Journal of Mathematical Modeling
Vol. 7, No. 1, 2019, pp. 35-48

JMM

A new two-parameter distribution: properties and applications

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Abstract. In this paper, a new two-parameter lifetime distribution called "the exponentiated Shanker distribution" is suggested. The new distribution has an increasing, decreasing and bathtub-shaped hazard rate function (hrf) for modeling lifetime data. Various mathematical and statistical properties of the proposed distribution including its hrf, complete and incomplete moments, skewness and kurtosis, mean deviations, Bonferroni and Lorenz curves are discussed. Estimation of its parameters is also discussed using the method of maximum likelihood estimation and a simulation study is given. Finally, two applications of the new distribution are presented using two real data sets. The results also confirmed the suitability of the proposed model for the real data sets.

Keywords: Exponentiated Shanker distribution, goodness of fit, lifetime data, mathematical and statistical characteristics, parameter estimation.
AMS Subject Classification: 60E05, 62F10.

1 Introduction

Modeling lifetime data is an important subject in areas like engineering,

[*]Corresponding author.
Received: 17 Mar 2018 / Revised: 27 November 2018 / Accepted: 28 November 2018.
DOI: 10.22124/jmm.2018.9994.1148

medicine, and finance. Recently, several lifetime distributions have been introduced in [6–8], that have some advantages and disadvantages in modeling lifetime data.

The probability density function (pdf) and the cumulative distribution function (cdf) of the Shanker distribution, introduced in [7], are

$$f(x;\theta) = \frac{\theta^2}{\theta^2+1}(\theta + x)\mathrm{e}^{-\theta x}, \quad x > 0, \quad \theta > 0, \tag{1}$$

and

$$F(x;\theta) = 1 - \frac{\theta x + \theta^2 + 1}{\theta^2+1}\,\mathrm{e}^{-\theta x}, \quad x > 0, \quad \theta > 0, \tag{2}$$

respectively.

Shanker [7] showed that the relation (1) is a two-component mixture of an exponential distribution and a gamma distribution. He has also presented various mathematical and statistical characteristics of this distribution. The Shanker and Lindley distributions have increasing hrfs. The purpose of this paper is to introduce a new two-parameter distribution, called the exponentiated Shanker ("E-Sh" for short) distribution, that can be established by the relationship (2) and has increasing, decreasing and bathtub-shaped hrfs and it is a very flexible model in lifetime phenomena. The exponentiated Shanker distribution can work better than some other distributions for modeling lifetime data.

The rest of the paper is organized as follows. The new model is introduced in Section 2. The hrf of the new model is also discussed in this section. The moments and incomplete moments are given in Section 3. Section 4 deals with the mean deviations and Bonferroni and Lorenz curves. Maximum likelihood (ML) estimation of the unknown parameters is discussed in Section 5. A simulation study is given in Section 6 to evaluate the ML estimators of the parameters. We provide two real data applications in Section 7. The paper ends with some concluding remarks.

2 The new model and its properties

Suppose that the random variable X has an E-Sh distribution, then its cdf is given by

$$F(x) = \left[1 - \frac{\theta x + \theta^2 + 1}{\theta^2+1}\,\mathrm{e}^{-\theta x}\right]^{\alpha}, \quad x > 0, \quad \alpha, \theta > 0. \tag{3}$$

The pdf of the E-Sh distribution is obtained by differentiating (3) and therefore we have

$$f(x)=\frac{\alpha\,\theta^2}{\theta^2+1}(\theta+x)\mathrm{e}^{-\theta x}\left[1-\frac{\theta x+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x}\right]^{\alpha-1},\quad x>0,\quad \alpha,\theta>0. \quad (4)$$

The hrf of the new distribution is given by

$$h(x)=\frac{\frac{\alpha\,\theta^2}{\theta^2+1}(\theta+x)\mathrm{e}^{-\theta x}\left[1-\frac{\theta x+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x}\right]^{\alpha-1}}{1-\left[1-\frac{\theta x+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x}\right]^{\alpha}}.$$

We plotted the pdfs of the new distribution for selected values of the parameters in Figure 1. We see that the pdf of this distribution can be either decreasing or unimodal depending on the values of the parameters. The plots of the hrfs of the E-Sh distribution for selected values of the parameters are given in Figure 2. From Figure 2, we observe that the hrf is decreasing, increasing or bathtub-shaped.

