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# Analysis of mixed priority retrial queueing system with two way communication and working breakdown

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**Abstract.** Incoming calls are arrive at the service system according to compound Poisson process. During the idle time, the server making an outgoing call with an exponentially distributed time. If the incoming call that finds the server busy will join an orbit. Here we use mixed priority services i.e., an arriving call may interrupt the service of an outgoing call or join the retrial queue (orbit). The server takes Bernoulli vacation. The server may become inactive due to normal as well as abnormal breakdown. After the completion of service, vacation and repair the server is in idle state. We consider reneging to occur at the orbit. Using supplementary variable technique, the stability condition is derived.

*Keywords*: Mixed priority queueing systems, two way communication, retrial queue, working breakdown, negative arrival, Bernoulli vacation.
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## 1 Introduction

Retrial queues are paid much attention because they have wide applications in performance analysis of various systems such as call centers, computer networks and tele-communication systems. Retrial queues are characterized

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by the fact that the customers (i.e., calls) that cannot receive service upon arrival enter into an orbit and retry for service again after some time.

In most literature on retrial queues, the server only serves incoming calls. After serving a call, the server waits either for the next arrival or for a retrial call. However, there exist real life situations where the servers have a chance to make outgoing phone calls. This queueing feature is known as the models of two way communication. Artalejo and Phung-Duc ( [1], [2]) describes about Markovian retrial queues with two way communication. Bhulai and Koole [4] propose a multi-server queueing model with infinite buffer for call centers where incoming and outgoing calls follow the same exponential distribution. Phung-Duc et al. [13] develop the retrial queues with balanced call blending analysis of single-server and multi-server model.

As we already know that the priority queueing system can be broadly classified into two categories such as non-pre-emptive and pre-emptive priority queueing system. Both priorities are used in real world queueing system. An arriving high priority customer may interrupt the ongoing service of the low priority customer(pre-emptive) or wait till the service completion of the low priority customer(non-preemptive) which is known as mixed priority services. Dimitriou [8] studied about the mixed priority queueing system with negative arrival.

Several authors studied about priority queueing system with retrial, where the customers are joined the orbit and retry for their service. The perfectly reliable servers are not possible in real world phenomena. In our paper, we have assumed that the server fails only in the operational state. The breakdown may cause due to normal as well as abnormal breakdown. The abnormal breakdown may cause due to negative arrival. It not only breakdown the system but also removes the positive customer currently in service. It is otherwise known as G-queue. It was first introduced by Gelenbe [9] in neural networks. Recently Bhagat and Jain [3] and Li and Zhang [12] extended the study about G-queues. During the normal breakdown period, the customer is in service station will get the remaining service at a slower rate. Ye and Liu [15] studied about the $MAP/M/1$ queue with working breakdowns. Recently, Li and Zhang [11] discussed discrete-time $Geo/Geo/1$ queue with negative customers and working breakdowns. Customer impatience can be viewed as a potential loss of customers. Yang and Wu [14] dealt with impatient customers.

The motivation for two way communication queueing model comes from call centers where an operator not only serves incoming calls but also makes outgoing calls when the server is free, as well as from other daily life situations arising when the server is a telephone attended by a person who is

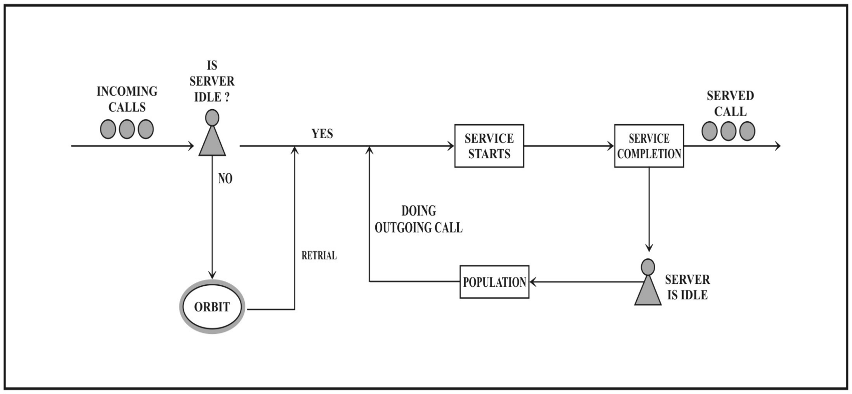


Figure 1: Two way communication.

also able to make his own calls. The schematic representation of two-way communication with single outgoing call is described in Figure 1.

In this paper, the term two way communication refers to the fact that the server is able to make outgoing calls while it is not engaged in conversation. We assume a retrial queue with constant retrial rate for incoming calls. We consider a single server mixed priority retrial queueing system with two way communication, working breakdown and Bernoulli vacation. Incoming calls arrive at the server according to general(arbitrary) distribution. If the server is idle, it starts making an outgoing call in an exponentially distributed time. Service times of these calls follow general distribution. An arriving call that finds the server is being busy with incoming call joins an orbit and retries to enter the service system after some general distributed time. Here we have used mixed priority services i.e. the arriving call may interrupt (pre-emptive) the service of an outgoing call or join (non pre-emptive) the retrial queue. The server takes vacation under Bernoulli vacation schedule. The server may become inactive when it is in operation due to normal as well as abnormal breakdown. During the normal breakdown period, the call currently in service will get the remaining service in a lower service rate. After completing this working breakdown period the repair process starts immediately. On completion of service, vacation and repair the server becomes idle. We consider reneging to occur at the orbit.

The summary of the paper is as follows. Section 1 is an introduction to mixed priority retrial queueing discipline and comprises literature review. Section 2 deals with model description, notations used, mathematical formulation and governing equations of the model. Section 3 elucidates the steady state solutions of the system. Section 4 demonstrates the performance measures of the model and concentrate on some special cases. In Section 5, the numerical results are computed and graphical studies are shown following which the conclusion is given.

