Journal of Mathematical Modeling
Vol. 6, No. 1, 2018, pp. 91-104

JMM

Modified Laplace decomposition method for fractional Volterra-Fredholm integro-differential equations

Ahmed A. Hamoud[†‡*] and Kirtiwant P. Ghadle[‡]

[†] *Department of Mathematics, Taiz University, Taiz, 96704, Yemen*
[‡] *Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University Aurangabad, 431004, India*
emails: drahmedselwi985@gmail.com, drkp.ghadle@gmail.com

Abstract. This paper successfully applies the Adomian decomposition and the modified Laplace Adomian decomposition methods to find the approximate solution of a nonlinear fractional Volterra-Fredholm integro-differential equation. The reliability of the methods and reduction in the size of the computational work give these methods a wider applicability. Also, the behavior of the solution can be formally determined by analytical approximate. Moreover, the paper proves the convergence and uniqueness of the solution. Finally, this study includes an example to demonstrate the validity and applicability of the proposed techniques.

Keywords: Laplace transform, Adomian decomposition method, fractional Volterra-Fredholm integro-differential equation, Caputo fractional derivative.
AMS Subject Classification: 44A10, 49M27, 45J05, 26A33.

1 Introduction

In this paper, we consider the nonlinear Caputo fractional Volterra-Fredholm integro-differential equations of the form:

$$ {}^cD^{\alpha}y(x) = g(x) + \int_0^x K_1(x,t)F_1(y(t))dt + \int_0^1 K_2(x,t)F_2(y(t))dt, \qquad (1) $$

[*]Corresponding author.
Received: 12 November 2017 / Revised: 6 May 2018 / Accepted: 9 May 2018.
DOI 10.22124/jmm.2018.2826

with the initial condition

$$y^{(i)}(0) = \delta_i, \quad i = 0, 1, 2, \ldots, n-1, \tag{2}$$

where $n - 1 < \alpha \le n$ and $n \in \mathbb{N}$, $y : [0, 1] \longrightarrow \mathbb{R}$ be the continuous function which has to be determined, $g : [0, 1] \longrightarrow \mathbb{R}$ and $K_i : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$, are continuous functions. $F_i : \mathbb{R} \longrightarrow \mathbb{R}, i = 1, 2$ are nonlinear terms and Lipschitz continuous functions. Here ${}^cD^\alpha$ stands for the Caputo fractional derivative.

The fractional integro-differential equations have attracted much more interest of mathematicians and physicists which provides an efficiency for the description of many practical dynamical arising in engineering and scientific disciplines such as, physics, biology, electrochemistry, chemistry, economy, electromagnetic, control theory and viscoelasticity [2,7,13–15,17,18]. In recent years, numerous papers have been concentrating on the development of numerical and analytical techniques for fractional integro-differential equations. For instance, Al-Samadi and Gumah [2] applied the homotopy analysis method for the fractional SEIR Epidemic Model, Yang and Hou [17] applied the Laplace decomposition method to solve the fractional integro-differential equations, Mittal and Nigam [14] applied the Adomian decomposition method to approximate solutions for fractional integro-differential equations, and Ma and Huang [13] applied hybrid collocation method to study integro-differential equations of fractional order. Moreover, several authors examined properties of the fractional integro-differential equations [5, 15, 18].

The main objective of the present paper is to study the behavior of the solution that can be formally determined by analytical approximated methods such as the Adomian decomposition method and the modified Laplace Adomian decomposition method. Moreover, the paper proves the uniqueness and convergence of the solution of nonlinear fractional Volterra-Fredholm integro-differential equation.

The rest of the paper is organized as follows: Section 2 recalls some preliminaries and basic definitions related to fractional calculus and Laplace transform. In Section 3, Adomian Decomposition Method is constructed for solving fractional Volterra-Fredholm integro-differential equations. In Section 4, modified Laplace Adomian Decomposition Method is constructed for solving Volterra-Fredholm integro-differential equations of fractional order. Section 5 proves the convergence and uniqueness of the solution. Section 6 presents an analytical example to illustrate the accuracy of the methods used in this study. The final Section 7 gives a report on the paper along with a brief conclusion.

