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Existence and continuation of solutions of Hilfer fractional differential equations

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Abstract. In the present paper we consider initial value problems for Hilfer fractional differential equations and for system of Hilfer fractional differential equations. By using equivalent integral equations and some fixed point theorems, we study the local existence of solutions. We extend these local existence results globally with the help of continuation theorems and generalized Gronwall inequality.

Keywords: Fractional differential equations, local existence, continuation theorem, global solutions.
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1 Introduction

Theory of fractional order systems has gained remarkable significance during last few decades due to its real world applications in ostensibly diverse and wide spread fields of applied mathematics, physics and engineering. The monographs [12, 14, 18, 20, 21] are devoted to such practical problems in control theory, modeling, relaxations and serve as a foundation of fractional order theory in physics and applied sciences. Recently, Hilfer [12, 13], Mainardi [20] discussed various applications of fractional differential equations (FDEs) in their works.

In the recent investigations, many researchers studied the existence and uniqueness of solution of nonlinear FDEs, see [1]- [8], [5, 9, 10, 16, 17, 21–23]

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and references therein. In this paper we mainly focus on developing the theory of existence and uniqueness. First we obtain the local existences followed by continuation theorems to extend the existence of solutions globally. The results obtained in this paper generalizes the existing results in the literature [15,19].

The rest of the paper is organized as follow. In Section 2, we collect all useful definitions and previously known results. Local existence of solutions are discussed in Section 3. Continuation and global existence of solution are obtained in Section 4. Concluding remarks are given in the last section.

2 Preliminaries

Let $C_{1-\gamma}(\Omega)$ be a complete metric space of all continuous functions mapping $\Omega = [0, T],\ T > 0$, into $\mathbb{R}$ with the metric d defined by [10]

$$d(x_1, x_2) = \|x_1 - x_2\|_{C_{1-\gamma}(\Omega)} := \max_{t\in\Omega} |t^{1-\gamma}[x_1(t) - x_2(t)]|,$$

where $C_{1-\gamma}(\Omega) = \{x(t) : (0, T] \to \mathbb{R} : t^{1-\gamma}x(t) \in C(\Omega)\}$. Further, $L^1(0, T)$ is the space of Lebesgue integrable functions on $(0, T)$.

Definition 1. *[16]* Let $(0, T]$ and $f : (0, \infty) \to \mathbb{R}$ be a real valued continuous function. The Riemann-Liouville fractional integral of a function f of order $\alpha \in \mathbb{R}^+$ is denoted as $I^\alpha_{0+}f$ and defined by

$$I^\alpha_{0+}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(r)dr}{(t-r)^{1-\alpha}}, \quad t > 0, \tag{1}$$

where $\Gamma(\alpha)$ is the Euler's Gamma function.

Definition 2. *[14]* Let $f : (0, \infty) \to \mathbb{R}$ be a real valued continuous function. The Riemann-Liouville fractional derivative of function f of order $\alpha \in \mathbb{R}^+_0 = [0, +\infty)$ is denoted as $D^\alpha_{0+}f$ and defined by

$$D^\alpha_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(r)dr}{(t-r)^{\alpha-n+1}}, \tag{2}$$

where $n = [\alpha] + 1$, and $[\alpha]$ means the integer part of α, provided the right hand side is pointwise defined on $(0, \infty)$.

Definition 3. *[14]* The Caputo fractional derivative of function f with order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$ is defined by

$${}^C D^\alpha_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(r)dr}{(t-r)^{\alpha-n+1}},\ t > 0. \tag{3}$$

Definition 4. *[12]* The Hilfer fractional derivative $D_{0^+}^{\alpha,\beta}$ of function $f \in L^1(0,T)$ of order $n-1<\alpha<n$ and type $0\le\beta\le 1$ is defined by

$$D_{0^+}^{\alpha,\beta} f(t) = I_{0^+}^{\beta(n-\alpha)} D^n I_{0^+}^{(1-\beta)(n-\alpha)} f(t), \tag{4}$$

where $I_{0^+}^{\alpha}$ and $D_{0^+}^{\alpha}$ are Riemann-Liouville fractional integral (Definiton 1) and derivative (Definition 2), respectively.

Remark 1. *(See [10]) Hilfer fractional derivative interpolates between the R-L (Definition 2) and Caputo (Definition 3) fractional derivatives since*

$$D_{0^+}^{\alpha,\beta} = \begin{cases} D I_{0^+}^{1-\alpha} = D_{0^+}^{\alpha}, & \beta = 0, \\ I_{0^+}^{1-\alpha} D = {}^C D_{0^+}^{\alpha}, & \beta = 1. \end{cases}$$

Let $0<\alpha<1, 0\le\beta\le 1$. First we consider the initial value problem (IVP)

$$\begin{cases} D_{0^+}^{\alpha,\beta} x(t) & = f(t,x), \quad t \in (0,+\infty), \\ I_{0^+}^{1-\gamma} x(0^+) & = x_0, \quad \gamma = \alpha+\beta-\alpha\beta, \end{cases} \tag{5}$$

for Hilfer FDEs and then IVP for the system of Hilfer FDEs

$$\begin{cases} D_{0^+}^{\alpha,\beta} x_1(t) & = f_1(t, x_1, x_2, \ldots, x_n), \\ D_{0^+}^{\alpha,\beta} x_2(t) & = f_2(t, x_1, x_2, \ldots, x_n), \\ & \vdots \\ D_{0^+}^{\alpha,\beta} x_n(t) & = f_n(t, x_1, x_2, \ldots, x_n), \\ I_{0^+}^{1-\gamma} x_i(0^+) & = x_0, \quad \gamma = \alpha+\beta-\alpha\beta, i = 1,2,\ldots,n, \end{cases} \tag{6}$$

where $f(t,x(t)) : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ in IVP (5), $f_i(t,x_i(t)) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ in IVP (6) have weak singularities with respect to t.

Furthermore, the equivalence of IVP (5) and integral equation

$$x(t) = \frac{x_0}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} f(r, x(r)) dr, \quad t \in (0,\infty), \tag{7}$$

is proved in following lemma [10].

Lemma 1. *[10] Let $\gamma = \alpha+\beta-\alpha\beta$ where $0<\alpha<1$ and $0\le\beta\le 1$. Assume that $f : (0,T]\times\mathbb{R} \to \mathbb{R}$ such that $f(t,x(t)) \in C_{1-\gamma}(\Omega)$ for any $x \in C_{1-\gamma}(\Omega)$. A function $x \in C_{1-\gamma}^{\gamma}(\Omega)$ satisfies IVP (5) if and only if x satisfies the integral equation (7).*

Lemma 2. *[15] If Q is a subset of $C_{1-\gamma}(\Omega), 0<\gamma\le 1$, then Q is precompact if and only if the following conditions hold:*

(i) $\{t^{1-\gamma}x(t) : x \in Q\}$ *is uniformly bounded,*

(ii) $\{t^{1-\gamma}x(t) : x \in Q\}$ *is equicontinuous on* Ω.

Lemma 3. *[16] Let* $a < b < c, 0 \leq \mu < 1$, $x \in C_\mu[a,b]$, $y \in C[b,c]$ *and* $x(b) = y(b)$. *Define*

$$z(t) = \begin{cases} x(t), & \text{if } \ t \in (a,b], \\ y(t), & \text{if } \ t \in [b,c], \end{cases}$$

then $z \in C_\mu[a,c]$.