## 2 Model description

The basic operation of the model is described in Figure 2:

**Arrival and retrial process:** Incoming calls arrive at the server according to compound Poisson process with arrival rate $\lambda(>0)$. Let $\lambda c_i\ dt$ $(i = 1, 2, 3, \ldots)$ be the first order probability that a batch of 'i' customers arrives at the system during a short interval of time $(t, t + dt)$, where for $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$. If the server is busy with an outgoing call then the arriving call may interrupt (pre-emptive) the service of an outgoing call with probability p or join (non pre-emptive) the retrial queue with probability 1-p. The incoming call on finding the server is being busy with an incoming call, are routed to a retrial queue (orbit) and they follow constant retrial policy that attempts to get the service. The retrial time is generally distributed with distribution function $I(s)$ and the density function $i(s)$. Let $\eta(x)dx$ be the conditional probability of completion of retrial during the interval $(x, x + dx]$ where $x$ is the elapsed retrial time.

**Service process**: During server's idle time, the server can make an outgoing call which is exponentially distributed with mean $1/\pi$. The service times of the incoming and outgoing calls are generally distributed with distribution functions $B_i(s)$ and the density functions $b_i(s)$, $i = 1, 2$ respectively. Let $\mu_i(x)dx$ be the conditional probability of completion of the high priority and low priority customer's service during the interval $(x, x+dx]$, where $x$ is the elapsed service time.

**Bernoulli Vacation**: After completing every service the server may take a vacation with probability $\theta$. Vacation time is generally distributed with distribution function $V(s)$ and the density function $v(s)$. Let $\beta(x)dx$ be the conditional probability of completion of vacation during the interval $(x, x + dx]$ where $x$ is the elapsed vacation time.

**Breakdown state**: The server may become inactive during busy period due to normal (with rate $\alpha$) as well as abnormal breakdown (with rate $\overline{\lambda}$). The abnormal breakdown causes due to the negative arrival. The negative arrival not only removes the positive customer but also destroys the server.

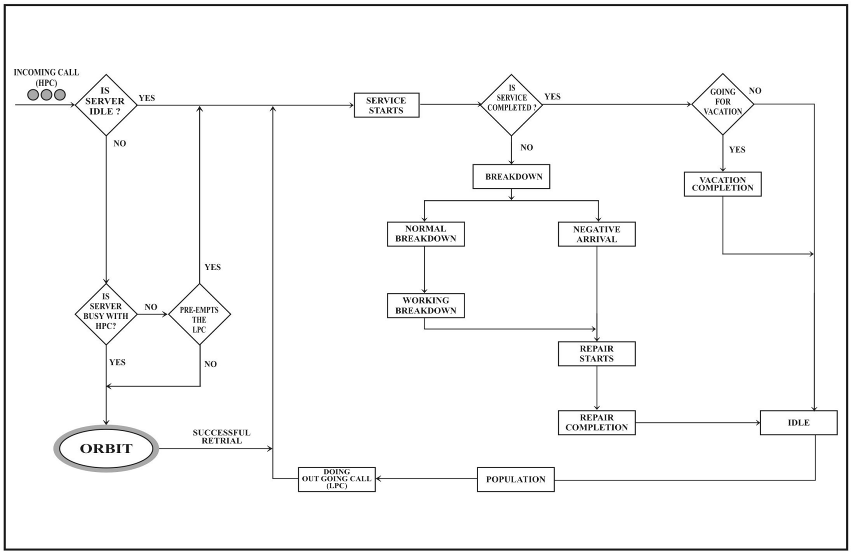


Figure 2: Flowchart of the model description.

**Working breakdown state**: If the server is inactive due to normal breakdown, then the call currently present in the service station will get the remaining service as a lower service rate. This state is known as working breakdown state.

**Repair Process**: After completing working breakdown state, the repair process starts immediately so as to regain its functionality. Repair time is exponentially distributed with rate $\gamma$.

**Reneging**: An incoming call may renege the system exponentially with rate $\xi$.

**Idle State**: After completing service, vacation and repair the server remains idle in the system.

## 2.1 Definitions and equations governing the system

Let $N(t)$ be the orbit size at time t, $B_1^0(t)$,$B_2^0(t)$, $V^0(t)$and $I^0(t)$ be the elapsed service time of the incoming, outgoing calls, elapsed vacation time and retrial time respectively at time t. Let S(t) denote the state of the server

$$S(t) = \begin{cases} 0, & \text{if the server is idle;} \\ 1, & \text{if the server is in retrial state;} \\ 2, & \text{if the server is providing an incoming service;} \\ 3, & \text{if the server is providing an outgoing service;} \\ 4, & \text{if the server is on vacation;} \\ 5, & \text{if the server is on working breakdown;} \\ 6, & \text{if the server is on repair;.} \end{cases}$$

We have $I(x)$, $B_i(x)$, $V(x)$ and $M(x)$ is continuous at $x = 0$, and $\eta(x)dx = \frac{dI(x)}{1 - I(x)}$, $\mu_i(x)dx = \frac{dB_i(x)}{1 - B_i(x)}$, i=1,2, $\beta(x)dx = \frac{dV(x)}{1 - V(x)}$, are the first order differential (hazard rate) functions of I(.), $B_i(.)$ and V(.) respectively.