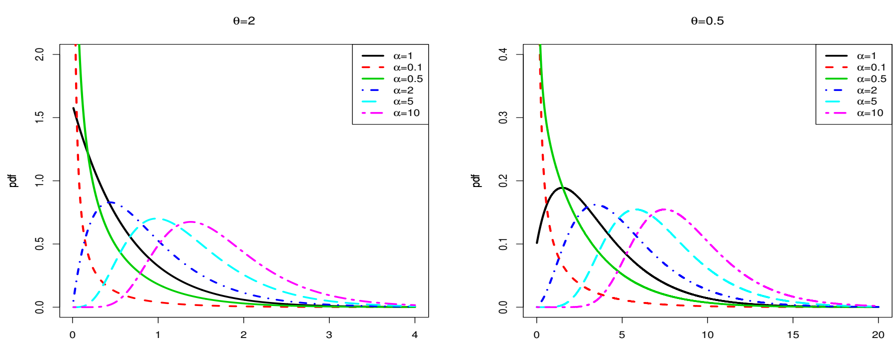

Figure 1: Pdfs of the E-Sh distribution for selected values of α and θ.

3 The moments and incomplete moments of the new distribution

Moments can be used to study the most important features and characteristics of a distribution. In this section, the complete and incomplete moments of the E-Sh distribution are presented. First, we present an expansion for $f(x)$. Using the generalized binomial expansion

$$(1-u)^a=\sum_{j=0}^{\infty}\binom{a}{j}(-1)^j\,u^j,\quad |u|<1,$$

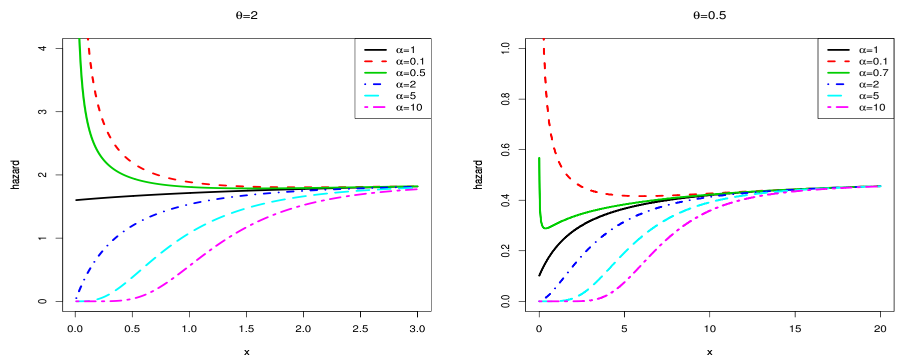

Figure 2: Hrfs of the E-Sh distribution for selected values of α and θ.

the pdf of the E-Sh can be expanded as

$$\begin{aligned} f(x) &= \frac{\alpha\,\theta^2}{\theta^2+1}(\theta+x)\mathrm{e}^{-\theta x}\left[1-\frac{\theta x+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x}\right]^{\alpha-1} \\ &= \frac{\alpha\theta^2}{\theta^2+1}(\theta+x)\mathrm{e}^{-\theta x}\sum_{j=0}^{\infty}\binom{\alpha-1}{j}(-1)^j\left(1+\frac{\theta x}{\theta^2+1}\right)^j\mathrm{e}^{-j\theta x} \\ &= \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{\alpha-1}{j}\binom{j}{k}\frac{(-1)^j\theta^{k+2}\,x^k}{(\theta^2+1)^{k+1}}(\theta+x)\mathrm{e}^{-(j+1)\theta x}. \end{aligned} \tag{5}$$

