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# Valid implementation of the Sinc-collocation method to solve linear integral equations by the CADNA library

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**Abstract.** The aim of this research is to apply the stochastic arithmetic (SA) for validating the Sinc-collocation method (S-CM) with single or double exponentially decay to find the numerical solution of second kind Fredholm integral equation (IE). To this end, the CESTAC(Controle et Estimation Stochastique des Arrondis de Calculs) method and the CADNA (Control of Accuracy and Debugging for Numerical Applications) library are applied. Using this method, the optimal iteration of S-CM, the optimal approximation, the absolute error and the numerical instabilities can be determined. A theorem is proved which shows the accuracy of the S-CM by means of the concept of common significant digits. Some IEs are presented and the numerical results of comparison between the single exponentially decay (SE) and the double exponentially decay (DE) are demonstrated in the tables.

*Keywords*: Stochastic arithmetic, CESTAC, Sinc-collocation method, CADNA library, Single exponentially decay, Double exponentially decay, Fredholm integral equations.
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## 1 Introduction

Consider the following second kind Fredholm IE

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$$y(s) = x(s) + \int_a^b H(s,t)y(t)dt, \quad a \le s,t \le b, \tag{1}$$

where $x(s)$ and $H(s,t)$ are given functions and $y(s)$ is an unknown function. In recent years, the collocation method (CM) with different types of basis functions has been applied to solve many problems which the results are computed based on the floating-point arithmetic (FPA) [10, 15, 17, 21, 30, 44, 45]. In the iterative schemes based on the FPA, usually the numerical results are obtained for special iteration or the following termination criterion is applied:

$$|y(s) - y_j(s)| \le \epsilon, \tag{2}$$

where the value $\epsilon$ is a given tolerance and $y_j(s)$ is the approximate solution of $y(s)$.

In this work, the Sinc basis functions with SE and DE precisions [6, 7, 16, 27–29, 31–34, 37, 38, 41, 42, 47] are considered to solve Eq. (1) based on the SA [9, 12, 22, 23] and validate the results by using the CESTAC method [1–5,8,14,18–20,35,36]. Also, the optimal iteration and the optimal approximation of S-CM with SE and DE precisions are found. Recently, the CESTAC method has been applied to implement the numerical methods for finding the approximate solution of different problems such as [1–5, 14, 18, 25]. In this method, instead of using the mathematical packages such as Matlab, Mathematica and the others, the CADNA library [1–5, 24, 25] is applied. Also, in this library the logical programs can be written by statements of C/C++, FORTRAN or ADA [13]. Some of the advantages of using the CESTAC method and CADNA library are:

- In the CESTAC method, not only the optimal numerical solution can be produced but also, the optimal iteration can be obtained.
- The CADNA library is able to detect any instability in mathematical operations, branching, functions and so on [43, 49, 50] but the FPA does not have these abilities.
- In the FPA, the termination criterion (2) depends on value $\epsilon$. For $\epsilon$ large enough, the iterations can be stopped before finding the suitable approximation and for small values of $\epsilon$ the unnecessary iterations can be produced without improving the accuracy of the results. In the SA, the numerical results do not depend on the value $\epsilon$ and a new stopping condition is replaced which is independent of $\epsilon$.

The remainder of this work is organized in the following form: In next section, some preliminaries about the SA, the CESTAC method and the

CADNA library are presented. Furthermore, algorithm of the CESTAC method is introduced. In section 3, the necessary definitions and properties of the Sinc function are presented. In section 4, the S-CM is applied to solve the second kind Fredholm IE. In this section, by presenting a theorem, the validation of results is considered. Some examples of Fredholm IEs are approximated by SE and DE S-CM in section 5. The optimal iteration of method, optimal approximation and absolute error are shown in several tables which are based on the proposed algorithm in the SA. Finally, in section 6 some conclusions are drawn.

## 2 Preliminaries

Let $y_j$ for $j \geq 1$ be the approximate solution of Eq. (1) which are produced by one of the numerical methods. In Eq. (2) which is applied in the FPA, for $\epsilon$ large enough, the suitable approximation can not be obtained and for small values of $\epsilon$ the unnecessary iterations can be produced.

In the SA, instead of Eq. (2), the following termination criterion is replaced:

$$|y_j(s) - y_{j+1}(s)| = @.0, \tag{3}$$

where @.0 means the computed result does not have any correct significant digits and it is called the informatical zero [1–5, 14, 18, 43].

Let $F$ be a set of real values which are reproduced by computer arithmetic and the arbitrary value $\psi$ is demonstrated as $\Psi \in F$. In the personal computer (PC), $\Psi$ with the binary FPA, $\rho$ mantissa bits and the rounding error term is shown by

$$\Psi = \psi - \varepsilon 2^{E-\theta}\rho, \tag{4}$$

where $2^{-\theta}\rho$ is the missing segment of mantissa which is obtained from round-off error, $\varepsilon$ is the sign of $\psi$ and $E$ is the binary power of the outcome. In order to apply the SA, we make the perturbation on the last mantissa bit of $\psi$. So the value $\rho$ can be considered as a random variable uniformly distributed on $[-1, 1]$. Therefore, the random result of $\Psi$ can be calculated where mean ($\mu$) and standard deviation ($\sigma$) are applied to guarantee the precision of results [8, 9, 22, 23, 49, 50].

In PC, for $\theta = 24, 53$ the results can be obtained by single and double precisions, respectively. By $m$ times performing the process for $\Psi_i, i = 1, \ldots, m$ the distribution of them is in the quasi Gaussian form. Therefore, the mean of them is equal to the exact value of $\psi$ and the values of $\mu$ and $\sigma$ can be estimated by these $m$ samples. The following algorithm of CESTAC method is presented where $\tau_\delta$ is the value of $T$ distribution with $m - 1$ degree of freedom and confidence interval $1 - \delta$.

**Algorithm 1:**

Step 1- Find $m$ samples for $\Psi$ as $\Phi = \{\Psi_1, \Psi_2, \ldots, \Psi_m\}$ by means of the perturbation of the last bit of mantissa.

Step 2- Compute $\Psi_{ave} = \frac{\sum_{i=1}^{m} \Psi_i}{m}$.

Step 3- Calculate $\sigma^2 = \frac{\sum_{i=1}^{m} (\Psi_i - \Psi_{ave})^2}{m-1}$.