2 Preliminaries

The mathematical definitions of fractional derivative and fractional integration are the subject of different approaches. The most frequently used definitions of the fractional calculus involve the Riemann-Liouville fractional derivative, Caputo derivative, Riesz derivative and Grunwald-Letnikov fractional derivative [5, 8, 14, 17, 18]. This study uses the Caputo's definition of fractional derivative.

Definition 1. (Riemann-Liouville fractional integral). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f is defined as

$$\begin{aligned} J^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \qquad x > 0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned} \tag{3}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2. (Caputo fractional derivative). The fractional derivative, introduced by Caputo in the late sixties, is called Caputo fractional derivative. The fractional derivative of $f(x)$ in the Caputo sense is defined by

$$\begin{aligned} {}^cD_t^{\alpha} f(x) &= J^{m-\alpha} D^m f(t) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \frac{d^m f(t)}{dt^m} dt, & m-1 < \alpha < m, \\ \dfrac{d^m f(x)}{dt^m}, & \alpha = m, \quad m \in N, \end{cases} \end{aligned} \tag{4}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered.

Hence, we have the following properties:

- $J^{\alpha} J^{v} f = J^{\alpha+v} f, \quad \alpha, v > 0.$
- $J^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}, \quad \alpha > 0, \beta > -1, \quad x > 0.$
- $J^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0, \quad m-1 < \alpha \leq m.$

Definition 3. (Riemann-Liouville fractional derivative). The Riemann-Liouville fractional derivative of order $\alpha > 0$ is normally defined as

$$D^{\alpha} f(x) = D^m J^{m-\alpha} f(x), \qquad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{5}$$

Definition 4. The Laplace transform of a function $f(x), x > 0$ is defined as

$$\mathcal{L}[f(x)] = F(s) = \int_0^{+\infty} f(x) e^{-sx} dx, \tag{6}$$

where s can be either real or complex.

Definition 5. Given two functions f and g, we define, for any $x > 0$,

$$(f * g)(x) = \int_0^x f(t) g(x-t) dt. \tag{7}$$

The function $f * g$ is called the convolution of f and g.

Theorem 1. (The convolution theorem).

$$\mathcal{L}[f * g] = \mathcal{L}[f(x)] \cdot \mathcal{L}[g(x)]. \tag{8}$$

Theorem 2. *The Laplace transform $\mathcal{L}[f(x)]$ of the Caputo derivative is given as*

$$\mathcal{L}[{}^cD^{\alpha} f(x)] = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1 < \alpha \leq n. \tag{9}$$

3 Adomian Decomposition method

We consider the equation (1) where the operator ${}^{c}D^{\alpha}$ was defined in (5). Operating with J^{α} on both sides of the equation (1), we get:

$$\begin{aligned} y(x) &= \sum_{k=0}^{n-1} y^{k}(0^{+})\frac{x^{k}}{k!} \\ &+J^{\alpha}\left(g(x)+\int_{0}^{x}K_{1}(x,t)F_{1}(y(t))dt+\int_{0}^{1}K_{2}(x,t)F_{2}(y(t))dt\right) \end{aligned} \tag{10}$$

The Adomian's method defines the solution $y(x)$ by the series

$$y=\sum_{n=0}^{\infty} y_{n}, \tag{11}$$

and the nonlinear function F is decomposed as

$$F_{1}=\sum_{n=0}^{\infty} A_{n}, \qquad F_{2}=\sum_{n=0}^{\infty} B_{n}, \tag{12}$$

where A_n and B_n are the Adomian polynomials given by

$$A_{n}=\frac{1}{n!}\left[\frac{d^{n}}{d\phi^{n}}(F_{1}\sum_{i=0}^{n}\phi^{i}y_{i})\right]_{\phi=0}, \tag{13}$$

$$B_{n}=\frac{1}{n!}\left[\frac{d^{n}}{d\phi^{n}}(F_{2}\sum_{i=0}^{n}\phi^{i}y_{i})\right]_{\phi=0}. \tag{14}$$