Similarly, the expansion of the cdf of the new distribution is

$$\begin{aligned} F(x) &= \left[1-\frac{\theta x+\theta^2+1}{\theta^2+1}\mathrm{e}^{-\theta x}\right]^{\alpha} = \sum_{j=0}^{\infty}\binom{\alpha}{j}(-1)^j\left(1+\frac{\theta x}{\theta^2+1}\right)^j\mathrm{e}^{-j\theta x} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{\alpha}{j}\binom{j}{k}\frac{(-1)^j\theta^k\,x^k}{(\theta^2+1)^k}\mathrm{e}^{-j\theta x}. \end{aligned} \tag{6}$$

So, the moment generating function of the model using (5) is given by

$$\begin{aligned} M_X(t) &= \int_0^{\infty}\mathrm{e}^{tx}\,f(x)\,\mathrm{d}x \\ &= \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\binom{\alpha-1}{j}\binom{j}{k}(-1)^j\theta^{k+2}k!}{(\theta^2+1)^{k+1}[(j+1)\theta-t]^{k+1}}\left(\theta+\frac{k+1}{(j+1)\theta-t}\right), \quad t<\theta. \end{aligned}$$

The r-th moment of the E-Sh distribution, from (5) is given by

$$\mu_r = E(X^r) = \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\binom{\alpha-1}{j}\binom{j}{k}(-1)^j\,\Gamma(k+r+1)}{(\theta^2+1)^{k+1}(j+1)^{k+r+1}\theta^r}\left(\theta^2+\frac{k+r+1}{j+1}\right). \tag{7}$$

Thus, the mean of the E-Sh distribution is

$$\mu = \mu_1 = E(X) = \alpha \sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{\binom{\alpha-1}{j}\binom{j}{k}(-1)^j\,\Gamma(k+2)}{(\theta^2+1)^{k+1}(j+1)^{k+2}\theta}\left(\theta^2+\frac{k+2}{j+1}\right).$$

Now, the skewness and kurtosis of the new distribution are given by

$$S = \frac{E\left[(X-E(X))^3\right]}{\left(E\left[(X-E(X))^2\right]\right)^{3/2}} = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{\left[\mu_2-\mu_1^2\right]^{3/2}},$$

and

$$K = \frac{E\left[(X-E(X))^4\right]}{\left(E\left[(X-E(X))^2\right]\right)^{2}} = \frac{\mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4}{\left[\mu_2-\mu_1^2\right]^{2}},$$

respectively, where μ_r is given in (7).

Next, we find the expressions for the incomplete moments of the new model. Let X be a random variable with the pdf given in (4), then the r-th incomplete moment of X is

$$\begin{aligned}\int_0^t x^r\, f(x)\,\mathrm{d}x &= \int_0^t x^r\,\frac{\alpha\,\theta^2}{\theta^2+1}(\theta+x)\mathrm{e}^{-\theta x}\left[1-\frac{\theta x+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x}\right]^{\alpha-1}\mathrm{d}x\\ &= \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{\alpha-1}{j}\binom{j}{k}\frac{(-1)^j\theta^{k+2}}{(\theta^2+1)^{k+1}}\\ &\quad\times\left[\int_0^t \theta\, x^{r+k}\mathrm{e}^{-(j+1)\theta x}\mathrm{d}x + \int_0^t x^{r+k+1}\mathrm{e}^{-(j+1)\theta x}\mathrm{d}x\right]\\ &= \alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{p=0}^{\infty}\binom{\alpha-1}{j}\binom{j}{k}\frac{(-1)^{j+p}\,\theta^{k+2+p}\,(j+1)^p\,t^{r+k+p+1}}{(\theta^2+1)^{k+1}\,p!}\\ &\quad\times\left[\frac{\theta}{r+k+p+1}+\frac{t}{r+k+p+2}\right].\end{aligned} \tag{8}$$

