & n = i, \\ 0, & n \neq i. \end{cases}$$

If $f \in C[0,1]$, then the series $f(t) = \sum_n 2^r \langle f, h_n \rangle h_n$ converges uniformly to f, where (see [19])

$$\langle f, h_n \rangle = \int_0^1 f(t) h_n(t) dt.$$

Also, the expansion of any $f \in C[0,1]$ by the RH function can be written as

$$f(x) = \sum_{i=0}^{m-1} l_i h_i(x) = \mathrm{C}^{\mathrm{T}} \mathrm{H}(x),$$

where vectors Λ and H are defined by

$$\Lambda = [l_0, l_1, \ldots, l_{m-1}]^T, \qquad \mathrm{H} = [h_0(x), h_1(x), \ldots, h_{m-1}(x)]^T.$$

Moreover, the RH function coefficients c_n are given by

$$l_0 = \int_0^1 f(x) h_0(x) dx, \qquad l_i = 2^r \int_0^1 f(x) h_i(x) dx.$$

Also, we have

$$\int_0^x \mathrm{H}(t) dt = \mathrm{PH}(x),$$

where P is an $m \times m$ operational matrix for integrating which is defined as

$$\mathrm{P}_{m\times\ m} = \frac{1}{2m} \begin{pmatrix} 2m\mathrm{P}_{\frac{m}{2}\times\frac{m}{2}} & -\widehat{\Phi}_{\frac{m}{2}\times\frac{m}{2}} \\ \widehat{\Phi}^{-1}_{\frac{m}{2}\times\frac{m}{2}} & 0 \end{pmatrix},$$

wherein $\Phi_{1\times1} = [1]$, $\mathrm{P}_{1\times1} = [\frac{1}{2}]$, and $\widehat{\Phi}_{m\times m}$ is given by

$$\widehat{\Phi}_{m\times m} = [H(\frac{1}{2m}), \mathrm{H}(\frac{3}{2m}), \ldots, \mathrm{H}(\frac{2m-1}{2m})].$$

The matrix form of the first eight RH functions as follows:

$$\hat{\Phi}_{8\times8} = \begin{pmatrix} \mathrm{h}_0(t) \\ \mathrm{h}_1(t) \\ \vdots \\ \mathrm{h}_7(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Also, generally we have

$$\widehat{\Phi}^{-1}_{m\times m} = \frac{1}{m} \widehat{\Phi}^T_{m\times m} \mathrm{diag}(1, 1, 2, 2, \underbrace{2^2, \ldots, 2^2}_{2^2}, \underbrace{2^3, \ldots, 2^3}_{2^3}, \ldots, \underbrace{\frac{m}{2}, \ldots, \frac{m}{2}}_{\frac{m}{2}}).$$

3 Numerical approximation of the solution

In this section, we propose an approach for computing the solution. In fact, we apply the successive approximations method for obtaining the approximate solution of (3), with initial condition $u_0 \in C[0,1]$. This iterative process is stopped as soon as an approximate solution with a prescribed accuracy is obtained. Thus for any $x, t \in [0,1]$, $r \geq 1$, and $m = 2^{r+1} \in \mathbb{N}$, we define the following recursive relation:

$$\psi_r(x,t) := Z(x,t,w_{r-1}(t)).$$

If Q_m be the orthogonal projection

$$Q_m(\psi)(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{ij} h_i(x) h_j(t),$$

by using the interpolation property we have

$$Q_m(\psi)(x,t) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} f_{ij} h_i(x) h_j(t). \tag{5}$$

It can be shown that the Eq. (5) can be written in the matrix form as

$$Q_m(\psi)(x,t) = \mathrm{H}^T(x)\mathrm{F}\mathrm{H}(t),$$

wherein $\mathrm{F} = [f_{lq}]_{m\times\ m}$ and

$$f_{lq} = 2^{\frac{i+j}{2}}\ \langle h_l(x), \langle \psi(x,t),\ h_q(t)\rangle\rangle,$$

for $i, j = 0, 1, \ldots,$ where

$$\begin{aligned} l &= 2^j + k, \qquad k = 0, 1, \ldots, 2^j - 1,\\ q &= 2^i + k', \qquad k' = 0, 1, \ldots, 2^i - 1. \end{aligned}$$