Lemma 4. *(Schauder Fixed Point Theorem [11]) Let* U *be a closed bounded convex subset of a Banach space* X *and* $T : U \to U$ *is completely continuous. Then* T *has a fixed point in* U.

Lemma 5. *[14] Assume that* $\alpha > 0$ *and* $0 \leq \mu < 1$. *If* $\mu > \alpha$, *then R-L fractional integrals* I^α_{0+} *are bounded from* $C_\mu(\Omega)$ *into* $C_{\mu-\alpha}(\Omega)$. *If* $\mu \leq \alpha$, *then R-L fractional integrals* I^α_{0+} *are bounded from* $C_\mu(\Omega)$ *into* $C(\Omega)$.

3 Local existence

In this section, we obtain the local existence of solutions of IVPs (5) and (6). To this end we make the following two hypotheses.

(H_1) $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ in IVP (5) is continuous and there exists a constant $0 \leq \theta < 1$ such that $(\Lambda x)(t) = t^\theta f(t, x(t))$ is a continuous bounded map from $C_{1-\gamma}(\Omega)$ into $C(\Omega)$.

(H_2) $f_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ in IVP (6) are continuous functions and there exists constants $0 \leq \theta_i < 1$, such that $(\Lambda_i x_i)(t) = t^{\theta_i} f_i(t, x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$ are continuous bounded maps from $C_{1-\gamma}(\Omega)$ into $C(\Omega)$.

Theorem 1. *Suppose that* (H_1) *holds. Then IVP* (5) *has at least one solution* $x \in C_{1-\gamma}(\Omega^*)$ *with* $\Omega^* = [0, h]$, $(T \geq)h > 0$.

Proof. Suppose that

$$X = \left\{ x \in C_{1-\gamma}(\Omega) : \left\| x - \frac{x_0}{\Gamma(\gamma)} t^{\gamma-1} \right\|_{C_{1-\gamma}(\Omega)} = \sup_{t\in\Omega} \left| t^{1-\gamma}x(t) - \frac{x_0}{\Gamma(\gamma)} \right| \leq b \right\},$$

where $b > 0$ is a constant. Since the operator Λ is bounded, there exists a constant $M > 0$, such that $\sup \left\{ |(\Lambda x)(t)| : t \in \Omega, x \in X \right\} \leq M$. We consider

$$X_h = \left\{ x : x \in C_{1-\gamma}(\Omega^*), \sup_{0\leq t\leq h} \left| t^{1-\gamma}x(t) - \frac{x_0}{\Gamma(\gamma)} \right| \leq b \right\},$$

where $h = \min\left\{\left(\frac{b\Gamma(\alpha-\theta+1)}{M\Gamma(1-\theta)}\right)^{\frac{1}{\alpha-\gamma-\theta+1}}, T\right\}$. Obviously, $X_h \subset C_{1-\gamma}(\Omega^*)$ is nonempty, closed bounded and convex subset.

Note that $h \leq T$ and we define the operator Θ by

$$(\Theta x)(t) = \frac{x_0}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1} f(r, x(r))dr, \quad t \in \Omega^*. \tag{8}$$

Clearly, it follows from (H_1) and Lemma 5 that $\Theta(C_{1-\gamma}(\Omega^*)) \subset C_{1-\gamma}(\Omega^*)$.

On the other hand, by relation (8), for any $x \in C_{1-\gamma}(\Omega^*)$ we have

$$\begin{aligned}\left|t^{1-\gamma}(Bx)(t) - \frac{x_0}{\Gamma(\gamma)}\right| &= \left|\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1} r^{-\theta}[r^\theta f(r, x(r))]dr\right| \\ &\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1} r^{-\theta} M dr \\ &\leq M t^{1-\gamma} I_{0+}^\alpha (t^{-\theta}) \leq \frac{M h^{\alpha-\gamma-\theta+1}\Gamma(1-\theta)}{\Gamma(\alpha-\theta+1)} \leq b,\end{aligned}$$

which means $\Theta(X_h) \subset X_h$.

Next we show that the operator Θ is continuous. Let $x_n, x \in X_h$, $\|x_n - x\|_{C_{1-\gamma}(\Omega^*)} \to 0$ as $n \to +\infty$. In view of continuity of Λ, we have $\|\Lambda x_n - Ax\|_{\Omega^*} \to 0$ as $n \to +\infty$. Note that

$$\begin{aligned}&\left|t^{1-\gamma}(\Theta x_n)(t) - t^{1-\gamma}(\Theta x)(t)\right| \\ &\quad = \left|\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1} f(r, x_n(r))dr - \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1} f(r, x(r))dr\right| \\ &\quad \leq \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1} r^{-\theta}[r^\theta |f(r, x_n(r)) - f(r, x(r))|]dr \\ &\quad \leq \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1} r^{-\theta}|(\Lambda x_n)(r) - (\Lambda x)(r)|dr \\ &\quad \leq \|(\Lambda x_n)(\cdot) - (\Lambda x)(\cdot)\|_{\Omega^*}\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1} r^{-\theta}dr,\end{aligned}$$

and therefore we have

$$\|(\Theta x_n)(t) - (\Theta x)(t)\|_{C_{1-\gamma}(\Omega^*)} \leq \|(\Lambda x_n)(\cdot) - (\Lambda x)(\cdot)\|_{\Omega^*}\frac{\Gamma(1-\theta)h^{\alpha-\gamma-\theta+1}}{\Gamma(\alpha-\theta+1)}.$$

Then $\|(\Theta x_n)(t) - (\Theta x)(t)\|_{C_{1-\gamma}(\Omega^*)} \to 0$ as $n \to +\infty$. Thus Θ is continuous. Furthermore, we shall prove that the operator $\Theta(X_h)$ is continuous.

Let $x \in X_h$, and $0 \leq t_1 < t_2 \leq h$. For any $\epsilon > 0$, note that

$$\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1} r^{-\theta}dr = \frac{\Gamma(1-\theta)}{\Gamma(\alpha-\theta+1)}t^{\alpha-\gamma-\theta+1} \to 0 \text{ as } t \to 0^+,$$

and hence there exists a $(h >)\theta_1 > 0$ such that for $t \in [0, \theta_1]$ we can write

$$\frac{2Mt^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} r^{-\theta} dr < \epsilon.$$

We know that $\Omega^* = [0, \theta_1] \cup [\frac{\theta_1}{2}, h]$. Suppose that $t_1, t_2 \in [0, \theta_1]$, we have

$$\begin{aligned} &\Big|\frac{t_1^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_1} (t_1-r)^{\alpha-1} f(r, x(r)) dr - \frac{t_2^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_2} (t_2-r)^{\alpha-1} f(r, x(r)) dr\Big| \\ &\leq \frac{Mt_1^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_1} (t_1-r)^{\alpha-1} r^{-\theta} dr + \frac{Mt_2^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_2} (t_2-r)^{\alpha-1} r^{-\theta} dr < \epsilon. \end{aligned} \tag{9}$$