4 Mean deviations and Bonferroni and Lorenz curves

Suppose X follows the E-Sh distribution with the pdf (4). Then, from (6) and (8), the mean deviation from the mean is given by

$$\begin{aligned}\delta_1(X) =& \int_0^\infty |x-\mu|\, f(x)\,\mathrm{d}x = 2\mu\, F(\mu) - 2I(\mu)\\ =& 2\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{\alpha}{j}\binom{j}{k}\frac{(-1)^j\theta^k\,\mu^{k+1}}{(\theta^2+1)^k}\mathrm{e}^{-j\theta\mu}\\ & -2\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{p=0}^{\infty}\binom{\alpha-1}{j}\binom{j}{k}\frac{(-1)^{j+p}\,\theta^{k+2+p}\,(j+1)^p\,\mu^{k+p+2}}{(\theta^2+1)^{k+1}\,p!}\\ & \times\left[\frac{\theta}{k+p+2}+\frac{\mu}{k+p+3}\right],\end{aligned}$$

where $I(b) = \int_0^b x\, f(x)\,\mathrm{d}x$ and $\mu = E(X)$.

Let M denote the median. Then the mean deviation from the median is similarly obtained as follows

$$\begin{aligned}\delta_2(X) =& \int_0^\infty |x-M|\, f(x)\,\mathrm{d}x = \mu - 2I(M)\\ =& \mu - 2\alpha\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{p=0}^{\infty}\binom{\alpha-1}{j}\binom{j}{k}\frac{(-1)^{j+p}\,\theta^{k+2+p}\,(j+1)^p\,M^{k+p+2}}{(\theta^2+1)^{k+1}\,p!}\\ & \times\left[\frac{\theta}{k+p+2}+\frac{M}{k+p+3}\right].\end{aligned}$$

Now, we present the formulas of Bonferroni and Lorenz curves, that have applications in economy, reliability, medicine, and insurance.

Suppose X has the pdf given in (4), then from (8), the Bonferroni curve can be obtained as follows

$$\begin{aligned}B_F\left[F(x)\right] =& \frac{1}{\mu F(x)}\int_0^x u\, f(u)\,\mathrm{d}u = \frac{\alpha}{\mu\left[1-\frac{\theta x+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x}\right]^\alpha}\\ & \times\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{p=0}^{\infty}\binom{\alpha-1}{j}\binom{j}{k}\frac{(-1)^{j+p}\,\theta^{k+2+p}\,(j+1)^p\,x^{k+p+2}}{(\theta^2+1)^{k+1}\,p!}\\ & \times\left[\frac{\theta}{k+p+2}+\frac{x}{k+p+3}\right].\end{aligned}$$

The Lorenz curve is also obtained as

$$
\begin{aligned}
L_F\,[F(x)] = B_F\,[F(x)]\,F(x) &= \frac{1}{\mu}\int_0^x u\,f(u)\,\mathrm{d}u \\
&= \frac{\alpha}{\mu}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{p=0}^{\infty}\binom{\alpha-1}{j}\binom{j}{k}\frac{(-1)^{j+p}\,\theta^{k+2+p}\,(j+1)^p\,x^{k+p+2}}{(\theta^2+1)^{k+1}\,p!} \\
&\quad\times\left[\frac{\theta}{k+p+2}+\frac{x}{k+p+3}\right].
\end{aligned}
$$

5 Maximum likelihood estimation

Let $x_1,\ldots,x_n$ be an observed random sample of size n from the E-Sh distribution. Then, the likelihood function is

$$
\mathcal{L}(\alpha,\theta) = \left(\frac{\alpha\,\theta^2}{\theta^2+1}\right)^n \mathrm{e}^{-\theta\sum_{i=1}^n x_i}\prod_{i=1}^n(\theta+x_i)\prod_{i=1}^n\left[1-\frac{\theta x_i+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x_i}\right]^{\alpha-1}.
$$

Therefore, the log-likelihood function is

$$
\begin{aligned}
\ell(\alpha,\theta) \;=\; \log\mathcal{L}(\alpha,\theta) &= n\log\left(\frac{\alpha\,\theta^2}{\theta^2+1}\right) - \theta\sum_{i=1}^n x_i + \sum_{i=1}^n \log(\theta+x_i) \\
&+(\alpha-1)\sum_{i=1}^n \log\left[1-\frac{\theta x_i+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x_i}\right].
\end{aligned}
$$