Step 4- Compute $C_{\Psi_{ave},\Psi} = \log_{10} \frac{\sqrt{m}\,|\Psi_{ave}|}{\tau_\delta \sigma}$, as the common significant digits between $\Psi$ and $\Psi_{ave}$.

Step 5- If $C_{\Psi_{ave},\Psi} \leq 0$ or $\Psi_{ave} = 0$, then write $\Psi =$@.0.

CADNA enables to create new numerical types with the others operators such as ADA, C/C++ or FORTRAN [13, 18]. The codes of CADNA library are similar to programs which are produced by these operators and by minor variations we can apply the CADNA programs. The process of algorithms is stopped when the informatical zero's sign @.0 is shown in the termination criterion. A sample program in order to run by CADNA library is shown in the following form:

```
program sample
# include <cadna.h>
cadna_init(-1)
double_st (float_st) value;
The Main Program
printf ("  value= %s  \n ",Strp(value));
cadna_end();
```

In order to show the significant digits by CADNA, the function "Strp" in the output instruction has been applied. Hence, if the number of significant digits becomes zero, it is shown with the notation @.0. It means that the value is an informatical zero. Also, if the values are in `double` or `float` precision, it must be placed type `double_st` or `float_st` at the related line. More information are presented in `www.cadna.lib6.fr`.

# 3 Sinc function

In this section, some definitions and properties of the Sinc approximation are given. Also, application of S-CM to solve the Fredholm IEs are considered. The Sinc function on the real line is defined as follows

$$\mathrm{Sinc}(s) = \begin{cases} \frac{\sin(\pi s)}{\pi s}, & s \neq 0, \\ 1, & s = 0. \end{cases} \tag{5}$$

The Whittaker cardinal for $f \in \mathbf{R}$ and the step size $\hbar > 0$ is defined as follows

$$\mathcal{C}a(f, \hbar)(s) = \sum_{k=-\infty}^{\infty} f(k\hbar)\mathcal{S}(k, \hbar)(s), \tag{6}$$

and the $l$-th order of Eq. (6) is

$$\mathcal{C}a_l(f, \hbar)(s) = \sum_{k=-l}^{l} f(k\hbar)\mathcal{S}(k, \hbar)(s), \tag{7}$$

whenever this series convergence, and the $M$-th order of Sinc function is defined in the following form

$$\mathcal{S}(M, \hbar)(s) = \frac{\sin[\pi(s - M\hbar)/\hbar]}{\pi(s - M\hbar)/\hbar}, \quad M = 0, \pm 1, \pm 2, \dots \ . \tag{8}$$

Now, the following function space should be introduced.

**Definition 1.** *[28]* Assume that $\alpha \in \mathbf{R}^+$ and for bounded and simply-connected domain $D$ we have $(a, b) \subset D$. The family of functions $f$ which is shown by $L_\alpha(D)$ satisfy in the following conditions:
(i) $f$ is analytic in $D$;
(ii) $\exists C' \in \mathbf{R}^+$ such that for all $s$ in $D$

$$|f(s)| \leq C'|(s - a)(b - s)|^\alpha. \tag{9}$$

The SE precision is presented in the following form

$$\phi^{SE}(t) = \frac{b - a}{2}\tanh(\frac{t}{2}) + \frac{b + a}{2}, \tag{10}$$

$$\{\phi^{SE}\}'(t) = \frac{1}{4}(b - a)\mathrm{sech}(\frac{t}{2})^2. \tag{11}$$

**Theorem 1.** *[46] Let $f \in \mathcal{L}_\alpha(\phi^{SE}(D_d))$ for $d$ with $0 < d < \frac{\pi}{2}$. Then for positive integer $N$ and*

$$\hbar = \sqrt{\frac{\pi d}{\alpha N}}, \tag{12}$$

*we have*

$$\left|\int_a^b f(s)ds - \hbar \sum_{k=-N}^{N} f(\phi^{SE}(k\hbar))\{\phi^{SE}\}'(k\hbar)\right| \leq W \exp\left(-\sqrt{\pi d \alpha N}\right), \tag{13}$$

*where $W$ is a constant value and independent of $N$.*

In order to apply the DE-transformation of Sinc function we get

$$\phi^{DE}(t) = \frac{b-a}{2}\tanh(\frac{\pi}{2}\sinh t) + \frac{a+b}{2}, \tag{14}$$

and

$$\{\phi^{DE}\}'(t) = \frac{b-a}{2}\frac{\frac{\pi}{2}\cosh(t)}{\cosh^2(\frac{\pi}{2}\sinh(t))}. \tag{15}$$

**Theorem 2.** *[39] Assume that $f \in \mathcal{L}_\alpha(\phi^{DE}(D_d))$ for $d$ with $0 < d < \frac{\pi}{2}$. Then for positive integer $N$ and*

$$\hbar = \frac{1}{N}\log(\frac{2dN}{\alpha}), \tag{16}$$

*we have*

$$\left|\int_a^b f(s)ds - \hbar \sum_{k=-N}^{N} f(\phi^{DE}(k\hbar))\{\phi^{DE}\}'(k\hbar)\right| \leq W \exp\left(\frac{-2\pi d N}{\log(\frac{2dN}{\alpha})}\right), \tag{17}$$

*where $W$ is a constant value and independent of $N$.*

## 4 Main idea

By using the SE and DE Sinc approximation for integral part of Eq. (1), the following relations are obtained

$$\int_a^b H(s,t)y(t)dt \approx \hbar \sum_{k=-N}^{N} H(s, \phi^{SE}(k\hbar))\{\phi^{SE}\}'(k\hbar)y_k, \tag{18}$$

$$\int_a^b H(s,t)y(t)dt \approx \hbar \sum_{k=-N}^{N} H(s, \phi^{DE}(k\hbar))\{\phi^{DE}\}'(k\hbar)y_k, \tag{19}$$

where $y_k = y(q_k), k = -N, \ldots, N$. By substituting Eqs. (18) and (19) into Eq. (1), we get

$$y(s) - \hbar \sum_{k=-N}^{N} H(s, \phi^{SE}(k\hbar))\{\phi^{SE}\}'(k\hbar)y_k \approx x(s), \tag{20}$$

$$y(s) - \hbar \sum_{k=-N}^{N} H(s, \phi^{DE}(k\hbar))\{\phi^{DE}\}'(k\hbar) y_k \approx x(s). \tag{21}$$