For $t_1, t_2 \in [\frac{\theta_1}{2}, h]$, we get

$$\begin{aligned} &\big|t_1^{1-\gamma}(\Theta x)(t_1) - t_2^{1-\gamma}(\Theta x)(t_2)\big| \\ &= \Big|\frac{t_1^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_1} (t_1-r)^{\alpha-1} f(r, x(r)) dr - \frac{t_2^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_2} (t_2-r)^{\alpha-1} f(r, x(r)) dr\Big| \\ &= \Big|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} [t_1^{1-\gamma}(t_1-r)^{\alpha-1} - t_2^{1-\gamma}(t_2-r)^{\alpha-1}] f(r, x(r)) dr \\ &\quad - \frac{t_2^{1-\gamma}}{\Gamma(\alpha)} \int_{t_1}^{t_2} t_2^{1-\gamma}(t_2-r)^{\alpha-1} f(r, x(r)) dr\Big|. \end{aligned}$$

If $0 \leq \mu_1 < \mu_2 \leq h$ then $\mu_1^{1-\gamma}(\mu_1 - r)^{\alpha-1} > \mu_2^{1-\gamma}(\mu_2 - r)^{\alpha-1}$ for $0 \leq r < \mu_1$ and we obtain

$$\begin{aligned} &\Big|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} [t_1^{1-\gamma}(t_1-r)^{\alpha-1} - t_2^{1-\gamma}(t_2-r)^{\alpha-1}] f(r, x(r)) dr\Big| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |[t_1^{1-\gamma}(t_1-r)^{\alpha-1} - t_2^{1-\gamma}(t_2-r)^{\alpha-1}] r^{-\theta}| r^{\theta} f(r, x(r)) dr \\ &\leq \frac{M}{\Gamma(\alpha)} \int_0^{\frac{\theta_1}{2}} |[t_1^{1-\gamma}(t_1-r)^{\alpha-1} - t_2^{1-\gamma}(t_2-r)^{\alpha-1}] r^{-\theta}| dr \\ &\quad + \left(\frac{\theta_1}{2}\right)^{-\theta} \frac{M}{\Gamma(\alpha)} \int_{\frac{\theta_1}{2}}^{t_1} [t_1^{1-\gamma}(t_1-r)^{\alpha-1} - t_2^{1-\gamma}(t_2-r)^{\alpha-1}] dr \\ &\leq \frac{2M(\frac{\theta_1}{2})^{1-\gamma}}{\Gamma(\alpha)} \int_0^{\frac{\theta_1}{2}} \big(\frac{\theta_1}{2} - r\big)^{\alpha-1} r^{-\theta} dr \\ &\quad + \frac{M(\frac{\theta_1}{2})^{-\theta}}{\Gamma(\alpha+1)} \Big[{t_2}^{1-\gamma}(t_2-t_1)^{\alpha} - {t_2}^{1-\gamma}\big(t_2 - \frac{\theta_1}{2}\big)^{\alpha} + {t_1}^{1-\gamma}\big(t_1 - \frac{\theta_1}{2}\big)^{\alpha} \Big] \\ &\leq \epsilon + \frac{M(\frac{\theta_1}{2})^{-\theta}}{\Gamma(\alpha+1)} \Big[h^{1-\gamma}(t_2-t_1)^{\alpha} + {t_2}^{1-\gamma}\big(t_2 - \frac{\theta_1}{2}\big)^{\alpha} + {t_1}^{1-\gamma}\big(t_1 - \frac{\theta_1}{2}\big)^{\alpha} \Big]. \end{aligned}$$

On the other hand

$$\begin{aligned}
\Big|\frac{{t_2}^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-r)^{\alpha-1}f(r,x(r))dr\Big| &\leq \frac{\left(\frac{\theta_1}{2}\right)^{-\theta}M}{\Gamma(\alpha)}\int_{t_1}^{t_2} t_2^{1-\gamma}(t_2-r)^{\alpha-1}dr\\
&= \frac{\left(\frac{\theta_1}{2}\right)^{-\theta}M}{\Gamma(\alpha+1)}t_2^{1-\gamma}(t_2-t_1)^{\alpha}\\
&\leq \frac{\left(\frac{\theta_1}{2}\right)^{-\theta}Mh^{1-\gamma}}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}.
\end{aligned}$$

Clearly, there exist a $\theta(\frac{\theta_1}{2}>\theta>0)$ such that, for $t_1,t_2\in[\frac{\theta_1}{2},h]$, $|t_1-t_2|<\theta$ implies

$$|{t_1}^{1-\gamma}(\Theta x)(t_1)-{t_2}^{1-\gamma}(\Theta x)(t_2)|<2\epsilon. \tag{10}$$

It follows from inequalities (9) and (10) that $\{t^{1-\gamma}(\Theta x)(t) : x\in X_h\}$ is equicontinuous. Obviously, it is clear that $\{t^{1-\gamma}(\Theta x)(t) : x\in X_h\}$ is uniformly bounded since $\Theta(X_h)\subset X_h$. By Lemma 2, $\Theta(X_h)$ is percompact. Therefore Θ is completely continuous. By Schauder fixed point theorem and Lemma 1, the IVP (5) has a local solution. □

Theorem 2. *Suppose that* (H_2) *holds. Then IVP* (6) *has at least one solution* $x_i\in C_{1-\gamma}(\Omega)$ *for some* $(T\geq 0)h>0$.

Proof. Suppose that

$$X_s=\Big\{x_i\in C_{1-\gamma}(\Omega):\|x_i-\frac{x_0t^{\gamma-1}}{\Gamma(\gamma)}\|_{C_{1-\gamma}(\Omega)}=\sup_{t\in\Omega}|t^{1-\gamma}x_i-\frac{x_0}{\Gamma(\gamma)}|\leq b_i\Big\},$$

for $b_i>0$ $(i=1,2,\ldots,n)$ are constants. Since operators Λ_i, $(i=1,2,\ldots,n)$ are bounded then there exists constants $M_i>0$, $(i=1,2,\ldots,n)$ such that

$$\sup\big\{|(\Lambda_i x_i)(t)| : t\in\Omega, x_i\in X_s\big\}\leq M_i,\ \ i=1,2,\ldots,n.$$

We consider

$$D_{ih}=\Big\{x_i : x_i\in C_{1-\gamma}(\Omega^*),\ \sup_{0\leq t\leq h}|t^{1-\gamma}x_i(t)-\frac{x_0}{\Gamma(\gamma)}|\leq b_i\Big\},$$

where

$$\begin{aligned}
h=\min\Bigg\{&\left(\frac{b_1\Gamma(\alpha-\theta_1+1)}{M_1\Gamma(1-\theta_1)}\right)^{\frac{1}{\alpha-\gamma-\theta_1+1}},\left(\frac{b_2\Gamma(\alpha-\theta_2+1)}{M_2\Gamma(1-\theta_2)}\right)^{\frac{1}{\alpha-\gamma-\theta_2+1}}\\
&\ldots,\left(\frac{b_n\Gamma(\alpha-\theta_n+1)}{M_n\Gamma(1-\theta_n)}\right)^{\frac{1}{\alpha-\gamma-\theta_n+1}},T\Bigg\}
\end{aligned}$$