The ML estimates of α and θ are obtained by maximizing the log-likelihood function with respect to the parameters. To do this, we take the derivatives of the log-likelihood function with respect to α and θ and then equate the results with zero. Therefore, we have

$$
\frac{\partial\ell(\alpha,\theta)}{\partial\alpha} = \frac{n}{\alpha}+\sum_{i=1}^n\log\left[1-\frac{\theta x_i+\theta^2+1}{\theta^2+1}\,\mathrm{e}^{-\theta x_i}\right] = 0, \tag{9}
$$

$$
\begin{aligned}
\frac{\partial\ell(\alpha,\theta)}{\partial\theta} &= \frac{2n}{\theta(\theta^2+1)} - \sum_{i=1}^n x_i + \sum_{i=1}^n\frac{1}{\theta+x_i} \\
&+(\alpha-1)\sum_{i=1}^n\frac{[2\theta+(\theta^2+1)(\theta+x_i)]\theta\,x_i\,\mathrm{e}^{-\theta x_i}}{(\theta^2+1)^2\left(1-\left[1+\frac{\theta x_i}{\theta^2+1}\right]\,\mathrm{e}^{-\theta x_i}\right)} = 0.
\end{aligned} \tag{10}
$$

We may use numerical methods to find the solutions of the above equations.

Next, we obtain the observed information matrix that is useful for finding approximate confidence intervals for the parameters. This matrix is defined as follows

$$I_n(\Omega) = -\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\theta} \\ I_{\alpha\theta} & I_{\theta\theta} \end{bmatrix},$$

where $\Omega = (\alpha, \theta)^T$ is the vector of parameters and

$$I_{\alpha\alpha} = \frac{\partial^2 \ell(\alpha,\theta)}{\partial \alpha^2} = \frac{-n}{\alpha^2},$$

$$I_{\alpha\theta} = \frac{\partial^2 \ell(\alpha,\theta)}{\partial \alpha\, \partial \theta} = \sum_{i=1}^{n} \frac{[2\theta + (\theta^2+1)(\theta + x_i)]\theta\, x_i\, \mathrm{e}^{-\theta x_i}}{(\theta^2+1)^2 \left(1 - \left[1 + \frac{\theta x_i}{\theta^2+1}\right] \mathrm{e}^{-\theta x_i}\right)},$$

$$\begin{aligned} I_{\theta\theta} = \frac{\partial^2 \ell(\alpha,\theta)}{\partial \theta^2} &= \frac{-2n(3\theta^2+1)}{[\theta(\theta^2+1)]^2} - \sum_{i=1}^{n} \frac{1}{(\theta + x_i)^2} \\ &-(\alpha-1)\sum_{i=1}^{n} \left(\frac{[2\theta + (\theta^2+1)(\theta + x_i)]\theta\, x_i\, \mathrm{e}^{-\theta x_i}}{(\theta^2+1)^2 \left(1 - \left[1 + \frac{\theta x_i}{\theta^2+1}\right] \mathrm{e}^{-\theta x_i}\right)} \right)^2 \\ &-(\alpha-1)\sum_{i=1}^{n} \frac{[\theta^4 x_i(\theta^2+\theta x_i+5)+2\theta^3(x_i^2+1)+\theta x_i(3\theta+x_i)-6\theta-x_i]x_i \mathrm{e}^{-\theta x_i}}{(\theta^2+1)^3 \left(1 - \left[1 + \frac{\theta x_i}{\theta^2+1}\right] \mathrm{e}^{-\theta x_i}\right)}. \end{aligned}$$

Let $J(\Omega)^{-1}$ be the inverse of the expected Fisher information matrix and $\widehat{\Omega}$ be the ML estimator of Ω. If the sample size n is large enough, then under the regularity conditions (see for example [3, Pages 461-463]), $\sqrt{n}(\widehat{\Omega} - \Omega)$ follows a two-variate normal distribution asymptotically with the mean $(0,0)^T$ and the variance-covariance matrix $J(\Omega)^{-1}$. We may replace $J(\Omega)^{-1}$ with $\left[\frac{1}{n} I_n(\Omega)\right]^{-1}$ as the asymptotic behavior remains valid. The unknown parameters that may appear in $\left[\frac{1}{n} I_n(\Omega)\right]^{-1}$ can be replaced by their corresponding ML estimates. We may obtain approximate (asymptotic) confidence intervals for the parameters and perform test hypotheses using the above explained normal approximation.