Now, by putting the collocation grids $s_i^{SE} = \phi^{SE}(i\hbar), i = -N, \ldots, N$ for SE precision and $s_i^{DE} = \phi^{DE}(i\hbar)$ for DE precision in Eqs. (20) and (21), the $(2N+1) \times (2N+1)$ system of equations can be produced as

$$y(s_i^{SE}) - \hbar \sum_{k=-N}^{N} H(s_i^{SE}, \phi^{SE}(k\hbar))\{\phi^{SE}\}'(k\hbar) y_k = x(s_i^{SE}), \tag{22}$$

$$y(s_i^{DE}) - \hbar \sum_{k=-N}^{N} H(s_i^{DE}, \phi^{DE}(k\hbar))\{\phi^{DE}\}'(k\hbar) y_k = x(s_i^{DE}), \tag{23}$$

for $i = -N, \ldots, N$. The matrix form of the systems (22) and (23) are given in the following form

$$(I - A^{SE})Y = X^{SE}, \tag{24}$$

$$(I - A^{DE})Y = X^{DE}, \tag{25}$$

where

$$A^{SE} = \Big[\hbar H(s_i^{SE}, \phi^{SE}(k\hbar))\{\phi^{SE}\}'(k\hbar)\Big]_{(2N+1)\times(2N+1)},$$

$$A^{DE} = \Big[\hbar H(s_i^{DE}, \phi^{DE}(k\hbar))\{\phi^{DE}\}'(k\hbar)\Big]_{(2N+1)\times(2N+1)},$$

and

$$Y = \Big[y_{-N}, \ldots, y_N\Big]^T_{(2N+1)\times(2N+1)},$$

$$X^{SE} = \Big[x(s_{-N}^{SE}), \ldots, x(s_N^{SE})\Big]^T_{(2N+1)\times(2N+1)},$$

$$X^{DE} = \Big[x(s_{-N}^{DE}), \ldots, x(s_N^{DE})\Big]^T_{(2N+1)\times(2N+1)}.$$

By solving the system of equations (24) (or (25)) and substituting the unknowns in

$$y_N^{SE}(s) \approx x(s) + \hbar \sum_{k=-N}^{N} H(s, \phi^{SE}(k\hbar))\{\phi^{SE}\}'(k\hbar) y_k, \tag{26}$$

$$y_N^{DE}(s) \approx x(s) + \hbar \sum_{k=-N}^{N} H(s, \phi^{DE}(k\hbar))\{\phi^{DE}\}'(k\hbar) y_k, \tag{27}$$

the numerical solution of Eq. (1) can be estimated.

**Theorem 3.** *( [28]) Assume $y(s)$ is the exact and $y_N^{SE}(s)$ is the numerical solutions of Eq. (1). Then*

$$\max_{s\in(a,b)} |y(s) - y_N^{SE}(s)| \leq W \mu_N^{SE} \log(N+1)\sqrt{N} \exp\left(-\sqrt{\pi d \alpha N}\right), \tag{28}$$

*where $\mu_N^{SE} = \|(I - A^{SE})^{-1}\|$ and $W$ is a constant and independent of $N$.*

**Theorem 4.** *( [28]) Assume $y(s)$ is the exact and $y_N^{DE}(s)$ is the numerical solutions of Eq. (1). Then*

$$\max_{s\in(a,b)} |y(s) - y_N^{DE}(s)| \leq W \mu_N^{DE} \log(N+1) \exp\left(\frac{-\pi d N}{\log(2dN/\alpha)}\right), \tag{29}$$

*where $\mu_N^{DE} = \|(I - A^{DE})^{-1}\|$ and $W$ is a constant and independent of $N$.*

**Definition 2.** *( [1, 2, 14])* The number of significant digits between two real numbers $\theta_1$ and $\theta_2$ is described as
(1) for $\theta_1 \neq \theta_2$,

$$C_{\theta_1,\theta_2} = \log_{10}\left|\frac{\theta_1+\theta_2}{2(\theta_1-\theta_2)}\right| = \log_{10}\left|\frac{\theta_1}{\theta_1-\theta_2} - \frac{1}{2}\right|. \tag{30}$$

(2) for all real numbers $\theta_1$, $C_{\theta_1,\theta_1} = +\infty$.

**Theorem 5.** *Let $y(s)$ be the exact solution and $y_N^{SE}(s)$ be the $N$-th order numerical solution of Eq. (1) which is produced by SE Sinc-collocation method. Then for arbitrary $s \in [a,b]$ we have*

$$C_{y_N^{SE}(s),y(s)} - C_{y_N^{SE}(s),y_{N+1}^{SE}(s)} = \mathcal{O}\left(\log(N+1)\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right)\right). \tag{31}$$

*Proof.* According to Definition 2

$$\begin{aligned}
C_{y_N^{SE}(s),y_{N+1}^{SE}(s)} &= \log_{10}\left|\frac{y_N^{SE}(s)+y_{N+1}^{SE}(s)}{2(y_N^{SE}(s)-y_{N+1}^{SE}(s))}\right| \\
&= \log_{10}\left|\frac{y_N^{SE}(s)}{y_N^{SE}(s)-y_{N+1}^{SE}(s)}-\frac{1}{2}\right| \\
&= \log_{10}\left|\frac{y_N^{SE}(s)}{y_N^{SE}(s)-y_{N+1}^{SE}(s)}\right| \\
&+ \log_{10}\left|1-\frac{1}{2y_N^{SE}(s)}(y_N^{SE}(s)-y_{N+1}^{SE}(s))\right| \\
&= \log_{10}\left|\frac{y_N^{SE}(s)}{y_N^{SE}(s)-y_{N+1}^{SE}(s)}\right| + \mathcal{O}\left(y_N^{SE}(s)-y_{N+1}^{SE}(s)\right).
\end{aligned}$$

Since

$$y_N^{SE}(s)-y_{N+1}^{SE}(s)=y_N^{SE}(s)-y(s)-(y_{N+1}^{SE}(s)-y(s))=E_n(s)-E_{n+1}(s),$$

thus

$$\begin{aligned}
\mathcal{O}\left(y_N^{SE}(s)-y_{N+1}^{SE}(s)\right) &= \mathcal{O}\left(E_n(s)-E_{n+1}(s)\right) \\
&= \mathcal{O}\left(\log(N+1)\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right)\right) \\
&\quad+\mathcal{O}\left(\log(N+2)\sqrt{N+1}\exp\left(-\sqrt{\pi d\alpha(N+1)}\right)\right) \\
&= \mathcal{O}\left(\log(N+1)\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right)\right).
\end{aligned}$$