Since the service time, vacation time and retrial time are not exponential, the process $\{S(t), N(t)\}$ is non Markovian. In such case we introduce supplementary variables corresponding to elapsed times to make it Markovian Cox [7]. Joint distributions of the server state and orbit size are defined as,

$$\overline{I}_0(s,t) = Pr\{S(t) = 0, N(t) = 0\}, \tag{1}$$
$$\overline{I}_m(x,s,t)dx = Pr\{S(t) = 1, N(t) = m \geq 1\}, \tag{2}$$
$$\overline{P}_m^{(1)}(x,s,t)dx = Pr\{S(t) = 2, x < B_1^0(t) \leq x + dx, N(t) = m \geq 0\}, \tag{3}$$
$$\overline{P}_m^{(2)}(x,s,t)dx = Pr\{S(t) = 3, x < B_2^0(t) \leq x + dx, N(t) = m \geq 0\}, \tag{4}$$
$$\overline{V}_m(x,s,t)dx = Pr\{S(t) = 4, x < V^0(t) \leq x + dx, N(t) = m\}, \tag{5}$$
$$\overline{Q}_m(s,t) = Pr\{S(t) = 5, N(t) = m \geq 0\}, \tag{6}$$
$$\overline{R}_m(x,s,t) = Pr\{S(t) = 6, N(t) = m \geq 0\}. \tag{7}$$

## 2.2 Kolmogorov equations

The Kolmogorov forward equations which governs the model:

1. The server is providing the service to the incoming call:

$$\frac{\partial}{\partial t}P_m^{(1)}(x,t) + \frac{\partial}{\partial x}P_m^{(1)}(x,t) = -(\lambda + \alpha + \overline{\lambda} + \xi + \mu_1(x))P_m^{(1)}(x,t) + \lambda\sum_{i=1}^{m} c_i P_{m-i}^{(1)}(x,t) + \xi P_{m+1}^{(1)}(x,t);\ m \geq 0, \tag{8}$$

2. The server is providing the service to the outgoing call:

$$\frac{\partial}{\partial t}P_m^{(2)}(x,t)+\frac{\partial}{\partial x}P_m^{(2)}(x,t)=-(\lambda+\alpha+\overline{\lambda}+\xi+\mu_2(x))P_m^{(2)}(x,t)$$
$$+\lambda(1-p)\sum_{i=1}^{m}c_iP_{m-i}^{(2)}(x,t)+\xi P_{m+1}^{(2)}(x,t);\ m\geq 0, \tag{9}$$

3. The server is on vacation state:

$$\frac{\partial}{\partial t}V_m(x,t)+\frac{\partial}{\partial x}V_m(x,t)=-(\lambda+\xi+\beta(x))V_m(x,t)$$
$$+\lambda\sum_{i=1}^{m}c_iV_{m-i}(x,t)+\xi V_{m+1}(x,t); m\geq 0, \tag{10}$$

4. The server is in working breakdown state with an incoming call:

$$\frac{d}{dt}Q_m^{(1)}(t)=-(\lambda+\mu_3+\xi)Q_m^{(1)}(t)+\lambda\sum_{i=1}^{m}c_iQ_{m-i}^{(1)}(t)$$
$$+\xi Q_{m+1}^{(1)}(t)+\alpha\int_0^\infty P_m^{(1)}(x,t)dx; m\geq 0, \tag{11}$$

5. The server is in working breakdown state with an outgoing call:

$$\frac{d}{dt}Q_m^{(2)}(t)=-(\lambda+\mu_4+\xi)Q_m^{(2)}(t)+\lambda\sum_{i=1}^{m}c_iQ_{m-i}^{(2)}(t)$$
$$+\xi Q_{m+1}^{(2)}(t)+\alpha\int_0^\infty P_m^{(2)}(x,t)dx; m\geq 0, \tag{12}$$

6. The server is in repair process:

$$\frac{d}{dt}R_m(t)=-(\lambda+\gamma+\xi)R_m(t)+\lambda\sum_{i=1}^{m}c_iR_{m-i}(t)$$
$$+\mu_3Q_m^{(1)}(t)+\mu_4Q_m^{(2)}(t)+\xi R_{m+1}(t)$$
$$+\overline{\lambda}\Big\{\int_0^\infty P_m^{(1)}(x,t)dx+\int_0^\infty P_m^{(2)}(x,t)dx\Big\};\ m\geq 0, \tag{13}$$

7. The server is in retrial state:

$$\frac{\partial}{\partial t}I_m(x,t)+\frac{\partial}{\partial x}I_m(x,t)=-(\lambda+\pi+\eta(x))I_m(x,t);\ m\geq 1, \tag{14}$$

8. The server is in idle state:

$$\frac{d}{dt}I_0(t) = -(\lambda+\pi)I_0(t) + \int_0^\infty V_0(x,t)\beta(x)dx + \gamma R_0(t) + (1-\theta)\big\{\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_0^{(2)}(x,t)\mu_2(x)dx\big\}. \quad (15)$$

The above set of equations are to be solved under the following boundary conditions at $x = 0$:

$$I_m(0,t) = \int_0^\infty V_m(x,t)\beta(x)dx + (1-\theta)\int_0^\infty P_m^{(1)}(x,t)\mu_1(x)dx + (1-\theta)\int_0^\infty P_m^{(2)}(x,t)\mu_2(x)dx + \gamma R_m(t);\ m \geq 1, \quad (16)$$

$$P_m^{(1)}(0,t) = \int_0^\infty I_{m+1}(x,t)\eta(x)dx + \lambda\sum_{i=1}^{m} c_i \int_0^\infty I_{m+1-i}(x,t)dx + \lambda c_i p \int_0^\infty P_{m+1-i}^{(2)}(x,t)dx + \lambda c_{m+1} I_0(t);\ m \geq 0, \quad (17)$$

$$P_m^{(2)}(0,t) = \pi\int_0^\infty I_m(x,t)dx;\ m \geq 0, \quad (18)$$

$$V_m(0,t) = \theta\big\{\int_0^\infty P_m^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_m^{(2)}(x,t)\mu_2(x)dx\big\};\quad m \geq 0. \quad (19)$$

We assume that initially there are no customers in the system and the server is idle. Then the initial conditions are,

$$P_m^{(i)}(0) = V_m(0) = Q_m^{(i)}(0) = R_m(0) = I_m(0) = 0 \ \text{ and } \ I_0(0) = 1, \quad (20)$$

for $m \geq 0$ and $i = 1, 2$.