6 A simulation study

In this section, we evaluate the performance of the ML estimators of the parameters of the E-Sh model by means of a simulation study. The inverse transform algorithm is used to generate random data from the E-Sh distribution. The precision of the ML estimators is discussed by means of the bias, mean squared error (MSE) and mean relative error (MRE). We generated $N = 10,000$ samples of sizes $n = 50, 150, 300$ from the E-Sh distribution with the following parameter combinations: $(\alpha,\, \theta) =$

$(2, 2)$, $(3, 5)$, $(0.5, 0.5)$ and $(2, 0.5)$. We obtained the ML estimates of the parameters for each generated sample and the standard errors of the ML estimators were obtained by inverting the observed information matrix. Let $\widehat{\alpha}$ be the ML estimator of α and $\widehat{\alpha}_i$ be the ML estimator of α that is obtained in the i-th iteration, then the estimated bias, MSE, and MRE of $\widehat{\alpha}$ can be obtained using the following equations

$$\widehat{Bias}_{\alpha}(N) = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\alpha}_i - \alpha), \quad \widehat{MSE}_{\alpha}(N) = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\alpha}_i - \alpha)^2,$$

and

$$\widehat{MRE}_{\alpha}(N) = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\widehat{\alpha}_i}{\alpha}\right),$$

respectively. We can obtain the estimated bias, MSE, and MRE of $\widehat{\theta}$ (the ML estimator of θ) similarly.

The numerical results of the simulation are given in Table 1. It is clear from Table 1 that the estimated biases and MSEs decrease when the sample size n increases. In addition, the estimated MREs of all parameters are close to one and approach this nominal value when the sample size increases.

7 Applications of the new model

In this section, we present two real data applications in order to show the flexibility of the E-Sh distribution. We compare the fit of the E-Sh distribution with those of some other lifetime distributions which are

1. The Shanker distribution, which is a special case of the E-Sh distribution when $\alpha = 1$.

2. The Lindley distribution with the following pdf

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)\,\mathrm{e}^{-\theta x}, \qquad x>0, \quad \theta>0.$$

3. The Weibull distribution with the following pdf

$$f(x;\alpha,\theta) = \alpha\theta\, x^{\alpha-1}\,\mathrm{e}^{-\theta x^{\alpha}}, \qquad x>0, \quad \alpha,\theta>0.$$

4. The new weighted Lindley (NWL) distribution, introduced in [1] with the following pdf

$$f(x;\alpha,\theta) = \frac{\theta^2(1+\alpha)^2\,(1+x)}{\alpha\,(\alpha\theta+\alpha+\theta+2)}\left(1-\mathrm{e}^{-\alpha\theta x}\right)\mathrm{e}^{-\theta x}, \quad x>0, \quad \alpha,\theta>0.$$

Table 1: The simulation results.

		$\alpha = 2,$	$\theta = 2$	
n	Parameters	Bias	MSE	MRE
50	α	0.182	0.348	1.091
	θ	0.088	0.092	1.044
150	α	0.064	0.081	1.032
	θ	0.029	0.024	1.015
300	α	0.027	0.036	1.014
	θ	0.013	0.011	1.007
		$\alpha = 3,$	$\theta = 5$	
n	Parameters	Bias	MSE	MRE
50	α	0.304	0.955	1.101
	θ	0.168	0.573	1.034
150	α	0.079	0.214	1.026
	θ	0.063	0.196	1.013
300	α	0.033	0.091	1.011
	θ	0.016	0.080	1.003
		$\alpha = 0.5,$	$\theta = 0.5$	
n	Parameters	Bias	MSE	MRE
50	α	0.083	0.136	1.041
	θ	0.020	0.018	1.014
150	α	0.049	0.062	1.024
	θ	0.011	0.009	1.007
300	α	0.020	0.041	1.010
	θ	0.005	0.005	1.003
		$\alpha = 2,$	$\theta = 0.5$	
n	Parameters	Bias	MSE	MRE
50	α	0.083	0.155	1.041
	θ	0.007	0.003	1.015
150	α	0.052	0.068	1.026
	θ	0.004	0.002	1.008
300	α	0.031	0.005	1.015
	θ	0.001	0.0006	1.003