Therefore,

$$C_{y_N^{SE}(s),y_{N+1}^{SE}(s)} = \log_{10}\left|\frac{y_N^{SE}(s)}{y_N^{SE}(s)-y_{N+1}^{SE}(s)}\right| + \mathcal{O}\left(\log(N+1)\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right)\right). \tag{32}$$

Furthermore,

$$\begin{aligned}
C_{y_N^{SE}(s),y(s)} &= \log_{10}\left|\frac{y_N^{SE}(s)+y(s)}{2(y_N^{SE}(s)-y(s))}\right| = \log_{10}\left|\frac{y_N^{SE}(s)}{y_N^{SE}(s)-y(s)}-\frac{1}{2}\right| \\
&= \log_{10}\left|\frac{y_N^{SE}(s)}{y_N^{SE}(s)-y(s)}\right| + \mathcal{O}(y_N^{SE}(s)-y(s)) \\
&= \log_{10}\left|\frac{y_N^{SE}(s)}{y_N^{SE}(s)-y(s)}\right| + \mathcal{O}\left(\log(N+1)\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right)\right).
\end{aligned} \tag{33}$$

Using Eqs. (32) and (33) we get
$C_{y_N^{SE}(s),y(s)} - C_{y_N^{SE}(s),y_{N+1}^{SE}(s)}$

$$\begin{aligned}
&= \log_{10}\left|\frac{y_N^{SE}(s)}{y_N^{SE}(s)-y(s)}\right| - \log_{10}\left|\frac{y_N^{SE}(s)}{y_N^{SE}(s)-y_{N+1}^{SE}(s)}\right| \\
&\quad +\mathcal{O}\left(\log(N+1)\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right)\right) \\
&= \log_{10}\left|\frac{y_N^{SE}(s)-y(s)}{y_N^{SE}(s)-y_{N+1}^{SE}(s)}\right| \\
&\quad +\mathcal{O}\left(\log(N+1)\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right)\right) \\
&= \log_{10}\left|\frac{\mathcal{O}\left(\log(N+1)\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right)\right)}{\mathcal{O}\left(\log(N+1)\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right)\right)}\right| \\
&\quad +\mathcal{O}\left(\log(N+1)\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right)\right) \\
&= \mathcal{O}\left(\log(N+1)\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right)\right),
\end{aligned}$$

and finally the following formula can be obtained

$$C_{y_N^{SE}(s),y(s)} - C_{y_N^{SE}(s),y_{N+1}^{SE}(s)} = \mathcal{O}\left(\log(N+1)\sqrt{N}\exp\left(-\sqrt{\pi d\alpha N}\right)\right).$$

□

When $N \to \infty$ the right hand side of above equation tends to zero and hence we obtain

$$C_{y_N^{SE}(s),y(s)} = C_{y_N^{SE}(s),y_{N+1}^{SE}(s)}.$$

Theorem 5 shows that the number of common significant digits between two successive approximations with SE precision is almost equal to the number of common significant digits between the exact and approximate solutions.

**Theorem 6.** *Let $y(s)$ be the exact solution and $y_N^{DE}(s)$ be the $N$-th order numerical solution of Eq. (1) which is produced by DE Sinc-collocation method. Then for arbitrary $s \in [a,b]$ we have*

$$C_{y(s),y_N^{DE}(s)} - C_{y_{N+1}^{DE}(s),y_N^{DE}(s)} = \log_{10}\left|1+\exp\left(\mathcal{O}\left(\frac{-\pi d}{\log(\frac{2d(N+1)}{\alpha})}\right)\right)\right|. \tag{34}$$

*Proof.* According to Definition 2

$$\begin{aligned}
C_{y(s),y_N^{DE}(s)} - C_{y_{N+1}^{DE}(s),y_N^{DE}(s)} &= \log_{10}\left|\frac{y(s)+y_N^{DE}(s)}{2(y(s)-y_N^{DE}(s))}\right| \\
&= \log_{10}\left|\frac{y_N^{SE}(s)}{y_N^{SE}(s)-y_{N+1}^{SE}(s)}\right| \\
&\quad - \log_{10}\left|\frac{y_{N+1}^{DE}(s)+y_N^{DE}(s)}{2(y_{N+1}^{DE}(s)-y_N^{DE}(s))}\right| \\
&= \log_{10}\left|\frac{y(s)+y_N^{DE}(s)}{y_{N+1}^{DE}(s)+y_N^{DE}(s)}\right| \\
&\quad + \log_{10}\left|\frac{y_{N+1}^{DE}(s)-y_N^{DE}(s)}{y(s)-y_N^{DE}(s)}\right|.
\end{aligned} \tag{35}$$

The first term of Eq. (35) can be neglected because when $N \to \infty$ the approximate solution is very close to exact solution. Therefore, the second term of Eq. (35) is written as follows

$$\begin{aligned}
\frac{y_{N+1}^{DE}(s)-y_N^{DE}(s)}{y(s)-y_N^{DE}(s)} &= \frac{(y_{N+1}^{DE}(s)-y(s))+(y(s)-y_N^{DE}(s))}{y(s)-y_N^{DE}(s)} \\
&= 1+\frac{y_{N+1}^{DE}(s)-y(s)}{y(s)-y_N^{DE}(s)}.
\end{aligned} \tag{36}$$

Also, by using Eq. (29) the following relation is obtained

$$\begin{aligned}
\frac{y_{N+1}^{DE}(s)-y(s)}{y(s)-y_N^{DE}(s)} &= \frac{\mathcal{O}\left[\log(N+2)\exp\left(\frac{-\pi d(N+1)}{\log(\frac{2d(N+1)}{\alpha})}\right)\right]}{\mathcal{O}\left[\log(N+1)\exp\left(\frac{-\pi dN}{\log(\frac{2dN}{\alpha})}\right)\right]} \\
&= \mathcal{O}\left[\frac{\log(N+2)}{\log N+1}\exp\left(\pi d\left(\frac{N}{\log(\frac{2dN}{\alpha})}-\frac{N+1}{\log(\frac{2d(N+1)}{\alpha})}\right)\right)\right].
\end{aligned} \tag{37}$$