$\alpha > \theta_i$, $i = 1, 2, \ldots, n$. Clearly, $D_{ih} \subset C_{1-\gamma}(\Omega^*)$ are nonempty, closed bounded and convex subsets. Note that $h \leq T, t \in \Omega^*$. Define operators Θ_i as

$$\begin{cases} (\Theta_1 x_1)(t) &= \frac{x_0 t^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} f_1(r, x_1(r), x_2(r), \ldots, x_n(r)) dr, \\ (\Theta_2 x_2)(t) &= \frac{x_0 t^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} f_2(r, x_1(r), x_2(r), \ldots, x_n(r)) dr, \\ &\vdots \\ (\Theta_n x_n)(t) &= \frac{x_0 t^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} f_n(r, x_1(r), x_2(r), \ldots, x_n(r)) dr. \end{cases} \tag{11}$$

By system (11), for $x_i \in C_{1-\gamma}(\Omega^*)$, we have

$$\begin{aligned} \left| t^{1-\gamma}(\Theta_1 x_1)(t) - \frac{x_0}{\Gamma(\gamma)} \right| &= \Big| \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} r^{-\theta_1} \\ &\qquad \times [r^{\theta_1} f_1(r, x_1(r), x_2, \ldots, x_n(r))] dr \Big| \\ &\leq M_1 t^{1-\gamma} {}_0I_t^{\alpha}(t^{-\theta_1}) = \frac{M_1 \Gamma(1-\theta_1)}{\Gamma(\alpha - \theta_1 + 1)} t^{\alpha - \theta_1 - \gamma + 1}, \end{aligned}$$

$$\begin{aligned} |t^{1-\gamma}(\Theta_1 x_1)(t) - \frac{x_0}{\Gamma(\gamma)}| &\leq \frac{M_1 \Gamma(1-\theta_1)}{\Gamma(\alpha - \theta_1 + 1)} h^{\alpha - \theta_1 - \gamma + 1} \leq b_1, \\ |t^{1-\gamma}(\Theta_2 x_2)(t) - \frac{x_0}{\Gamma(\gamma)}| &\leq \frac{M_2 \Gamma(1-\theta_2)}{\Gamma(\alpha - \theta_2 + 1)} h^{\alpha - \theta_2 - \gamma + 1} \leq b_2, \\ &\vdots \\ |t^{1-\gamma}(\Theta_n x_n)(t) - \frac{x_0}{\Gamma(\gamma)}| &\leq \frac{M_n \Gamma(1-\theta_n)}{\Gamma(\alpha - \theta_n + 1)} h^{\alpha - \theta_n - \gamma + 1} \leq b_n, \end{aligned}$$

which show that $\Theta_i(D_{ih}) \subset D_{ih}, i = 1, 2, \ldots, n$.

Next we show that operators Θ_i are continuous. Let $x_m, x_i \in D_{ih}$, $i = 1, 2, \ldots, n$, $m > n$ such that $\|x_m - x_i\| \to 0$ as $m \to +\infty$. In view of continuity of operators Λ_i, we have $\|\Lambda_i x_m - \Lambda_i x_i\|_{\Omega^*} \to 0$ as $m \to +\infty$. Note that

$$\begin{aligned} &|t^{1-\gamma}(\Theta_i x_m)(t) - t^{1-\gamma}(\Theta_i x_i)(t)| \\ &\quad = \Big|\frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} f_i(r, x_m(r)) dr - \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} f_i(r, x_i(r)) dr\Big| \\ &\quad \leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} |f_i(r, x_m(r)) - f_i(r, x_i(r))| dr \\ &\quad \leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} r^{-\theta_i} |\Lambda_i(x_m)(r) - \Lambda_i(x_i)(r)| dr \\ &\quad \leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} r^{-\theta_i} dr \|\Lambda_i(x_m)(r) - \Lambda_i(x_i)(r)\|_{\Omega^*}. \end{aligned}$$

We have

$$\|(\Theta_i x_m)(r) - (\Theta_i x_i)(r)\|_{\Omega^*} \leq \frac{\Gamma(1-\theta_i)}{\Gamma(\alpha-\theta_i+1)} \times h^{\alpha-\theta_i-\gamma+1}\|\Lambda_i(x_m)(t) - \Lambda_i(x_i)(t)\|_{\Omega^*}.$$

Then $\|(\Theta_i x_m)(r) - (\Theta_i x_i)(r)\|_{\Omega^*} \to 0$ as $m \to +\infty$. Thus Θ_i are continuous. Furthermore, we prove that operators $\Theta_i(D_{ih})$ are continuous. Let $x_i \in D_{ih}$ and $0 \leq t_1 < t_2 \leq h$. For any $\epsilon > 0$, note that

$$\frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} r^{-\theta_i} dr = \frac{\Gamma(1-\theta_i)}{\Gamma(\alpha-\theta_i+1)} t^{\alpha-\theta_i-\gamma+1} \to 0,$$

as $t \to 0^+$, where $0 \leq \theta_i < 1$. There exists $\tilde{\theta}_i > 0$ such that for $t \in \Omega^*$,

$$\frac{2M_i t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} r^{-\theta_i} dr < \epsilon,$$

holds.

We know $\Omega^* = [0, \theta_i] \cap [\frac{\theta_i}{2}, h]$. For $t_1, t_2 \in [0, \tilde{\theta}_i]$, we have

$$\begin{aligned}
&|\frac{{t_1}^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_1} (t_1-r)^{\alpha-1} f_i(r, x_i(r)) dr - \frac{{t_2}^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_2} (t_2-r)^{\alpha-1} f_i(r, x_i(r)) dr| \\
&\leq \frac{M_i {t_1}^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_1} (t_1-r)^{\alpha-1} r^{-\theta_i} dr + \frac{M_i {t_2}^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_2} (t_2-r)^{\alpha-1} r^{-\theta_i} dr < \epsilon. \quad (12)
\end{aligned}$$

For $t_1, t_2 \in [\frac{\tilde{\theta}_i}{2}, h]$, we get

$$\begin{aligned}
&|{t_1}^{1-\gamma}(\Theta_i x_i)(t_1) - {t_2}^{1-\gamma}(\Theta_i x_i)(t_2)| \\
&=|\frac{{t_1}^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_1} (t_1-r)^{\alpha-1} f_i(r, x_i(r)) dr \\
&\quad - \frac{{t_2}^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_2} (t_2-r)^{\alpha-1} f_i(r, x_i(r)) dr| \\
&\leq|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} [{t_1}^{1-\gamma}(t_1-r)^{\alpha-1} - {t_2}^{1-\gamma}(t_2-r)^{\alpha-1}] f_i(r, x_i(r)) dr| \\
&\quad + |\frac{{t_2}^{1-\gamma}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-r)^{\alpha-1} f_i(r, x_i(r)) dr|. \quad (13)
\end{aligned}$$