The Adomian polynomials were introduced in [1,3,6,11,12,14,16] as:

$$\begin{aligned} A_{0} &= F_{1}(y_{0}), \\ A_{1} &= y_{1}F_{1}^{'}(y_{0}), \\ A_{2} &= y_{2}F_{1}^{'}(y_{0})+\frac{1}{2}y_{1}^{2}F_{1}^{''}(y_{0}), \\ A_{3} &= y_{3}F_{1}^{'}(y_{0})+y_{1}y_{2}F_{1}^{''}(y_{0})+\frac{1}{3}y_{1}^{3}F_{1}^{'''}(y_{0}), \\ &\vdots \end{aligned} \tag{15}$$

and

$$\begin{aligned} B_{0} &= F_{2}(y_{0}), \\ B_{1} &= y_{1}F_{2}^{'}(y_{0}), \\ B_{2} &= y_{2}F_{2}^{'}(y_{0})+\frac{1}{2}y_{1}^{2}F_{2}^{''}(y_{0}), \\ B_{3} &= y_{3}F_{2}^{'}(y_{0})+y_{1}y_{2}F_{2}^{''}(y_{0})+\frac{1}{3}y_{1}^{3}F_{2}^{'''}(y_{0}), \\ &\vdots \end{aligned} \tag{16}$$

The components $y_0,\ y_1,\ y_2,\ldots$ are determined recursively by

$$y_0 = \sum_{k=0}^{n-1} y^k(0^+)\frac{x^k}{k!} + J^\alpha g(x), \tag{17}$$

$$y_{k+1} = J^\alpha\left(\int_0^x K_1(x,s)A_k ds + \int_0^1 K_2(x,s)B_k ds\right). \tag{18}$$

Having defined the components $y_0,\ y_1,\ y_2,\ldots$, the solution y in a series form defined by (11) follows immediately. It is important to note that the decomposition method suggests that the 0^{th} component y_0 be defined by the initial conditions and the function $g(x)$ as described above. The other components namely $y_1,\ y_2,\ldots$, are derived recurrently.

4 Modified Laplace decomposition method

We apply the Laplace transform to both sides of Eq. (1):

$$\mathcal{L}[{}^cD^\alpha y(x)] = \mathcal{L}[g(x)] + \mathcal{L}[\int_0^x K_1(x,s)F_1(y(s))ds + \int_0^1 K_2(x,s)F_2(y(s))ds]. \tag{19}$$

Using the differentiation property of the Laplace transform (9) we get

$$s^\alpha\mathcal{L}[y(x)] - c = \mathcal{L}[g(x)] + \mathcal{L}[\int_0^x K_1(x,s)F_1(y(s))ds + \int_0^1 K_2(x,s)F_2(y(s))ds], \tag{20}$$

where $c = \sum_{k=0}^{m-1} x^{\alpha-k-1}y^{(k)}(0)$. Thus, the given equation is equivalent to

$$\begin{aligned}\mathcal{L}[y(x)] &= \frac{c}{s^\alpha} + \frac{1}{s^\alpha}\mathcal{L}[g(x)]\\ &+\frac{1}{s^\alpha}\mathcal{L}[\int_0^x K_1(x,s)F_1(y(s))ds + \int_0^1 K_2(x,s)F_2(y(s))ds].\end{aligned} \tag{21}$$

Substituting (11) and (12) into (21), we will get

$$\begin{aligned}\mathcal{L}\left[\sum_{n=0}^\infty y_n\right] &= \frac{c}{s^\alpha} + \frac{1}{s^\alpha}\mathcal{L}\left[g(x)\right] + \frac{1}{s^\alpha}\\ &\times\mathcal{L}\left[\int_0^t K_1(x,s)\sum_{n=0}^\infty A_n ds + \int_0^1 K_2(x,s)\sum_{n=0}^\infty B_n ds\right].\end{aligned} \tag{22}$$