Since, $\lim_{N\to\infty}\frac{\log(N+2)}{\log(N+1)} = 1$, we can write

$$\frac{y_{N+1}^{DE}(s)-y(s)}{y(s)-y_N^{DE}(s)} = \mathcal{O}\left[\exp\left(\pi d\left(\frac{N}{\log(\frac{2dN}{\alpha})}-\frac{N+1}{\log(\frac{2d(N+1)}{\alpha})}\right)\right)\right], \tag{38}$$

and

$$\pi d\left(\frac{N}{\log(\frac{2dN}{\alpha})}-\frac{N+1}{\log(\frac{2d(N+1)}{\alpha})}\right)=\pi d\left(\frac{\log(\frac{2d(N+1)}{\alpha})^{N}-\log(\frac{2dN}{\alpha})^{(N+1)}}{\log(\frac{2dN}{\alpha})\log(\frac{2d(N+1)}{\alpha})}\right). \tag{39}$$

When $N\to\infty$, the numerator of Eq. (39) is written as follows

$$\log\frac{(\frac{2d(N+1)}{\alpha})^{N}}{(\frac{2dN}{\alpha})^{N+1}}=\log\frac{(\frac{2d(N+1)}{\alpha})^{N}}{(\frac{2dN}{\alpha})^{N}}\frac{1}{\frac{2dN}{\alpha}}=\log\frac{e}{\frac{2dN}{\alpha}}=-\log\frac{2dN}{e\alpha}, \tag{40}$$

and by using Eqs. (39) and (40) the following equation is obtained

$$-\pi d\left(\frac{\log\frac{2dN}{e\alpha}}{\log(\frac{2dN}{\alpha})\log(\frac{2d(N+1)}{\alpha})}\right)=\mathcal{O}\left(\frac{-\pi d}{\log(\frac{2d(N+1)}{\alpha})}\right). \tag{41}$$

By considering Eqs. (35), (36) and (41) and for $N$ large enough, we have

$$\begin{aligned}C_{y(s),y_N^{DE}(s)}-C_{y_{N+1}^{DE}(s),y_N^{DE}(s)}&=\log_{10}\left|\frac{y_{N+1}(s)-y_N(s)}{y(s)-y_N(s)}\right|\\&=\log_{10}\left|1+\frac{y_{N+1}^{DE}(s)-y(s)}{y(s)-y_N^{DE}(s)}\right|\\&=\log_{10}\left|1+\exp\left(\mathcal{O}\left(\frac{-\pi d}{\log(\frac{2d(N+1)}{\alpha})}\right)\right)\right|.\end{aligned} \tag{42}$$

□

When $N$ increases, in the right hand side of Eq. (42) we get

$$\exp\left(\mathcal{O}\left(\frac{-\pi d}{\log(\frac{2d(N+1)}{\alpha})}\right)\right)\ll 1.$$

Therefore

$$C_{y(s),y_N^{DE}(s)}=C_{y_{N+1}^{DE}(s),y_N^{DE}(s)}.$$

Theorems 5 and 6 illustrate the accuracy of the S-CM in SE and DE precisions respectively and permit us to apply an optimal termination criterion like Eq. (3) in the CESTAC method. Also, based on Eqs. (31) and (34), the DE S-CM is faster and more accurate than the SE precision.

## 5 Numerical experiments

In this section, some examples are given; the Love's IE, the Lichtenstein-Gershgorin IE and the Fredholm IE with weakly singular kernel. The numerical results are obtained based on the SE and DE S-CM. These results are calculated by the CESTAC method. The following algorithm is based on the CADNA library and the stopping condition has been chosen according to Theorem 5 for SE S-CM. This algorithm can be written similarly for DE precision. Optimal step of SE and DE S-CM, the optimal approximate solutions and the absolute errors in companion with the difference between two sequential results are presented in the tables. Furthermore, comparison between the exact and the approximate solutions and figures of the absolute error functions are given.

**Algorithm 2:**

Step 1- Let $N = 1$.
Step 2- Define the functions $\phi^{SE}(s), \{\phi^{SE}\}'(s), H(s,t), x(s)$.
Step 3- Enter the parameters $a, b, s, \alpha, d$.
Step 4- Do the following steps while $|y_{N+1}^{SE} - y_N^{SE}| \neq$ @.0
{
Step 4-1- Let $\hbar = \frac{1}{N}\log(\frac{2dN}{\alpha})$.
Step 4-2- Calculate the collocation points.
Step 4-3- Approximate the coefficient matrix $A^{SE}$.
Step 4-4- Create the matrix $(I - A^{SE})$ and $(I - A^{SE})^{-1}$.
Step 4-5- Construct the right hand side matrix $X$.
Step 4-6- Solve the system of equations (24) by using the obtained results of steps (4-4) and (4-5).
Step 4-7- Apply Eqs. (26) to find the numerical solution Eq. (1).
Step 4-8- Print $y_N^{SE}$, $\left|y_{N+1}^{SE} - y_N^{SE}\right|$ and $\left|y - y_N^{SE}\right|$.
Step 4-9- $N = N + 1$.
}

**Example 1.** The Love's IE is given as

$$y(s) - \frac{\xi}{\pi}\int_{-1}^{1}\frac{y(t)}{\xi^2 + (s-t)^2}dt = x(s),$$

where $\xi = -1$ and $x(s) = 1 + \frac{1}{\pi}(\arctan(1+s) + \arctan(1-s))$. It appears in electrostatic [26, 40]. The results of SE precision are presented in Table 1. The optimal iteration of this method is $N = 71$ and the optimal approximation is $y_{71}^{SE}(0.2) = 0.1000008E+001$. Also, the numerical results which are based on the DE S-CM are shown in Table 2. The optimal iteration for DE precision is $N = 7$ and the optimal approximate solution for $s = 0.2$ is $y_7^{DE}(0.2) = 0.1000001E + 001$. According to numerical results, the DE precision is faster and more accurate than the SE S-CM.