The Probability Generating Function(PGF) of this model is

$$A(x,z,t) = \sum_{m=1}^{\infty} z^m A_m(x,t),$$

where $A = I, V, P^{(i)}, Q^{(i)}$ and $R$, for $i = 1, 2$.

By taking Laplace transforms from equation (8) to (15) and solving the equations

$$\overline{I}_0(x,s,z) = \overline{I}_0(0,s,z)[1-\overline{I}(\varphi(a,s))]e^{-\varphi(a,s)x}, \tag{21}$$

$$\overline{P}^{(1)}(x,s,z) = \overline{P}^{(1)}(0,s,z)[1-\overline{B}_1(\varphi_1(s,z))]e^{-\varphi_1(s,z)x}, \tag{22}$$

$$\overline{P}^{(2)}(x,s,z) = \overline{P}^{(2)}(0,s,z)[1-\overline{B}_2(\varphi_2(s,z))]e^{-\varphi_2(s,z)x}, \tag{23}$$

$$\overline{V}(x,s,z) = \overline{V}(0,s,z)[1-\overline{V}(\varphi_3(s,z))]e^{-\varphi_3(s,z)x}, \tag{24}$$

$$\overline{Q}^{(1)}(s,z) = \frac{\alpha\overline{P}^{(1)}(0,s,z)[1-\overline{B}_1(\varphi_1(s,z))]e^{-\varphi_1(s,z)x}}{\varphi_4(s,z)}, \tag{25}$$

$$\overline{Q}^{(2)}(s,z) = \frac{\alpha\overline{P}^{(2)}(0,s,z)[1-\overline{B}_2(\varphi_2(s,z))]e^{-\varphi_2(s,z)x}}{\varphi_5(s,z)}, \tag{26}$$

$$\overline{R}(s,z) = \frac{\left\{\begin{array}{l}\{\overline{\lambda}\{\overline{P}^{(1)}(0,s,z)[1-\overline{B}_1(\varphi_1(s,z))]e^{-\varphi_1(s,z)x}+\overline{P}^{(2)}(0,s,z)\\ \times[1-\overline{B}_2(\varphi_2(s,z))]e^{-\varphi_2(s,z)x}\}+\mu_3\overline{Q}^{(1)}(s,z)+\mu_4\overline{Q}^{(2)}(s,z)\}\end{array}\right\}}{z\varphi_6(s,z_2)}, \tag{27}$$

where

$$\begin{aligned}
\varphi(a,s) &= s+\lambda+\pi,\\
\varphi_1(s,z) &= s+\lambda[1-C(z)]+\alpha+\overline{\lambda}+\xi[1-\frac{1}{z}],\\
\varphi_2(s,z) &= s+\lambda[1-(1-p)C(z)]+\alpha+\overline{\lambda}+\xi[1-\frac{1}{z}],\\
\varphi_3(s,z) &= s+\lambda[1-C(z)]+\xi[1-\frac{1}{z}],\\
\varphi_4(s,z) &= s+\lambda[1-C(z)]+\mu_3+\xi[1-\frac{1}{z}],\\
\varphi_5(s,z) &= s+\lambda[1-C(z)]+\mu_4+\xi[1-\frac{1}{z}],\\
\varphi_6(s,z) &= s+\lambda[1-C(z)]+\gamma+\xi[1-\frac{1}{z}].
\end{aligned}$$

By solving the above equations, we get,

$$\overline{P}^{(1)}(0,s,z) = \frac{\left\{\begin{array}{l}(1-(s+\lambda+\pi)\overline{I}_0(s))\overline{\zeta}_1(s,z)+\lambda C(z)\overline{I}_0(s)\\ \times[1-\pi[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}]\overline{\zeta}_3(s,z)]\end{array}\right\}}{\left\{z\left\{1-\pi[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}]\overline{\zeta}_2(s,z)\right\}-\overline{\zeta}_1(s,z)\overline{\zeta}_2(s,z)\right\}} \tag{28}$$

$$\overline{P}^{(2)}(0,s,z) = \frac{\begin{Bmatrix} \pi\big\{(1-(s+\lambda+\pi)\overline{I}_0(s))\overline{\zeta}_1(s,z) + \lambda C(z)\overline{I}_0(s) \\ \times [1-\pi[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}]\overline{\zeta}_3(s,z)]\big\} \end{Bmatrix}}{\Big\{ z\ \{\ 1-\pi[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}]\overline{\zeta}_2(s,z)\ \}\ -\overline{\zeta}_1(s,z)\overline{\zeta}_2(s,z) \Big\}} \tag{29}$$

$$\overline{I}(0,s,z) = \frac{\{\ (1-(s+\lambda+\pi)\overline{I}_0(s))z + \lambda C(z)\overline{I}_0(s)\overline{\zeta}_2(s,z)\ \ \}}{\Big\{ z\ \{\ 1-\pi[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}]\overline{\zeta}_3(s,z)\ \}\ -\overline{\zeta}_1(s,z)\overline{\zeta}_2(s,z) \Big\}} \tag{30}$$