If $0 \leq \mu_1 < \mu_2 \leq h$ then $\mu_1^{1-\gamma}(\mu_1-r)^{\alpha-1} > \mu_2^{1-\gamma}(\mu_2-r)^{\alpha-1}$for $0 \leq r < \mu_1$, and from the first term on right hand side of inequality (13) we obtain that

$$|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} [{t_1}^{1-\gamma}(t_1-r)^{\alpha-1} - {t_2}^{1-\gamma}(t_2-r)^{\alpha-1}] f_i(r, x_i(r)) dr|$$

$$
\begin{aligned}
&\leq \left| \frac{M_i}{\Gamma(\alpha)} \int_0^{t_1} [{t_1}^{1-\gamma}(t_1-r)^{\alpha-1} - {t_2}^{1-\gamma}(t_2-r)^{\alpha-1}] r^{-\theta_i} dr \right| \\
&\leq \left| \frac{M_i}{\Gamma(\alpha)} \int_0^{\frac{\tilde{\theta}_i}{2}} |[{t_1}^{1-\gamma}(t_1-r)^{\alpha-1} - {t_2}^{1-\gamma}(t_2-r)^{\alpha-1}] r^{-\theta_i} \right| dr \\
&\quad + \frac{(\frac{\tilde{\theta}_i}{2})^{-\theta_i}}{\Gamma(\alpha)} \int_{(\frac{\tilde{\theta}_i}{2})}^{t} |[{t_1}^{1-\gamma}(t_1-r)^{\alpha-1} - {t_2}^{1-\gamma}(t_2-r)^{\alpha-1}] r^{-\theta_i}| dr \\
&\leq \left| \frac{2M_i(\frac{\tilde{\theta}_i}{2})^{1-\gamma}}{\Gamma(\alpha)} \int_0^{(\frac{\tilde{\theta}_i}{2})} \left(\frac{\tilde{\theta}_i}{2} - r\right)^{\alpha-1} r^{-\theta_i} \right| dr + \frac{M_i(\frac{\tilde{\theta}_i}{2})^{-\theta_i}}{\Gamma(\alpha+1)} \\
&\qquad \times \left[{t_2}^{1-\gamma}(t_2-t_1)^{\alpha} - {t_2}^{1-\gamma}\left(t_2 - \frac{\tilde{\theta}_i}{2}\right)^{\alpha} + {t_1}^{1-\gamma}\left(t_2 - \frac{\tilde{\theta}_i}{2}\right)^{\alpha} \right] \\
&\leq \epsilon + \frac{M_i(\frac{\tilde{\theta}_i}{2})^{-\theta_i}}{\Gamma(\alpha+1)} [h^{1-\gamma}(t_2-t_1)^{\alpha} \\
&\qquad + \left| {t_2}^{1-\gamma}\left(t_2 - \frac{\tilde{\theta}_i}{2}\right)^{\alpha} + {t_1}^{1-\gamma}\left(t_2 - \frac{\tilde{\theta}_i}{2}\right)^{\alpha} \right|].
\end{aligned}
$$

On the other hand, from the second term on right hand side of inequality (13), we have

$$
\begin{aligned}
\left|\frac{{t_2}^{1-\gamma}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-r)^{\alpha-1} f_i(r, x_i(r)) dr\right| &\leq \frac{M_i(\frac{\tilde{\theta}_i}{2})^{-\theta_i}}{\Gamma(\alpha)} \int_{t_1}^{t_2} {t_2}^{1-\gamma}(t_2-r)^{\alpha-1} dr \\
&= \frac{M_i(\frac{\tilde{\theta}_i}{2})^{-\theta_i}}{\Gamma(\alpha+1)} {t_2}^{1-\gamma}(t_2-t_1)^{\alpha}.
\end{aligned}
$$

From the above discussion, there exists a $\lambda, (\frac{\tilde{\theta}_i}{2} > \lambda > 0)$ such that for $t_1, t_2 \in [\frac{\tilde{\theta}_i}{2}, h]$ and $|t_1 - t_2| < \lambda$, we have

$$
|{t_1}^{1-\gamma}(\Theta_i x_i)(t_1) - {t_2}^{1-\gamma}(\Theta_i x_i)(t_2)| < 2\epsilon. \tag{14}
$$

It follows from inequalities (12) and (14) that $\{t^{1-\gamma}(\Theta_i x_i)(t) : x_i \in D_{ih}\}$ are equicontinuous. It is also clear that $\{t^{1-\gamma}(\Theta_i x_i)(t) : x_i \in D_{ih}\}$ is uniformly bounded since $\Theta_i(D_{ih}) \subset D_{ih}$. Therefore $\Theta_i(D_{ih})$ are precompact. So the operators Θ_i are completely continuous. By Schauder fixed point theorem and Lemma 1, the IVP (6) has a local solution. □

Remark 2. *If we let $\beta = 0$ in IVP (5), then Theorem 1 yields the local existence ([15], Theorem 3.1) associated with R-L IVP ([15], Eq. (1)).*

Remark 3. *If we let $\beta = 1$ in IVP (5), then Theorem 1 yields the local existence ([19], Theorem 3.1) associated with Caputo IVP ([19], Eq. (1.1)).*

4 Continuation and global existence

In this section, we concerned with continuation of solution of IVP (5) and then we obtain the global existence. Initially, we need the following definition.

Definition 5. *[15] Let $x(t)$ on $(0,\nu)$ and $\tilde{x}(t)$ on $(0,\tilde{\nu})$ both are solutions of IVP (5). If $\nu < \tilde{\nu}$ and $x(t) = \tilde{x}(t)$ for $t \in (0,\nu)$, we say that $\tilde{x}(t)$ is continuation of $x(t)$ or $x(t)$ can be continued to $(0,\tilde{\nu})$. A solution $x(t)$ is noncontinuable if it has no continuation. The existing interval of noncontinuable solution $x(t)$ is called the maximum existing interval of $x(t)$.*

Lemma 6. *[15] Let $0 < \alpha < 1, \nu > 0, h > 0, 0 \le \sigma < 1, u \in C_\sigma[0, \frac{\nu}{2}]$ and $v \in [\frac{\nu}{2}, h]$. Then*

$$I_1(t) = \int_0^{\frac{\nu}{2}} (t-r)^{\alpha-1} u(r)dr, \quad I_2(t) = \int_{\frac{\nu}{2}}^{\nu} (t-r)^{\alpha-1} v(r)dr$$

are continuous on $[\nu, \nu+h]$.

Theorem 3. *[Continuation Theorem I] Assume that (H_1) holds. Then $x(t), t \in (0,\nu)$ is noncontinuable if and only if for some $\tau \in (0, \frac{\nu}{2})$ and any bounded closed subset $D \subset [\tau, +\infty) \times \mathbb{R}$, there exists a $t^* \in [\tau, \nu)$ such that $(t^*, x(t^*)) \notin D$.*

Proof. We prove this theorem by contradiction. If possible, suppose that $x(t)$ is continuable. Then there exists a solution $\tilde{x}(t)$ defined on $(0,\tilde{\nu})$, such that $x(t) = \tilde{x}(t)$ for $t \in (0,\nu)$, which implies $\lim\limits_{t\to\nu^-} x(t) = \tilde{x}(\nu)$. Define $x(\nu) = \tilde{x}(\nu)$. Evidently, $K = \{(t, x(t)) : t \in [\tau,\nu)\}$ is a compact subset of $[\tau, +\infty) \times \mathbb{R}$. However, there exists no $t^* \in [\tau,\nu)$ such that $(t^*, x(t^*)) \notin K$. This contradiction implies $x(t)$ is noncontinuable.