Table 1: Numerical results of SE S-CM of Example 1 when $s = 0.2$, $\alpha = 1$, $d = \frac{\pi}{6}$.

| $N$ | $y_N^{SE}(s)$ | $\lvert y_{N+1}^{SE}(s) - y_N^{SE}(s)\rvert$ | $\lvert y(s) - y_N^{SE}(s)\rvert$ |
|---|---|---|---|
| 1 | 0.1060901E+001 | 0.1060901E+001 | 0.6090104E-001 |
| 2 | 0.1044062E+001 | 0.1683819E-001 | 0.4406285E-001 |
| 3 | 0.1031736E+001 | 0.1232624E-001 | 0.3173661E-001 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 30 | 0.1000327E+001 | 0.40E-004 | 0.327E-003 |
| 31 | 0.1000292E+001 | 0.3528594E-004 | 0.292E-003 |
| 32 | 0.1000261E+001 | 0.31E-004 | 0.2614259E-003 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 69 | 0.1000009E+001 | 0.8344650E-006 | 0.9775161E-005 |
| 70 | 0.1000009E+001 | 0.7152557E-006 | 0.9059906E-005 |
| 71 | 0.1000008E+001 | @.0 | 0.86E-005 |

Table 2: Numerical results of DE S-CM for Example 1 when $s = 0.2$, $\alpha = 1$, $d = \frac{\pi}{6}$.

| $N$ | $y_N^{DE}(s)$ | $\lvert y_{N+1}^{DE}(s) - y_N^{DE}(s)\rvert$ | $\lvert y(s) - y_N^{DE}(s)\rvert$ |
|---|---|---|---|
| 1 | 0.1400815E+001 | 0.1400815E+001 | 0.400815E+000 |
| 2 | 0.1013534E+001 | 0.387281E+000 | 0.13534E-001 |
| 3 | 0.1001183E+001 | 0.12350E-001 | 0.1183E-002 |
| 4 | 0.1000113E+001 | 0.1069E-002 | 0.1139640E-003 |
| 5 | 0.1000009E+001 | 0.103E-003 | 0.99E-005 |
| 6 | 0.1000002E+001 | 0.77E-005 | 0.2264976E-005 |
| 7 | 0.1000001E+001 | @.0 | 0.1E-005 |

**Example 2.** *( [48])* Consider the following Lichtenstein-Gershgorin IE

$$\Phi(\ell) - \frac{1}{\pi}\int_0^{2\pi} \frac{n\Phi(y)}{(n^2+1) - (n^2-1)\cos(\ell+y)}dy$$

$$= 2\arctan\left[\frac{n\sin\ell}{n^2(\cos\ell - \cos^2\ell) - \sin^2\ell}\right], \quad 0 \le \ell \le 2\pi.$$

for This equation has applications in physics and engineering [40]. By substituting

$$\ell = \pi(1+s), \quad y = \pi(1+t), \quad \Phi(\ell) = \upsilon(s),$$

we have

$$\upsilon(s) - \int_{-1}^{1} \frac{ny(t)}{(n^2+1)-(n^2-1)\cos\pi(s+t)}dt$$
$$= 2\arctan\left[\frac{n\sin\pi s}{n^2(\cos\pi s - \cos^2\pi s)+\sin^2\pi s}\right], \qquad -1 \le s \le 1. \tag{43}$$

Now, we apply the S-CM with SE and DE decays to find the approximate solution of Eq. (43), for $n = 1.2$. Because the exact solution of Eq. (43) is unavailable therefore we evaluate the results of Table 4 with results of [40]. The numerical solution of IE (43) for $s = 0.5$ is $y(0.5) \approx 1.5707963267$. In Tables 3 and 4, sign @.0 shows the optimal iteration of SE and DE precisions for arbitrary value $s = 0.5$. By comparison between the numerical results of Tables 3 and 4, the SE optimal iteration is $N = 17$ and the DE optimal iteration is $N = 5$.

Table 3: Numerical results of SE S-CM for Example 2 when $s = 0.5$, $\alpha = 1$, $d = \frac{\pi}{6}$.

| $N$ | $y_N^{SE}(s)$ | $\lvert y_{N+1}^{SE}(s) - y_N^{SE}(s)\rvert$ | $\lvert y(s) - y_N^{SE}(s)\rvert$ |
|---|---|---|---|
| 1 | 0.160437E+001 | 0.160437E+001 | 0.33574E-001 |
| 2 | 0.1593363E+001 | 0.11007E-001 | 0.22567E-001 |
| 3 | 0.158435E+001 | 0.9005E-002 | 0.1356E-001 |
| 4 | 0.1578845E+001 | 0.5512E-002 | 0.8049E-002 |
| 5 | 0.157570E+001 | 0.3142E-002 | 0.4906E-002 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 14 | 0.15708E+001 | 0.6E-004 | 0.8E-004 |
| 15 | 0.157082E+001 | 0.5E-004 | 0.3E-004 |
| 16 | 0.15707E+001 | 0.4E-004 | @.0 |
| 17 | 0.15707E+001 | @.0 | @.0 |

**Example 3.** Let us consider the following singular Fredholm integral equation [11]

$$y(s) = \sqrt{s} - \frac{\pi}{2} + \int_0^1 \frac{y(t)}{\sqrt{1-t}}dt, \qquad 0 \le s \le 1, \tag{44}$$

Table 4: Numerical results of DE S-CM for Example 2 when $s = 0.5$, $\alpha = 1$, $d = \frac{\pi}{6}$.

| $N$ | $y_N^{DE}(s)$ | $\lvert y_{N+1}^{DE}(s) - y_N^{DE}(s)\rvert$ | $\lvert y(s) - y_N^{DE}(s)\rvert$ |
|---|---|---|---|
| 1 | 0.1751564E+001 | 0.1751564E+001 | 0.1807684E+000 |
| 2 | 0.156827E+001 | 0.18329E+000 | 0.252E-002 |
| 3 | 0.15651E+001 | 0.31E-002 | 0.56E-002 |
| 4 | 0.1566E+001 | 0.1E-002 | 0.44E-002 |
| 5 | 0.15E+001 | @.0 | @.0 |

Table 5: Numerical results of SE S-CM for Example 3 when $s = 0.5$, $\alpha = 1$, $d = \frac{\pi}{6}$.