The first real data set, reported by [2, Page 105], is related to the relief times (in hours) of 20 patients receiving an analgesic. The data that are denoted by D I here, are given by

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

The second data, which are denoted by D II in this paper, are taken from [4] and they are related to the endurance test of deep groove ball bearings. Each datum represents the number of revolutions in hundreds of millions to failure for a ball bearing. The data are

0.1788, 0.2892, 0.3300, 0.4152, 0.4212, 0.4560, 0.4848, 0.5184,
0.5196, 0.5412, 0.5556, 0.6780, 0.6864, 0.6864, 0.6888, 0.8412,
0.9312, 0.9864, 1.0512, 1.0584, 1.2792, 1.2804, 1.734.

We use the Kolmogorov-Smirnov (K-S) test statistic, the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) in order to compare the fits. The computed ML estimates, K-S test statistics and the values of AIC and BIC for both data sets are given in Table 2. These criteria are widely utilized to check how closely a specified cdf fits the empirical distribution of a given data set. It is well-known that the smaller values of AIC, BIC, and K-S test statistic mean a better fit to a data set. Here, it is observed from Table 2 that the E-Sh model outperforms all the other considered models in the sense of the considered criteria.

In the sequel we discuss the existence of the solutions of the likelihood equations (9) and (10). We have plotted the profile log-likelihood functions with respect to the parameters for D I and D II in Figure 3. The plots confirm that the solutions exist and each pair of solutions maximize their corresponding log-likelihood function.

To compare the fits visually, the fitted pdfs of the considered models as well as the histograms are plotted for D I and D II, see Figure 4. It can be seen that the E-Sh is the most suitable distribution for both data sets among the other considered models.

We also make use of the likelihood ratio (LR) tests to compare the E-Sh distribution with its sub-model. Here, performing the test of $H_0 : \alpha = 1$ against $H_1 : \alpha \neq 1$ means that we want to compare the E-Sh distribution with its sub-model, i.e. the Shanker distribution. The following statistic may be used to perform the mentioned LR test

$$LR = 2\left[\ell(\widehat{\alpha},\widehat{\theta}) - \ell(1,\widehat{\theta}_0)\right],$$

where $\widehat{\theta}_0$ is the ML estimator of θ obtained under H_0. The LR test statistic converges to a chi-square distribution with one degree of freedom. The results of the LR tests are presented in Table 3. As the p-values are too small, we may reject H_0 for both data sets and reach the conclusion that the E-Sh model is superior to its sub-model, the Shanker distribution, in this example. We note that the figure plotting and computations of this paper have been performed using R [5]. Moreover, the package *survival* was used [9].