Now, by utilizing the RH function vector $\mathrm{H}(t)$ and the definition of the matrix $\widehat{\boldsymbol{\Phi}}_{m\times m}$, we get

$$\mathrm{F} = (\widehat{\Phi}^{-1}_{m\times m})^T \widehat{\mathrm{F}} (\widehat{\Phi})^{-1}_{m\times m},$$

where $\widehat{\mathrm{F}} = [\hat{f}_{ij}]_{m\times m}$. In fact, for $i, j = 1, 2, \ldots, m$, $\hat{f}_{ij}$ is defined as

$$\hat{f}_{ij} = \psi(\ \frac{2i-1}{2m},\ \frac{2j-1}{2m}), \qquad i, j = 1, 2, \ldots, m.$$

Thus, for the integral equation (3), we have

$$w_r(x) := f(x) + \int_0^x Q_m(\psi_{r-1})(x,t)dt, \quad r = 1, 2, \ldots. \tag{6}$$

4 Error analysis

In this section, by using the Banach fixed point theorem, we obtain an upper bound for the error of the proposed method. Also, the order of the convergence is analyzed. To do so, we need to introduce the integral operator $T : (C([0,1]), \|\cdot\|_\infty) \to (C([0,1]), \|\cdot\|_\infty)$, and so for Eq. (3) we have

$$Tw(x) = f(x) + \int_0^x Z(x,t,w(t))dt. \tag{7}$$

Lemma 4.1. *Let $Z : [0,1]^2 \times \mathbb{R} \to \mathbb{R}$, be a continuous and Lipschitzian function with the Lipschitz constant L, namely*

$$|Z(x,t,w_1(t)) - Z(x,t,w_2(t))| \leq L|w_1(t) - w_2(t)|,$$

Then T has a unique fixed point and for all $w_0 \in C([0,1])$,

$$\|w - T^r(w_0)\|_\infty \leq \|T(w_0) - w_0\|_\infty \times \sum_{j=r}^{\infty} L^j, \tag{8}$$

where $L < 1$ and w is the fixed point of T.

Proof. If $w_1(x), w_2(x) \in C([0,1])$, then we have

$$\begin{aligned} |Tw_1(x) - Tw_2(x)| &= \left| \int_0^x (Z(x,t,w_1(t)) - Z(x,t,w_2(t))dt \right| \\ &\leq \int_0^x |Z(x,t,w_1(t)) - Z(x,t,w_2(t))dt|dt \\ &\leq L \int_0^x |w_1(t) - w_2(t)|dt \leq L\|w_1 - w_2\|_\infty. \end{aligned}$$

Induction on $n \in \mathbb{N}$ gives $|\ T^n(w_1) - T^n(w_2)|_\infty \leq L^n\|w_1 - w_2\|_\infty$. Now, since $L < 1$, so we get $\sum_{n=1}^{\infty} \|T^n(w_1) - T^n(w_2)\|_\infty < \infty$. Thus, the integral equation (3) has a unique solution and the inequality (8) follows from the Banach fixed-point theorem, and T has a unique fixed point. □

Theorem 4.2. *Assume that $\psi_{r-1} \in C([0,1]^2)$ and $Z : [0,1]^2 \times \mathbb{R} \to \mathbb{R}$ be a continuous and Lipschitzian function with Lipschitz constants L. Then*

$$\|w - w_r\|_\infty \leq \|T(w_0) - w_0\|_\infty \times \sum_{j=r}^{\infty} L^j + \sum_{j=1}^{r} L^{r-j}\varepsilon_j,$$

where ε_j, for $j = 1, 2, \ldots, r$, is a constant.

Proof. Suppose that

$$M_{r-1} = \max\left\{ \left\| \frac{\partial \psi_{r-1}}{\partial t} \right\|_\infty, \left\| \frac{\partial \psi_{r-1}}{\partial s} \right\|_\infty \right\}, \quad r = 0, 1, \dots.$$

According to the integral equation (3),we have

$$\begin{aligned} \|T(w_{r-1}) - w_r\|_\infty &\leq \left\| \int_0^x \varphi_{r-1}(t,x) - Q_m(\varphi_{r-1})(t,x)dt \right\|_\infty \\ &\leq \|\varphi_{r-1} - Q_m(\varphi_{r-1})\|_\infty. \end{aligned}$$