| $N$ | $y_N^{SE}(s)$ | $\lvert y_{N+1}^{SE}(s) - y_N^{SE}(s)\rvert$ | $\lvert y(s) - y_N^{SE}(s)\rvert$ |
|---|---|---|---|
| 1 | 0.128159E+001 | 0.128159E+001 | 0.57449E+000 |
| 2 | 0.108988E+001 | 0.19171E+000 | 0.382782E+000 |
| 3 | 0.9726340E+000 | 0.11725E+000 | 0.2655271E+000 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 23 | 0.715821E+000 | 0.106E-002 | 0.8714E-002 |
| 24 | 0.714891E+000 | 0.929E-003 | 0.7785E-002 |
| 25 | 0.71407E+000 | 0.81E-003 | 0.697E-002 |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 47 | 0.708021E+000 | 0.7E-004 | 0.914E-003 |
| 48 | 0.707952E+000 | 0.69E-004 | 0.845E-003 |
| 49 | 0.70789E+000 | 0.6E-004 | 0.78E-003 |
| 50 | 0.7078E+000 | @.0 | 0.7E-003 |

where $y(s) = \sqrt{s}$. The optimal iteration, optimal approximation, difference between two successive approximation and the absolute error for SE and DE precisions are shown in Tables 5 and 6. According to obtained results, the presented algorithm can be stopped when difference between two successive approximations is equal with the informatical zero sign @.0. In this example, the optimal iteration of SE decay is $N = 50$ and the optimal approximation is $y_{50}^{SE}(0.5) = 0.7078E{+}000$. But, the optimal iteration of DE S-CM is $N = 10$ and the optimal approximation is $y_{10}^{DE}(0.5) = 0.707108E + 000$. Therefore, the DE S-CM is better and faster in comparison with the SE precision.

Table 6: Numerical results of DE S-CM for Example 3 when $s = 0.5$, $\alpha = 1$, $d = \frac{\pi}{6}$.

| $N$ | $y_N^{DE}(s)$ | $\lvert y_{N+1}^{DE}(s) - y_N^{DE}(s)\rvert$ | $\lvert y(s) - y_N^{DE}(s)\rvert$ |
|---|---|---|---|
| 1 | 0.198464E+001 | 0.198464E+001 | 0.127754E+001 |
| 2 | 0.773888E+000 | 0.121076E+001 | 0.66781E-001 |
| 3 | 0.718046E+000 | 0.55841E-001 | 0.1094E-001 |
| 4 | 0.7092914E+000 | 0.8755E-002 | 0.2184E-002 |
| 5 | 0.707582E+000 | 0.1708E-002 | 0.476E-003 |
| 6 | 0.707216E+000 | 0.366E-003 | 0.10E-003 |
| 7 | 0.707133E+000 | 0.82E-004 | 0.26E-004 |
| 8 | 0.707113E+000 | 0.19E-004 | 0.6E-005 |
| 9 | 0.707109E+000 | 0.4E-005 | 0.2E-005 |
| 10 | 0.707108E+000 | @.0 | 0.1E-005 |

# 6 Conclusions

The validation of Sinc-collocation method with SE and DE precisions to find the optimal iteration and the optimal approximation of Fredholm IE is illustrated. To this aim, the CESTAC method which is based on the SA was applied. In order to implement the CESTAC method, the CADNA library is used. The accuracy of the Sinc method is proved which allows us to apply the CESTAC method. According to the sample examples, the DE S-CM is faster and more accurate in comparison with the SE precision. Consequently, it is suggested to apply the proposed scheme to validate the algorithms based on the Sinc method for solving IEs to find the optimal results computationally.

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# References

[1] S. Abbasbandy and M.A. Fariborzi Araghi, *The use of the stochastic arithmetic to estimate the value of interpolation polynomial with optimal degree*, Appl. Numer. Math. **50** (2004) 279–290.

[2] S. Abbasbandy and M.A. Fariborzi Araghi, *A reliable method to determine the ill-condition functions using stochastic arithmetic*, Southwest J. Pure. Appl. Math. **1** (2002) 33–38.

[3] S. Abbasbandy and M.A. Fariborzi Araghi, *Numerical solution of improper integrals with valid implementation*, Math. Comput. Appl. **7** (2002) 83–91.

[4] S. Abbasbandy and M.A. Fariborzi Araghi, *The valid implementation of numerical integration methods*, Far East J. Appl. Math. **8** (2002) 89–101 .

[5] S. Abbasbandy and M.A. Fariborzi Araghi, *A stochastic scheme for solving definite integrals*, Appl. Numer. Math. **55** (2005) 125–136.

[6] A. Abdrabou and M. El-Gamel, *On the sinc-Galerkin method for triharmonic boundary-value problems*, Comput. Math. Appl. **76** (2018) 520–533.

[7] M.S. Akel and H.S. Hussein, *Numerical treatment of solving singular integral equations by using Sinc approximations*, Appl. Math. Comput. **218** (2011) 3565–3573.

[8] N. Alliot, *Data error analysis in ATA Error analysis in unconstrained optimization problems with the CESTAC method*, Math. Comput. Simul. **50** (1988) 531–539.

[9] R. Alt, J.L. Lamotte and S. Markov, *Stochastic arithmetic, Theory and experiments*, Serdica J. Computing **4** (2010) 1–10.

[10] Z. Avazzadeh, M. Heydari and G.B. Loghmani, *A comparison between solving two dimensional integral equations by the traditional collocation method and radial basis functions*, Appl. Math. Sci. **5** (2011) 1145–1152.

[11] R. Behzadi, E. Tohidi and F. Toutounian, *Numerical solution of weakly singular Fredholm integral equations via generalization of the Euler-Maclaurin summation formula*, J. Taibah Univ. Sci. **8** (2014) 199–205.

[12] J.-M. Chesneaux, *Stochastic arithmetic properties*, Computational and Applied Mathematics, Algorithms and theory, C Brezinski (editor), North Holland, (1992) 81-91.

[13] J.M. Chesneaux, *CADNA, an ADA tool for round-off error analysis and for numerical debugging*, in: Proc. Congress on ADA in Aerospace, Barcelona, 1990.

[14] J.M. Chesneaux, F. Jézéquel, *Dynamical control of computations using the Trapezoidal and Simpson's rules*, J. Universal Comput. Sci. **4** (1998) 2–10.

[15] N. Ebrahimi, J. Rashidinia, Spline Collocation for system of Fredholm and Volterra integro-differential equations, J. Math. Model. **3** (2016) 189–218.

[16] M.A. Fariborzi Araghi and G. Kazemi Gelian, *Numerical solution of nonlinear Hammerstein integral equations via Sinc collocation method based on double exponential transformation*, Math. Sci. **7** (2013) 1–7.

[17] M.A. Fariborzi Araghi and S. Noeiaghdam, *Fibonacci-regularization method for solving Cauchy integral equations of the first kind*, Ain Shams Eng. J. **8** (2017) 363–369.

[18] M.A. Fariborzi Araghi and S. Noeighdam, *Dynamical control of computations using the Gauss-Laguerre integration rule by applying the CADNA library*, Adv. Appl. Math. Sci. **16** (2016) 1–18.