$$\overline{V}(0,s,z) = \theta\overline{P}^{(1)}(0,s,z)\overline{B_1}(\varphi_1(s,z)) + \theta\overline{P}^{(2)}(0,s,z)\overline{B_2}(\varphi_2(s,z)). \tag{31}$$

where

$$\overline{\zeta}_1(s,z) = \overline{I}(\varphi(a,s)) + \lambda C(z)\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}\big\{1+p\pi\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}\big\},$$

$$\overline{\zeta}_2(s,z) = [1-\theta+\theta\overline{V}(\varphi_3(s,z))]\overline{B}_1(\varphi_1(s,z)) + [\frac{1-\overline{B}_1(\varphi_1(s,z))}{\varphi_1(s,z)\varphi_6(s,z)}](\frac{\overline{\lambda}}{z} + \frac{\alpha\mu_3\gamma}{z\varphi_4(s,z)}),$$

$$\overline{\zeta}_3(s,z) = [1-\theta+\theta\overline{V}(\varphi_3(s,z))]\overline{B}_2(\varphi_2(s,z)) + [\frac{1-\overline{B}_2(\varphi_2(s,z))}{\varphi_2(s,z)\varphi_6(s,z)}](\frac{\overline{\lambda}}{z} + \frac{\alpha\mu_4\gamma}{z\varphi_5(s,z)}).$$

## 2.3 Stability condition

**Theorem 1.** *The inequality*

$$P^{(1)}(1) + P^{(2)}(1) + Q^{(1)}(1) + Q^{(2)}(1) = \rho < 1,$$

*is a necessary and sufficient condition for the system to be stable and under this condition the marginal PGF of the server's state and orbit size distributions are given by*

$$\overline{I}(s,z) = \overline{I}(0,s,z)[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}], \tag{32}$$

$$\overline{P}^{(1)}(s,z) = \overline{P}^{(1)}(0,s,z)[\frac{1-\overline{B_1}(\varphi_1(s,z))}{\varphi_1(s,z)}], \tag{33}$$

$$\overline{P}^{(2)}(s,z) = \overline{P}^{(2)}(0,s,z)[\frac{1-\overline{B_2}(\varphi_2(s,z))}{\varphi_2(s,z)}], \tag{34}$$

$$\begin{aligned}\overline{V}(s,z) = \theta\big\{&\overline{P}^{(1)}(0,s,z)\overline{B_1}(\varphi_1(s,z)) + \overline{P}^{(2)}(0,s,z)\overline{B}_2(\varphi_1(s,z))\big\} \\ &\times[\frac{1-\overline{V}(\varphi_3(s,z))}{\varphi_3(s,z)}],\end{aligned} \tag{35}$$

$$\overline{Q}^{(1)}(s,z) = \frac{\alpha\overline{P}^{(1)}(0,s,z)[1-\overline{B}_1(\varphi_1(s,z_1))]e^{-\varphi_1(s,z)x}}{\varphi_4(s,z)}, \tag{36}$$

$$\overline{Q}^{(2)}(s,z) = \frac{\alpha \overline{P}^{(2)}(0,s,z)[1-\overline{B}_2(\varphi_2(s,z_1))]e^{-\varphi_2(s,z)x}}{\varphi_5(s,z)}, \tag{37}$$

$$\overline{R}(s,z) = \frac{\left\{ \begin{array}{l} \{(\overline{\lambda}\{\overline{P}^{(1)}(0,s,z)[1-\overline{B}_1(\varphi_1(s,z))]e^{-\varphi_1(s,z)x} + \overline{P}^{(2)}(0,s,z) \\ \times [1-\overline{B}_2(\varphi_2(s,z))]e^{-\varphi_2(s,z)x}\} + \mu_3\overline{Q}^{(1)}(s,z) + \mu_4\overline{Q}^{(2)}(s,z)\} \end{array} \right\}}{z\varphi_6(s,z_2)}. \tag{38}$$

## 3 Steady state analysis

By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t),$$

to the above equations, we obtain the steady- state solutions of this model. In order to determine $I_0$, we use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + V(1) + Q^{(1)}(1) + Q^{(2)}(1) + R(1) + I(1) + I_0 = 1.$$

For this, let $P_q(z)$ be the probability generating function of the queue size irrespective of the state of the system. Then adding all the steady state equations, we obtain,

$$P_q(z) = P^{(1)}(z) + P^{(2)}(z) + V(z) + Q^{(1)}(z) + Q^{(2)}(z) + R(z) + I(z), \tag{39}$$

$$P_q(z) = P^{(1)}(0,z)\frac{\psi_1(z)}{\omega_1(z)} + I(0,z)[\frac{1-\overline{I}(\varphi(a))}{\varphi(a)}]\frac{\psi_2(z)}{\omega_2(z)},$$

where

$$\begin{aligned}
\psi_1(z) ={}& \{\ [1-\overline{B}_1(\varphi_1(z))]\varphi_3(z)\{z\varphi_4(z)\varphi_6(z) + \overline{\lambda}\varphi_4(z) + \alpha z\varphi_6(z) \\
&+ \alpha\gamma\mu_3\} + \theta\overline{B}_1(\varphi_1(z))(1-\overline{V}(\varphi_3(z)))z\varphi_1(z)\varphi_4(z)\varphi_6(z)\ \},\\
\psi_2(z) ={}& \{\ z\overline{B}_2(\varphi_2(z))\varphi_2(z)\varphi_5(z)\varphi_6(z)[\pi\varphi_3(z) + \theta(1-\overline{V}(\varphi_3(z)))] \\
&+ z\varphi_2(z)\varphi_3(z)\varphi_5(z)\varphi_6(z) + +[1-\overline{B_2}(\varphi_2(z))]\varphi_3(z) \\
&\times \{\alpha z\varphi_6(z) + \overline{\lambda}\varphi_5(z) + \alpha\gamma\mu_4\}\ \},\\
\omega_1(z) ={}& \varphi_1(z)\varphi_3(z)\varphi_4(z)\varphi_6(z), \quad \omega_2(z) = z\varphi_2(z)\varphi_3(z)\varphi_5(z)\varphi_6(z).
\end{aligned}$$