We prove converse in two steps. Suppose that there exists a compact subset $\Omega \subset [\tau, +\infty) \times \mathbb{R}$, such that $\{(t, x(t)) : t \in [\tau,\nu)\} \subset \Omega$. The compactness of Ω implies $\nu < +\infty$. By (H_1), there exists a $C_1 > 0$ such that $\sup_{(t,x)\in\Omega} |f(t,x)| \le C_1$.

Step 1. We now show that the $\lim\limits_{t\to\nu^-} x(t)$ exists. Let

$$G(r,t) = \left|\frac{x_0}{\Gamma(\gamma)} r^{\gamma-1} - \frac{x_0}{\Gamma(\gamma)} t^{\gamma-1}\right|, \quad (r,t) \in [2\tau,\nu] \times [2\tau,\nu], \tag{15}$$

$$J(t) = \int_0^\tau (t-r)^{\alpha-1} r^{-\theta} dr, \quad t \in [2\tau, \nu]. \tag{16}$$

Clearly, $G(r,t)$ and $J(t)$ are uniformly continuous on $[2\tau,\nu]\times[2\tau,\nu]$ and $[2\tau,\nu]$, respectively.

For all $t_1, t_2 \in [2\tau,\nu), t_1 < t_2$, by Eq. (15) we have

$$\begin{aligned}
|x(t_1) - x(t_2)| &= |\frac{x_0}{\Gamma(\gamma)} {t_1}^{\gamma-1} - \frac{x_0}{\Gamma(\gamma)} {t_2}^{\gamma-1} + \int_0^{t_1} \frac{(t_1-r)^{\alpha-1}}{\Gamma(\alpha)} f(r,x(r))dr \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-r)^{\alpha-1} f(r,x(r))dr| \\
&\le G(t_1,t_2) + |\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-r)^{\alpha-1} f(r,x(r))dr \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-r)^{\alpha-1} f(r,x(r))dr| \\
&\le G(t_1,t_2) + |\frac{1}{\Gamma(\alpha)} \int_0^{\tau} [(t_1-r)^{\alpha-1} - (t_2-r)^{\alpha-1}] \\
&\qquad r^{-\theta}[r^\theta f(r,x(r))]dr| \\
&\quad + |\frac{1}{\Gamma(\alpha)} \int_\tau^{t_1} [(t_1-r)^{\alpha-1} - (t_2-r)^{\alpha-1}] f(r,x(r))dr| \\
&\quad + |\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-r)^{\alpha-1} f(r,x(r))dr| \\
&\le G(t_1,t_2) + |\int_0^{\tau} \frac{[(t_1-r)^{\alpha-1} - (t_2-r)^{\alpha-1}]}{\Gamma(\alpha)} r^{-\theta}(\Lambda x)(r)dr| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_\tau^{t_1} [(t_1-r)^{\alpha-1} - (t_2-r)^{\alpha-1}] |f(r,x(r))|dr \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-r)^{\alpha-1} |f(r,x(r))|dr \\
&\le G(t_1,t_2) + \frac{\|\Lambda x\|_{[0,\tau]}}{\Gamma(\alpha)} \int_0^{\tau} [(t_1-r)^{\alpha-1} - (t_2-r)^{\alpha-1}] r^{-\theta} dr \\
&\quad + \frac{C_1}{\Gamma(\alpha)} \int_\tau^{t_1} [(t_1-r)^{\alpha-1} - (t_2-r)^{\alpha-1}] dr \\
&\quad + \frac{C_1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-r)^{\alpha-1} dr \\
&\le G(t_1,t_2) + \|\Lambda x\|_{[0,\tau]} |J(t_1) - J(t_2)| \\
&\quad + \frac{C_1}{\Gamma(\alpha+1)} \Big[\big[(t_1-r)^\alpha - (t_2-r)^\alpha\big]\Big|_\tau^{t_1} + \big[(t_2-r)^\alpha\big]\Big|_{t_1}^{t_2} \Big]
\end{aligned}$$

$$\leq G(t_1,t_2)+\|\Lambda x\|_{[0,\tau]}|J(t_1)-J(t_2)| \\ +\frac{C_1}{\Gamma(\alpha+1)}\big[2(t_2-t_1)^{\alpha}+(t_1-\tau)^{\alpha}-(t_2-\tau)^{\alpha}\big].$$

From continuity of $G(r,t)$ and $J(t)$ together with the Cauchy convergence criterion, we obtain $\lim\limits_{t\to\nu^-} x(t)=x^*$.

Step 2. Now we show that $x(t)$ is continuable. Since Ω is closed subset, we have $(\nu,x^*)\in\Omega$. Define $x(\nu)=x^*$. Then $x(t)\in C_{1-\gamma}[0,\nu]$. We define operator

$$(Ny)(t)=x_1(t)+\frac{1}{\Gamma(\alpha)}\int_{\nu}^{t}(t-r)^{\alpha-1}f(r,x(r))dr, \qquad t\in[\nu,\nu+1],$$

where $y\in[\nu,\nu+1]$ and

$$x_1(t)=\frac{x_0}{\Gamma(\gamma)}t^{1-\gamma}+\frac{1}{\Gamma(\alpha)}\int_{0}^{\nu}(t-r)^{\alpha-1}f(r,x(r))dr, \quad t\in[\nu,\nu+1].$$

In view of Lemma 5 and Lemma 6, we have $N(C[\nu,\nu+1])\subset C[\nu,\nu+1]$. Suppose $X_b=\big\{(t,y):\nu\leq t\leq\nu+1,|y|\leq\max\limits_{\nu\leq t\leq\nu+1}|x(t)|+b\big\}$, $b>0$. In view of continuity of f on X_b, we denote $M=\max\limits_{(t,y)\in X_b}|f(t,y)|$. Consider $X_h=\big\{y\in[\nu,\nu+h]:\max_{t\in[\nu,\nu+h]}|y(t)-x_1(t)|\leq b, y(\nu)=x_1(\nu)\big\}$, where $h=\min\left\{1,\big(\frac{\Gamma(\alpha+1)b}{M}\big)^{\frac{1}{\alpha}}\right\}$.

We claim that N is completely continuous on X_h. First we show that operator N is continuous. In fact, let $\{y_n\}\subseteq C[\nu,\nu+h]$, $\|y_n-y\|_{[\nu,\nu+h]}\to 0$ as $n\to+\infty$. Then we have

$$\begin{aligned}|(Ny_n)(t)-(Ny)(t)|&=\Big|\frac{1}{\Gamma(\alpha)}\int_{\nu}^{t}(t-r)^{\alpha-1}[f(r,y_n(r))-f(r,y(r))]dr\Big|\\&\leq\frac{1}{\Gamma(\alpha)}\int_{\nu}^{t}(t-r)^{\alpha-1}|f(r,y_n(r))-f(r,y(r))|dr\\&\leq\|f(\cdot,y_n(\cdot))-f(\cdot,y(\cdot))\|_{[\nu,\nu+h]}\frac{1}{\Gamma(\alpha)}\int_{\nu}^{t}(t-r)^{\alpha-1}dr\\&\leq\|f(\cdot,y_n(\cdot))-f(\cdot,y(\cdot))\|_{[\nu,\nu+h]}\frac{h^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned}$$

By virtue of continuity of f on X_b, we have $\|Ny_n-Ny\|_{[\nu,\nu+h]}\to 0$ as $n\to+\infty$, which implies that N is a continuous operator.