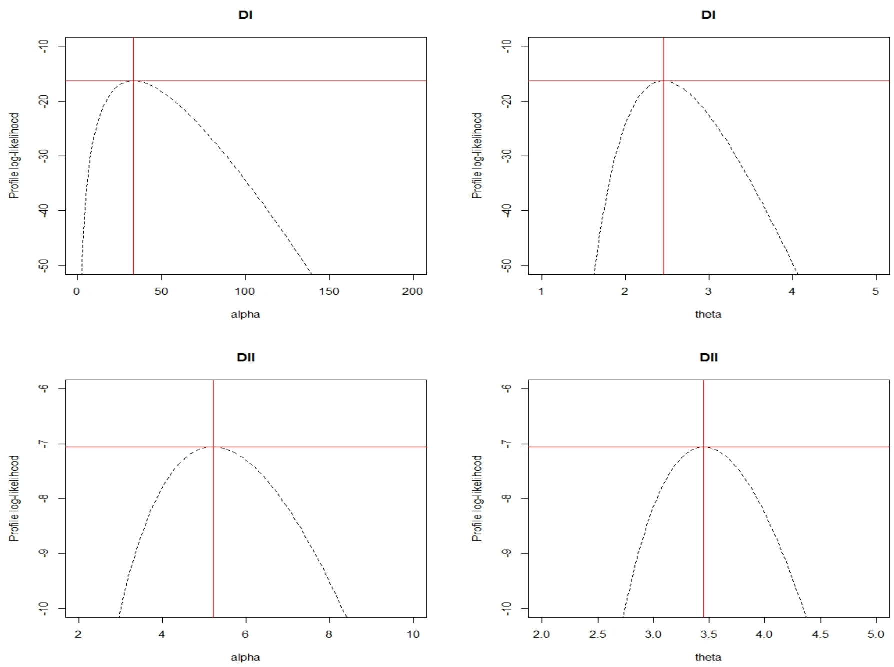

Figure 3: The profile log-likelihood function plots of the E-Sh distribution for D I and D II.

8 Concluding remarks and future works

In this paper, we introduced a new distribution and discussed its properties. It has only two parameters and its cdf and hrf have simple forms. The hrf of the new model can be decreasing, increasing and bathtub-shaped depending on the values of the parameters. We can claim that the proposed distribution provides a quite flexible model for fitting many positive data sets that may appear in the areas like engineering, survival analysis, hydrology, economics, and so on, in comparison with some other distributions. There exist many other properties of the new distribution, like the reliability parameter, Kullback-Leibler divergence and so on, that have not been discussed in this paper. Moreover, some interesting inferential topics related to the new distribution are the Bayesian estimation of the parameters, prediction of future observations and estimation based on censored samples.

Table 2: The ML estimates of the parameters (standard errors in parentheses), K-S test statistics and the values of AIC and BIC for D I and D II.

Data set	Models	Parameter Estiamtes α	θ	K-S	AIC	BIC
D I	Lindley		1.123 (0.154)	0.540	77.361	78.357
	Shanker		0.804 (0.119)	0.365	61.783	62.779
	Weibull	2.787 (0.427)	2.130 (0.182)	0.185	45.173	47.164
	NWL	0.0003 (0.553)	1.365 (0.419)	0.293	53.718	55.710
	E-Sh	33.305 (23.710)	2.452 (0.423)	0.135	36.664	38.655
D II	Lindley		2.123 (0.359)	0.339	21.090	22.225
	Shanker		1.703 (0.265)	0.287	31.947	33.082
	Weibull	2.102 (0.329)	0.819 (0.086)	0.151	19.545	21.816
	NWL	0.0002 (0.672)	3.292 (1.184)	0.176	22.466	24.737
	E-Sh	5.204 (2.043)	3.452 (0.616)	0.106	18.113	20.384

Acknowledgements

We would like to thank the two anonymous reviewers for their valuable comments.

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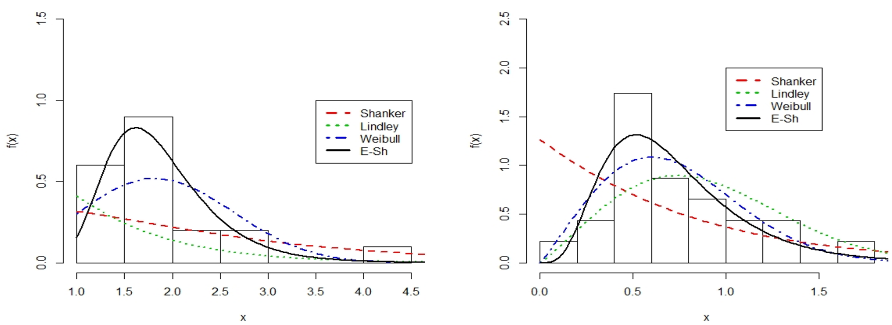

Figure 4: The plots of the fitted pdfs of the considered distributions as well as the histograms for D I (left) and D II (right).

Table 3: The results of LR tests for D I and D II.

Data set		Hypotheses	LR	p-value
D I	E-Sh versus Shanker	$H_0 : \alpha = 1$	27.120	< 0.001
D II	E-Sh versus Shanker	$H_0 : \alpha = 1$	15.833	< 0.001

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