In order to obtain the probability of idle time $I_0$, we use the normalizing condition, $P_q(1) + I_0 = 1$, from which we can have,

$$I_0 = \frac{\omega_1'(1)\omega_2'(1)}{\omega_1'(1)\omega_2'(1) + P^{(1)}(0,1)\psi_1'(1)\omega_2'(1) + I(0,1)\psi_2'(1)\omega_2'(1)}. \tag{40}$$

## 4 Performance measures

**Theorem 2.** *If the system is in steady state condition, then we have* $(i)$ *The probability that the server is idle* $= I_{0,0}$*, where* $I_{0,0}$ *has been found above.*
$(ii)$ *The probability that the server is being busy*

$$P^{(1)}(1)+P^{(2)}(1)+Q^{(1)}(1)+Q^{(2)}(1) = \frac{(\alpha+\mu_3)P^{(1)}(0,1)[1-\overline{B}_1(\alpha+\overline{\lambda})]}{(\alpha+\overline{\lambda})\mu_3} + \frac{(\alpha+\mu_4)P^{(2)}(0,1)[1-\overline{B}_2(\lambda p+\alpha+\overline{\lambda})]}{(\lambda p+\alpha+\overline{\lambda})\mu_4},$$

$(iii)$ *The probability that the server is on vacation*

$$V(1) = \theta E(V)\big\{\overline{P}^{(1)}(0,1)\overline{B}_1(\overline{\lambda}+\alpha)+\overline{P}^{(2)}(0,1)\overline{B}_2(\lambda p+\overline{\lambda}+\alpha)\big\}.$$

*Proof.* Note that

$$\begin{aligned} P^{(1)}(1)+P^{(2)}(1) &= \lim_{z\to 1}[P^{(1)}(z)+P^{(2)}(z)], \\ V(1) &= \lim_{z\to 1} V(z), \\ Q(1) &= \lim_{z\to 1} Q(z), \end{aligned}$$

and by direct calculation we get the above formulae. □

### 4.1 The average orbit length and waiting time

The Mean number of customers in the orbit under the steady state condition is

$$L_q = \frac{d}{dz}P_q(z)|_{z=1}. \tag{41}$$

then $L_q = \frac{Nr}{Dr}$, where

$$\begin{aligned} Nr &= 2\omega_1'(1)\omega_2'(1)\big\{6P^{(1)}(0,1)\psi_1'(1)\omega_2'(1)+3P^{(1)}(0,1)\psi_1'(1)\omega_2'(1) \\ &\quad +3P^{(1)}(0,1)\psi_1'(1)\omega_2'(1)+6I(0,1)\psi_2'(1)\omega_2'(1)+3I(0,1)\psi_2'(1)\omega_2'(1) \\ &\quad +I(0,1)\psi_2'(1)\omega_2'(1)\big\}-\big\{3\omega_1''(1)\psi_1'(1)2(\omega_1'(1))^2+3\omega_2'(1)\psi_2''(1) \\ &\quad -\omega_2''(1)\psi_2'(1)\big\}\big\{3P^{(1)}(0,1)\psi_1'(1)\omega_2'(1)+6I(0,1)\psi_2'(1)\omega_2'(1)\big\}, \\ Dr &= \big\{3(\omega_2'(1))\omega_1'(1)^2\big\}. \end{aligned}$$

By Little's Law, average waiting time of a customer in the orbit is

$$W_q = \frac{L_q}{\lambda}. \tag{42}$$

### 4.2 Particular cases

**Case: 1** $M/G/1$ Queueing model:
If there is no outgoing call, no vacation, no working breakdown, no reneging, the retrial rate tends to infinity and the arrival is single, then the model under study becomes classical $M/G/1$ queueing system. In this case, the PGF of the busy state is given as,

$$P(z) = \frac{(1-\overline{B}(\lambda-\lambda z))I_0}{\overline{B}(\lambda-\lambda z)-z}. \tag{43}$$

The result coincides with the result of Gross and Harris [10].

**Case: 2** $M/G/1$ Queueing model with single vacation:
If there is no outgoing call, no working breakdown, no reneging, retrial rate tends to infinity and the arrival is single, then the model under study becomes classical $M/G/1$ queueing system with vacation. In this case, the PGF of the busy state and vacation state are given as,

$$P(z) = \frac{(1-\overline{B}(\lambda-\lambda z))I_0}{\overline{B}(\lambda-\lambda z)(1-\theta+\theta\overline{V}(\lambda-\lambda z))-z} \tag{44}$$

$$V(z) = \theta\frac{\overline{B}(\lambda-\lambda z)(1-\overline{V}(\lambda-\lambda z))I_0}{\overline{B}(\lambda-\lambda z)(1-\theta+\theta\overline{V}(\lambda-\lambda z))-z} \tag{45}$$

**Case: 3** $M^X/G/1$ Retrial Queueing model:
If there is no outgoing call, no working breakdown, no reneging and no vacation, then the model under study becomes classical $M^X/G/1$ retrial queueing system. In this case, the PGF of the busy state and retrial state are given as

$$P(z) = \frac{(1-\overline{B}(\lambda-\lambda z))I_0\overline{I}_\lambda}{\overline{B}(\lambda-\lambda z)[C(z)(1-\overline{I}_\lambda)+\overline{I}_\lambda]-z},$$

$$I(z) = \frac{[z-C(z)\overline{B}(\lambda-\lambda z)](1-\overline{I}(\lambda)I_0}{\overline{B}(\lambda-\lambda z)[C(z)(1-\overline{I}_\lambda)+\overline{I}_\lambda]-z}. \tag{46}$$

The result coincides with the results of Corral [6].