By defining $g(t, s) := \psi_{r-1} - Q_m(\psi_{r-1})$ and using the mean-value theorem for two variables t_i and s_j and applying the interpolation property we have

$$\begin{aligned} t_i &= \frac{1}{2^{n_1+1}} + \frac{v_1}{2^{n_1}}, \quad i = 2^{n_1} + v_1, \quad n_1, n_2 \geq 1, \\ s_j &= \frac{1}{2^{n_2+1}} + \frac{v_2}{2^{n_2}}, \quad j = 2^{n_2} + v_2, \end{aligned}$$

where $s_0 = t_0 = 0$. Therefore,

$$\begin{aligned} \|\psi_{r-1} - Q_m(\psi_{r-1})\|_\infty &= \left\| g(t_i, s_j) + \frac{\partial g}{\partial t}(\alpha, \beta)(\alpha - t_i) + \frac{\partial g}{\partial s}(\alpha, l)(\beta - s_j) \right\|_\infty \\ &= \left\| (I - Q_m)\frac{\partial \psi_{r-1}}{\partial t}(\alpha, \beta) + (I - Q_m)\frac{\partial \psi_{r-1}}{\partial s}(\alpha, \beta) \right\|_\infty \\ &\quad \times \max\{\|\alpha - t_i\|_\infty, \|\beta - s_j\|_\infty\} \\ &\leq \frac{2}{2^r}\|(I - Q_m)\|_\infty \left\| \frac{\partial \psi_{r-1}}{\partial t}(\alpha, \beta) + \frac{\partial \psi_{r-1}}{\partial s}(\alpha, \beta) \right\|_\infty. \end{aligned}$$

Thus

$$\|T(w_{r-1}) - w_r\|_\infty \leq \frac{4M_{r-1}}{2^r}.$$

If $\frac{4M_{k-1}}{2^k} < \varepsilon_k$ for $k = 1, 2, \dots, r$ such that $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r > 0$ for $r \geq 1$, then

$$\|T(w_{r-1}) - w_r\|_\infty < \varepsilon_r.$$

Now, by applying the triangle inequality, we achieve

$$\|w - w_r\|_\infty \leq |w - T^r(w_0)|_\infty + \sum_{j=1}^{r} L^{r-j}\|T(w_{j-1}) - w_j\|_\infty. \tag{9}$$

From (8) and (9), we conclude that

$$\|w - w_r\|_\infty \leq \|T(w_0) - w_0\|_\infty \sum_{j=r}^{\infty} L^j + \sum_{j=1}^{r} L^{r-j}\varepsilon_j.$$

So, the proof is completed. □

5 A numerical example

In this section, we give an example to compare the proposed method with some other ones. The implementation has been done in Maple 2017 in a machine with Intel core i7-4710HQ, CPU 2.5 GHz and 8 GB RAM. By utilizing the recursive relation presented in (6), finally the integral equation (3) is solved.

Example 5.1. Let us consider the first Painlevé equation (1). We compare the numerical results of the proposed method with those of the Finite element method (FEM), Adomian's decomposition method (ADM) and the variational iteration method (VIM). Numerical results have been presented in Table 1. As we see, there is not any significant difference between the computed solutions, however the CPU time of the proposed method is less than the other methods. Also, Fig. 1 displays the approximate solution computed by the proposed method.

Table 1: Numerical results for Example 5.1

x_i	FEM $r=4$	ADM [2] $r=9$	VIM [5] $r=4$	RH wavelet $r=6$
0.1	0.100216	0.100215	0.100216	0.100211
0.2	0.202139	0.202117	0.202139	0.202125
0.3	0.308630	0.308630	0.308630	0.308671
0.4	0.423988	0.423985	0.423986	0.424303
0.5	0.554370	0.554335	0.554339	0.517173
0.7	0.900935	0.899217	0.899229	0.894911
0.9	1.526678	1.481201	1.481778	1.477958
CPU	81.391	0.640	0.969	0.612

6 Conclusion

We have proposed a numerical method to approximate the solution of the first Painlevé equation (1). The method is based on the expansion of the solution as a series of the Haar functions. We reformulated the problem into an integral equation and by proposing a successive method for approximating (6), the problem (1) was solved. Reported results show effectiveness of the proposed method.

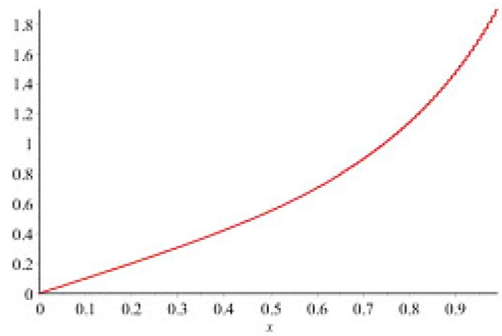

Figure 1: The numerical solution for Example 5.1.

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