Secondly, we prove that $N(X_h)$ is euqicontinuous. For any $y\in X_h$, for which $(Ny)(\nu)=x_1(\nu)$ and

$$|(Ny)(t)-x_1(t)|=\Big|\frac{1}{\Gamma(\alpha)}\int_{\nu}^{t}(t-r)^{\alpha-1}f(r,x(r))dr\Big|$$

$$\le \frac{1}{\Gamma(\alpha)}\int_{\nu}^{t}(t-r)^{\alpha-1}|f(r,x(r))|dr$$
$$\le \frac{M}{\Gamma(\alpha)}\int_{\nu}^{t}(t-r)^{\alpha-1}dr$$
$$\le \frac{M(t-\nu)^{\alpha}}{\Gamma(\alpha+1)} \le \frac{Mh^{\alpha}}{\Gamma(\alpha+1)} \le b.$$

Thus $N(X_h) \subset X_h$.

Set

$$I(t) = \frac{1}{\Gamma(\alpha)}\int_{0}^{\nu}(t-r)^{\alpha-1}f(r,x(r))dr.$$

By Lemma 6, $I(t)$ is continuous on $[\nu, \nu+h]$. For every $y \in X_h$, $\nu \le t_1 < t_2 \le \nu + h$, we have

$$\begin{aligned}
&|(Ny)(t_1) - (Ny)(t_2)| \\
&\quad \le G(t_1,t_2) + \frac{1}{\Gamma(\alpha)}|\int_{0}^{\nu}[(t_1-r)^{\alpha-1} - (t_2-r)^{\alpha-1}]f(r,x(r))dr| \\
&\qquad + \frac{1}{\Gamma(\alpha)}|\int_{\nu}^{t_1}[(t_1-r)^{\alpha-1} - (t_2-r)^{\alpha-1}]f(r,x(r))dr| \\
&\qquad + \frac{1}{\Gamma(\alpha)}|\int_{t_1}^{t_2}(t_2-r)^{\alpha-1}f(r,y(r))dr| \\
&\quad \le G(t_1,t_2) + \frac{1}{\Gamma(\alpha)}|\int_{0}^{\nu}[(t_1-r)^{\alpha-1} - (t_2-r)^{\alpha-1}]f(r,x(r))dr| \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_{\nu}^{t_1}[(t_1-r)^{\alpha-1} - (t_2-r)^{\alpha-1}]|f(r,x(r))|dr \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-r)^{\alpha-1}|f(r,y(r))|dr \\
&\quad \le G(t_1,t_2) + |I(t_1) - I(t_2)| \\
&\qquad + \frac{M}{\Gamma(\alpha+1)}\big[\big|(t_1-\nu)^{\alpha} + (t_2-t_1)^{\alpha} - (t_2-\nu)^{\alpha} + (t_2-t_1)^{\alpha}\big|\big] \\
&\quad \le G(t_1,t_2) + |I(t_1) - I(t_2)| \\
&\qquad + \frac{M}{\Gamma(\alpha+1)}\big[2(t_2-t_1)^{\alpha} + (t_1-\nu)^{\alpha} - (t_2-\nu)^{\alpha}\big]
\end{aligned} \tag{17}$$

By uniform continuity of $I(t)$ on $[\nu, \nu+h]$ and relation (17), we conclude that $\{(Ny)(t) : y \in X_h\}$ is equicontinuous. Thus N is completely continuous. By Schauder fixed point theorem, N has a fixed point $\tilde{x}(t) \in X_h$. i.e.

$$\tilde{x}(t) = x_1(t) + \frac{1}{\Gamma(\alpha)}\int_{\nu}^{t}(t-r)^{\alpha-1}f(r,\tilde{x}(r))dr$$

$$= \frac{x_0}{\Gamma(\gamma)} t^{1-\gamma} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} f(r, \bar{x}(r)) dr$$

where

$$\bar{x}(t) = \begin{cases} x(t), & \text{if } \ t \in (0, \nu], \\ \tilde{x}(t), & \text{if } \ t \in [\nu, \nu + h]. \end{cases}$$

It follows from Lemma 3, that $\bar{x} \in C_{1-\gamma}[0, \nu + h]$ and

$$\bar{x}(t) = \frac{x_0}{\Gamma(\gamma)} t^{1-\gamma} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} f(r, \bar{x}(r)) dr.$$

Therefore, in view of Lemma 5, $\bar{x}(t)$ is a solution of IVP (5) on $(0, \nu + h]$. This yields contradiction since $x(t)$ is noncontinuable. □

Now we present another continuation theorem which is more convenient for application purpose.

Theorem 4. *[Continuation Theorem II] Assume that (H_1) holds. Then $x(t), t \in (0, \nu)$, is noncontinuable if and only if*

$$\lim_{t \to \nu^-} \sup |M(t)| = +\infty, \tag{18}$$

where $M(t) = (t, x(t))$ and $|M(t)| = (t^2 + x^2(t))^{\frac{1}{2}}$.

Proof. We prove this theorem by contradiction. If possible, suppose $x(t)$ is continuable. Then there exists a solution $\tilde{x}(t)$ of IVP (5) defined on $(0, \tilde{\nu})$, $\nu < \tilde{\nu}$, such that $x(t) = \tilde{x}(t)$ for $t \in (0, \nu)$, which implies $\lim_{t \to \nu^-} x(t) = \tilde{x}(\nu)$. Thus $|M(t)| \to |M(\nu)|$, as $t \to \nu^-$, which is a contradiction.

Conversely, suppose that relation (18) is not true. Then there exists a sequence $\{t_n\}$ and constant $L > 0$ such that

$$\begin{aligned} & t_n < t_{n+1}, \ n \in \mathbb{N}, \\ & \lim_{n \to \infty} t_n = \nu, \ |M(t_n)| \leq L, \\ & i.e. \ {t_n}^2 + x^2(t_n) \leq L^2. \end{aligned} \tag{19}$$

Since $\{x(t_n)\}$ is bounded convergent subsequence, without loss of generality, we set

$$\lim_{n \to +\infty} x(t_n) = x^*. \tag{20}$$

Now we show that, for any given $\epsilon > 0$, there exists $T \in (0, \nu)$, such that $|x(t) - x^*| < \epsilon, \ t \in (T, \nu)$, we have

$$\lim_{t \to \nu^-} x(t_n) = x^*. \tag{21}$$

For sufficiently small $\tau > 0$, let

$$X_1 = \big\{(t,x) : t \in [\tau,\nu], |x| \leq \sup_{t\in[\tau,\nu)} |x(t)|\big\}.$$