**Case: 4** $M^X/G/1$ Retrial queueing model with vacation:
If there is no outgoing call, no reneging, no working breakdown and no negative arrival then our model reduces to $M^{[X]}/G/1$ retrial queueing system with general retrial times under Bernoulli vacation for unreliable server and repair.

$$I(z) = \frac{I_0\{z - C(z)\overline{B}(\phi[z])[1-\theta+\theta\overline{V}(\psi[z])]\}[1-\overline{I}(\lambda)]}{\overline{B}(\phi[z])[1-\theta+\theta\overline{V}(\psi[z])]\{C(z)[1-\overline{I}(\lambda)]+\overline{I}(\lambda)\}-z}, \tag{47}$$

$$P(z) = \frac{\lambda I_0\overline{I}(\lambda)[1-C(z)]\{1-\overline{B}(\phi[z])\}}{Dr}, \tag{48}$$

$$V(z) = \frac{\theta\lambda I_0\overline{I}(\lambda)[1-C(z)]\overline{B}(\phi[z])\{1-\overline{V}(\psi(z))\}}{Dr}, \tag{49}$$

where, Dr $=[\overline{B}(\phi[z])[1-\theta+\theta\overline{V}(\psi[z])]\{C(z)[1-\overline{I}(\lambda)]+\overline{I}(\lambda)\}-z]\phi[z]$.
If we add the delaying repair variable and the repair time is in general distribution, then the result coincides with Choudhury and Ke [5].

## 5 Numerical results

In order to see the effect of different parameters on the different states of the server we compute some numerical results. We consider the service time, vacation time and retrial time to be exponentially distributed to numerically illustrate the feasibility of our results. By giving the following suitable values for the parameters which satisfy the stability condition, we compute the table values.

For Table 1 we choose that $\pi$ takes the values 1.1 to 2.0 with increment 0.1, $\lambda = 4$, $\mu_1 = 4$, $\mu_2 = 5$, $\mu_3 = 1$, $\mu_4 = 3$, $\eta = 5$, $\theta = 0.3$, $\alpha = 1$, $\overline{\lambda} = 1$, $\gamma = 1$, $\xi = 0.2$, —$\beta = 0.5$ and $p = 0.5$.

For Table 2 we choose that $\eta$ takes the values 1 to 10 with increment 1, $\lambda = 5$, $\mu_1 = 5$, $\mu_2 = 5$, $\mu_3 = 3$, $\mu_4 = 3$, $\pi = 8$, $\theta = 0.75$, $\alpha = 3$, $\overline{\lambda} = 0.5$, $\gamma = 1$, $\xi = 10$, $\beta = 10$ and $p = 0.8$.

For Table 3 we choose that $\lambda$ takes the values 1.1 to 2 with increment 0.1, $\eta = 1, \mu_1 = 20, \mu_2 = 1, \mu_3 = 0.5, \mu_4 = 0.5, \pi = 1, \theta = 0.5, \alpha = 1, \overline{\lambda} = 1, \gamma = 1, \xi = 0.1, \beta = 1$ and $p = 0.5$.

Table 1 and Figure 3 clearly shows that as long as the outgoing call rate increases the servers idle time decreases. And the average queue length for incoming call increases. Simultaneously the utilisation factor is increases. Table 2 and Figure 4 reveals that as long as the retrial rate increases the servers idle time decreases. Simultaneously the utilisation factor is increases and the average queue length for incoming calls are decreases. Table 3 and Figure 5 clearly shows that as long as the incoming call rate increases the

Table 1: Effect of Outgoing call rate.

| $\pi$ | $I_0$ | $\rho$ | $L_q$ | $W_q$ |
|---|---|---|---|---|
| 1.1 | 0.1064 | 0.8936 | 3.7834 | 0.9459 |
| 1.2 | 0.0915 | 0.9085 | 4.0957 | 1.0239 |
| 1.3 | 0.0799 | 0.9201 | 4.3270 | 1.0817 |
| 1.4 | 0.0706 | 0.9294 | 4.5008 | 1.1252 |
| 1.5 | 0.0629 | 0.9371 | 4.6326 | 1.1581 |
| 1.6 | 0.0565 | 0.9435 | 4.7327 | 1.1832 |
| 1.7 | 0.0512 | 0.9488 | 4.8087 | 1.2022 |
| 1.8 | 0.0466 | 0.9534 | 4.8657 | 1.2164 |
| 1.9 | 0.0426 | 0.9574 | 4.9076 | 1.2269 |
| 2.0 | 0.0392 | 0.9608 | 4.9375 | 1.2344 |

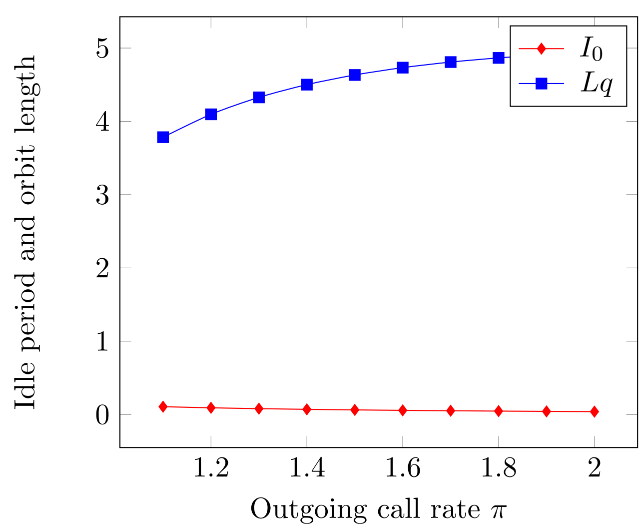


Figure 3: Average orbit size Vs Outgoing call rate $\pi$.

servers idle time decreases. And the average queue length for incoming call increases. Simultaneously the utilisation factor is increases.