[19] M.A. Fariborzi Araghi and S. Noeiaghdam, *A Valid Scheme to Evaluate Fuzzy Definite Integrals by Applying the CADNA Library*, Intern. J. Fuzzy Syst. Appl. **6** (2017) 1–20.

[20] M.A. Fariborzi Araghi and E. Zarei, *Dynamical control of computations using the iterative methods to solve fully fuzzy linear systems*, Adv. Fuzzy Logic Tech. **2017** (2017) 55–68.

[21] Z. Gouyandeh, T. Allahviranloo and A. Armand, *Numerical solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via Tau-collocation method with convergence analysis*, J. Comput. Appl. Math. **15** (2016) 435–446.

[22] S. Graillat, F. Jézéquel and R. Picot, *Numerical Validation of Compensated Summation Algorithms with Stochastic*, Electron. Notes. Theor. Comput. Sci. Arithmetic **317** (2015) 55–69.

[23] S. Graillat, F. Jézéquel, S. Wang and Y. Zhu, *Stochastic arithmetic in multi precision*, Math. Comput. Sci. **5** (2011) 359–375.

[24] F. Jézéquel, *A dynamical strategy for approximation methods*, Compt. Rend. Meca. **334** (2006) 362–367.

[25] D. Khojasteh Salkuyeh and F. Toutounian, *Numerical accuracy of a certain class of iterative methods for solving linear system*, Appl. Math. Comput. **176** (2006) 727–738.

[26] E.R. Love, *The Electrostatic field of two equal circular conducting disks*, Q. J. Mech. Appl. Math. **2** (1949) 420–451.

[27] K. Maleknejad, R. Mollapourasl and A. Ostadi, *Convergence analysis of Sinc-collocation methods for nonlinear Fredholm integral equations with a weakly singular kernel*, J. Comput. Appl. Math. **278** (2015) 1–11.

[28] K. Maleknejad and K. Nedaiasl, *Application of Sinc-collocation method for solving a class of nonlinear Fredholm integral equations*, Comput. Math. Appl. **62** (2011) 3292–3303.

[29] H. Mesgarani and R. Mollapourasl, *Theoretical investigation on error analysis of Sinc approximation for mixed Volterra-Fredholm integral equation*, Comput. Math. Math. Phy. **53 (5)** (2013) 530–539.

[30] F. Mirzaee and A.A. Hoseini, *Numerical solution of nonlinear Volterra-Fredholm integral equations using hybrid of block-pulse functions and Taylor series*, Alexandria Eng. J. **52** (2013) 551–555.

[31] A. Mohsen and M. El-Gamel, *On the Galerkin and collocation methods for two-point boundary value problems using sinc bases*, Comput. Math. Appl. **56(4)** (2008) 930–941.

[32] M. Mori, A. Nurmuhammad and M. Muhammad, *DE-sinc method for second order singularly perturbed boundary value problems*, Japan J. Indust. Appl. Math. **26 (41)** (2009) 41–63.

[33] M. Mori, A. Nurmuhammad and T. Murai, *Numerical solution of Volterra integral equations with weakly singular kernel based on the DE-sinc method*, Japan J. Indust. Appl. Math. **25** (2008) 165–183.

[34] M. Nabati, S. Nikmanesh and M. Jalalvand, Solution of Troesche's problem by double exponential Sinc collocation method, J. Math. Model. **6** (2018) 77–90.

[35] S. Noeiaghdam and M.A. Fariborzi Araghi, *Finding optimal step of fuzzy Newton-Cotes integration rules by using the CESTAC method*, J. Fuzzy Set Val. Anal. **2017** (2017) 62–85.

[36] S. Noeiaghdam, M.A. Fariborzi Araghi and S. Abbasbandy, *Finding optimal convergence control parameter in the homotopy analysis method to solve integral equations based on the stochastic arithmetic*, Numer. Algor., https://doi.org/10.1007/s11075-018-0546-7, 2018.

[37] T. Okayama, T. Matsuo and M. Sugihara, *Sinc-collocation methods for weakly singular Fredholm integral equations of the second kind*, J. Comput. Appl. Math. **234** (2010) 1211–1227.

[38] T. Okayama, T. Matsuo and M. Sugihara, *Improvement of a Sinc-collocation method for Fredholm integral equations of the second kind*, BIT Numer. Math. **51** (2011)339–366.

[39] T. Okayama, T. Matsuo and M. Sugihara, *Error estimates with explicit constants for Sinc approximation, Sinc quadrature and Sinc indefinite integration*, Numer. Math. **124** (2013) 361–394.

[40] S. Panda, S.C. Martha and A. Chakrabarti, *A modified approach to numerical solution of Fredholm integral equations of the second kind*, Appl. Math. Comput. **271** (2015) 102–112.

[41] J. Rashidinia and M. Zarebnia, *Numerical solution of linear integral equations by using Sinc collocation method*, Appl. Math. Comput. **168** (2005) 806–822 .

[42] R. Revelli and L. Ridolfi, *Sinc collocation-interpolation method for the simulation of nonlinear waves*, Comput. Math. Appl. **46** (2003) 1443–1453.

[43] N.S. Scott, F. Jézéquel, C. Denis and J.M. Chesneaux, *Numerical 'health check' for scientific codes: the CADNA approach*, Comput. Phys. Comm. **176** (2007) 507–521.

[44] S.R. Shesha, A.L. Nargund and N.M. Bujurke, Numerical solution of non-planar Burgers equation by Haar wavelet method, J. Math. Model. **5** (2017) 89–118.

[45] S. Sohrabi, H. Ranjbar and M. Saei, *Convergence analysis of the Jacobi-collocation method for nonlinear weakly singular Volterra integral equations*, Appl. Math. Comput. **299** (2017) 141–152.

[46] F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions,* New York, Springer-Verlag, 1993.

[47] M. Sugihara, *Near optimality of the Sinc approximation*, Math. Comp. **71** (2002) 767–786.

[48] D. Elliott, *A Chebyshev series method for the numerical solution of Fredholm integral equations*, Comput. J. **6** (1963) 102–112.

[49] J. Vignes, *A stochastic arithmetic for reliable scientific computation*, Math. Comput. Simul., **35** (1993) 233–261.

[50] J. Vignes, *Discrete stochastic arithmetic for validating results of numerical software*, Numer. Algor. **37** (2004) 377–390.