Since f is continuous on X_1, denote $M = \max_{(t,y)\in X_1} |f(t,y)|$. It follows from Eqs. (19) and (20) that there exists n_0 such that $t_{n_0} > \tau$ and for $n \geq n_0$, we have $|x(t_n) - x^*| \leq \frac{\epsilon}{2}$. If Eq. (20) is not true, then for $n \geq n_0$, there exists $\eta_k \in (t_n, \nu)$ such that for $t \in (t_n, \eta_n)$, $|x(t) - x^*| < \epsilon$ and $|x(\eta_n) - x^*| \geq \epsilon$. Thus

$$\begin{aligned}
\epsilon &\leq |x(\eta_n) - x^*| \leq |x(t_n) - x^*| + |x(\eta_n) - x(t_n)| \\
&\leq \frac{\epsilon}{2} + \Big|\int_0^{t_n} \frac{(t_n - r)^{\alpha-1}}{\Gamma(\alpha)} f(r,x(r))dr - \int_0^{\eta_n} \frac{(\eta_n - r)^{\alpha-1}}{\Gamma(\alpha)} f(r,x(r))dr\Big| \\
&\leq \frac{\epsilon}{2} + \frac{1}{\Gamma(\alpha)}\Big|\int_0^{\tau} [(t_n - r)^{\alpha-1} - (\eta_n - r)^{\alpha-1}] f(r,x(r))dr\Big| \\
&\quad + \frac{1}{\Gamma(\alpha)}\Big|\int_\tau^{t_n} [(t_n - r)^{\alpha-1} - (\eta_n - r)^{\alpha-1}] f(r,x(r))dr\Big| \\
&\quad + \frac{1}{\Gamma(\alpha)}\Big|\int_{t_n}^{\eta_n} (\eta_n - r)^{\alpha-1} f(r,x(r))dr\Big| \\
&\leq \frac{\epsilon}{2} + \frac{\|\Lambda x\|_{[0,\tau]}}{\Gamma(\alpha)} |J(t_n) - J(\eta_n)| \\
&\quad + \frac{M}{\Gamma(\alpha+1)}\Big[(t_n - r)^\alpha - (\eta_n - r)^\alpha\Big|_\tau^{t_n} + (\eta_n - r)^\alpha\Big|_{t_n}^{\eta_n}\Big] \\
&\leq \frac{\epsilon}{2} + \frac{\|\Lambda x\|_{[0,\tau]}}{\Gamma(\alpha)} |J(t_n) - J(\eta_n)| \\
&\quad + \frac{M}{\Gamma(\alpha+1)}\Big[2(\eta_n - t_k)^\alpha + (t_n - \tau)^\alpha - (\eta_n - \tau)^\alpha\Big],
\end{aligned}$$

where $J(t)$ is defined by (16). In view of the continuity of $J(t)$ on $[t_{n_0}, \nu]$, for sufficiently large $n \geq n_0$, we have

$$\frac{\|\Lambda x\|_{[0,\tau]}}{\Gamma(\alpha)} |J(t_n) - J(\eta_n)| + \frac{M}{\Gamma(\alpha+1)}\Big[2(\eta_n - t_n)^\alpha + (t_n - \tau)^\alpha - (\eta_n - \tau)^\alpha\Big] < \frac{\epsilon}{2},$$

that implies $\epsilon \leq |x(\eta_n) - x^*| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This contradicts and $\lim_{t\to\nu^-} x(t)$ exists.

By using similar arguments as in the proof of Theorem 3, we can easily find the continuation of $x(t)$. □

Remark 4. *For $\beta = 0$, above continuation Theorems I and II reduces to continuation Theorems for R-L IVP ([15], Theorem 4.1, Theorem 4.2, respectively).*

Remark 5. *For $\beta = 1$, above continuation Theorems I and II reduces to continuation Theorems for Caputo IVP ([19], Theorem 4.2, Theorem 4.4, respectively).*

Now we discuss the existence of global solutions of IVP (5) based on the results obtained above.

Applying continuation Theorem II, we have the following conclusion about the existence of global solution for IVP (5).

Theorem 5. *Suppose that (H_1) holds and $x(t)$ is a solution of IVP (5) on $(0, \nu)$. If $x(t)$ is bounded on $[\tau, \nu)$ for some $\tau > 0$ then $\nu = +\infty$.*

We need following lemma for our further discussion.

Lemma 7. *[24] Let $v : \Omega \to [0, +\infty)$ be a real function, and $w(\cdot)$ be a nonnegative locally integrable function on Ω. For $0 < \alpha < 1$, there exists $a > 0$ such that*

$$v(t) \leq w(t) + a \int_0^t (t-r)^{-\alpha} v(r) dr.$$

Then there exists a constant $k = k(\alpha)$ such that for $t \in \Omega$, we have

$$v(t) \leq w(t) + ka \int_0^t (t-r)^{-\alpha} w(r) dr.$$

Theorem 6. *Suppose that (H_1) is satisfied and there exist three non-negative continuous functions $l(t), m(t), p(t) : [0, \infty) \to [0, \infty)$ such that $|f(t, x)| \leq l(t) m(|x|) + p(t)$, where $m(r) \leq r$ for $r \geq 0$. Then IVP (5) has a solution in $C_{1-\gamma}[0, \infty)$.*

Proof. The existence of a local solution $x(t)$ of IVP (5) follows from Theorem 1. By Lemma 1, $x(t)$ satisfies the integral equation (7). Suppose that $[0, \nu)$, $\nu < +\infty$, is the maximum existing interval of $x(t)$. Then

$$\begin{aligned}
|t^{1-\gamma} x(t)| &= \left| \frac{x_0}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} f(r, x(r)) dr \right| \\
&\leq \frac{x_0}{\Gamma(\gamma)} + \frac{\nu^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} [l(r) m(r^{1-\gamma}|x(r)|) + p(r)] dr \\
&\leq \frac{x_0}{\Gamma(\gamma)} + \frac{\nu^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} l(r) m(r^{1-\gamma}|x(r)|) dr
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\nu^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}p(r)dr\\
\le\ &\frac{x_0}{\Gamma(\gamma)}+\frac{\nu^{1-\gamma}}{\Gamma(\alpha)}\|l\|_{[0,\nu]}\int_0^t (t-r)^{\alpha-1}m(r^{1-\gamma}|x|)dr\\
&+\frac{\nu^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}p(r)dr.
\end{aligned}
$$

If we take $v(t)=t^{1-\gamma}|x(t)|$, $a=\frac{\nu^{1-\gamma}\|l\|_{[0,\nu]}}{\Gamma(\alpha)}$ and

$$
w(t)=\frac{x_0}{\Gamma(\gamma)}+\frac{\nu^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}p(r)dr,
$$

by Lemma 7, we know that $v(t)=t^{1-\gamma}|x(t)|$ is bounded on $[0,\nu)$. Thus for any $\tau\in(0,\nu)$, $x(t)$ is bounded on $[\tau,\nu)$. By Theorem 5, IVP (5) has a solution $x(t)$ on $[0,\infty)$. $\square$

The following result guarantees the existence and uniqueness of global solution of IVP (5) on $\mathbb{R}^+$.

Theorem 7. *Suppose that (H_1) is satisfied and there exists a nonnegative continuous function $l(t)$ defined on $[0,\infty)$ such that*

$$
|f(t,x)-f(t,y)|\le l(t)|x-y|.
$$

Then IVP (5) has a unique solution in $C_{1-\gamma}[0,\infty)$.

The existence of global solution can be obtained by the arguments similar as above. From the Lipschitz-type condition and Lemma 7, we can conclude the uniqueness of global solution. We omit the proof here.

5 Conclusion

The global existence of a unique solution of nonlinear IVP with Hilfer fractional derivative is proved by using fixed point technique and Gronwall inequality. Our results in this paper generalizes the existing results in the literature.

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