## 6 Conclusion

In this paper we have analysed mixed priority retrial system under Bernoulli vacation subject to two way communication, working breakdown, negative arrival, repair and Bernoulli vacation also investigated. A practical problem arising in the call center literature is that incoming calls have priority over

Table 2: Effect of Retrial rate.

| $\eta$ | $I_0$ | $\rho$ | $L_q$ | $W_q$ |
|---|---|---|---|---|
| 1 | 0.0099 | 0.9901 | 9.3105 | 1.8621 |
| 2 | 0.0099 | 0.9901 | 9.2145 | 1.8429 |
| 3 | 0.0098 | 0.9902 | 9.1305 | 1.8261 |
| 4 | 0.0098 | 0.9902 | 9.0562 | 1.8112 |
| 5 | 0.0098 | 0.9902 | 8.9902 | 1.7980 |
| 6 | 0.0097 | 0.9903 | 8.9310 | 1.7862 |
| 7 | 0.0097 | 0.9903 | 8.8778 | 1.7756 |
| 8 | 0.0097 | 0.9903 | 8.8295 | 1.7659 |
| 9 | 0.0097 | 0.9903 | 8.7857 | 1.7571 |
| 10 | 0.0097 | 0.9903 | 8.7456 | 1.7491 |

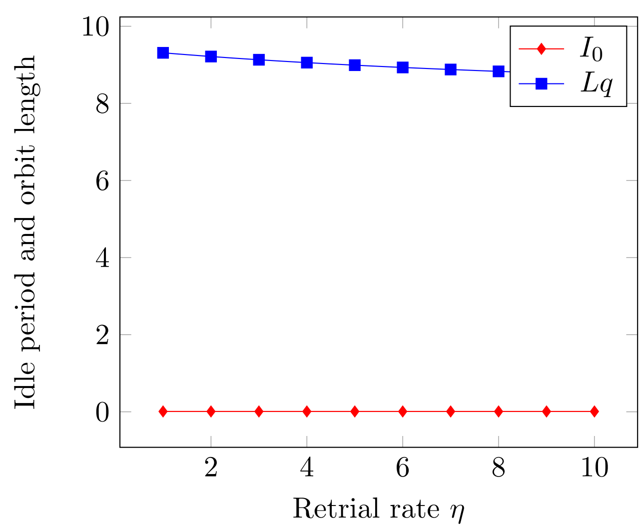


Figure 4: Average orbit size Vs Retrial rate $\eta$.

the outgoing calls. The server does not know how many calls are present in the orbit. As a result, the server may make an outgoing call even when there are some incoming calls in the orbit. This comment shows that the incoming calls preserve some priority over the outgoing calls. In addition, the effect of impatient behaviour of the customer on a service system is studied. Numerical examples have been carried out to observe the trend of the mean number of customers in the system for varying parametric values. This paper analyzes a single-server retrial queue with constant retrial policy. The novelty of this investigation is the discussion of the constant retrial policy, mixed priority service, two way communication with retrial queueing

Table 3: Effect of Incoming Call rate.

| $\lambda$ | $I_0$ | $\rho$ | $L_q$ | $W_q$ |
|---|---|---|---|---|
| 1.1 | 0.0480 | 0.9520 | 2.2042 | 2.0038 |
| 1.2 | 0.0472 | 0.9528 | 2.5022 | 2.0852 |
| 1.3 | 0.0466 | 0.9534 | 2.8150 | 2.1654 |
| 1.4 | 0.0460 | 0.9540 | 3.1450 | 2.2464 |
| 1.5 | 0.0454 | 0.9546 | 3.4936 | 2.3291 |
| 1.6 | 0.0449 | 0.9551 | 3.8620 | 2.4138 |
| 1.7 | 0.0444 | 0.9556 | 4.2508 | 2.5005 |
| 1.8 | 0.0440 | 0.9560 | 4.6603 | 2.5891 |
| 1.9 | 0.0436 | 0.9564 | 5.0909 | 2.6794 |
| 2.0 | 0.0432 | 0.9568 | 5.5426 | 2.7713 |

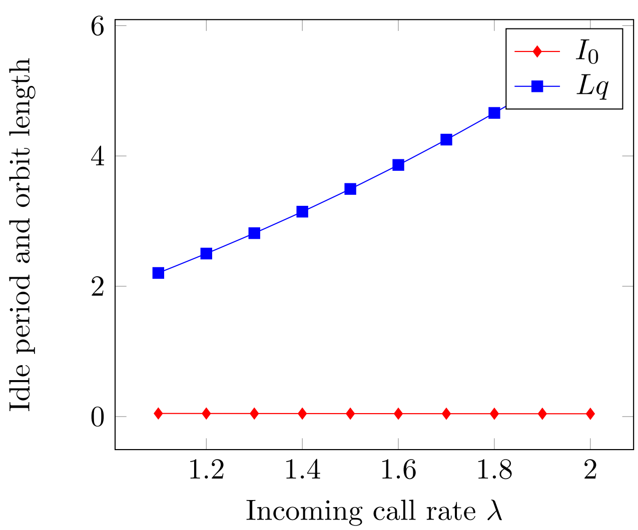


Figure 5: Average orbit size Vs Incoming call rate $\lambda$

system. This makes the system more complex though realistic.

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