Matching both sides of (22) yields the following iterative algorithm [4,8–10,17]:

$$\begin{aligned}
\mathcal{L}[y_0] &= \frac{c}{s^\alpha} + \frac{1}{s^\alpha}\mathcal{L}\left[g(x)\right]\\
\mathcal{L}[y_1] &= \frac{1}{s^\alpha}\mathcal{L}\left[\int_0^x K_1(x,s)A_0 ds + \int_0^1 K_2(x,s)B_0 ds\right],\\
\mathcal{L}[y_2] &= \frac{1}{s^\alpha}\mathcal{L}\left[\int_0^x K_1(x,s)A_1 ds + \int_0^1 K_2(x,s)B_1 ds\right],\\
&\vdots\\
\mathcal{L}[y_{n+1}] &= \frac{1}{s^\alpha}\mathcal{L}\left[\int_0^x K_1(x,s)A_n ds + \int_0^1 K_2(x,s)B_n ds\right],
\end{aligned} \tag{23}$$

5 Uniqueness and convergence

In this section, we shall give existence and uniqueness of the solution for equation (1) with the initial condition (2) and prove it. Before starting and proving the main results, we introduce the following hypotheses:

(A1) There exists two constants $L_{F_1}, L_{F_2} > 0$ such that, for any $y_1, y_2 \in C(J,\mathbb{R})$

$$|F_1(y_1(x)) - F_1(y_2(x))| \leq L_{F_1}\,|y_1 - y_2|\,,$$

and

$$|F_2(y_1(x)) - F_2(y_2(x))| \leq L_{F_2}\,|y_1 - y_2|\,;$$

(A2) There exists two functions $K_1^*, K_2^* \in C(D,\mathbb{R}^+)$, the set of all positive and continuous functions on $D = \{(x,t) \in \mathbb{R}\times\mathbb{R} : 0 \leq t \leq x \leq 1\}$ such that

$$K_1^* = \sup_{x\in[0,1]} \int_0^x |K_1(x,t)|\, dt < \infty, \quad K_2^* = \sup_{x\in[0,1]} \int_0^x |K_2(x,t)|\, dt < \infty;$$

(A3) The function $g : J \to \mathbb{R}$ is continuous.

Lemma 1. *If $y_0(x) \in C(J,\mathbb{R})$, then $y(x) \in C(J,\mathbb{R}^+)$ is a solution of the problem (1) − (2) iff y satisfies*

$$\begin{aligned}
y(x) &= c + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} g(s) ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\\
&\quad \times \left(\int_0^s K_1(s,\tau)F_1(y(\tau))d\tau + \int_0^1 K_2(s,\tau)F_2(y(\tau))d\tau\right) ds,
\end{aligned}$$

for $x \in J$.

Our first result is based on the Banach contraction principle.

Theorem 3. *Assume that (A1), (A2) and (A3) hold. If*

$$\left(\frac{K_1^* L_{F_1} + K_2^* L_{F_2}}{\Gamma(\alpha+1)}\right) < 1, \tag{24}$$

then there exists a unique solution $y(x) \in C(J)$ to (1) − (2).

Proof. By Lemma 1 we know that a function y is a solution to (1) − (2) iff y satisfies

$$\begin{aligned} y(x) &= c + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \\ &\quad \times \left(\int_0^s K_1(s,\tau) F_1(y(\tau)) d\tau + \int_0^1 K_2(s,\tau) F_2(y(\tau)) d\tau \right) ds. \end{aligned}$$

Let the operator $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be defined by

$$\begin{aligned} (Ty)(x) &= c + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \\ &\quad \times \left(\int_0^s K_1(s,\tau) F_1(y(\tau)) d\tau + \int_0^1 K_2(s,\tau) F_2(y(\tau)) d\tau \right) ds. \end{aligned}$$

We can see that, If $y \in C(J,\mathbb{R})$ is a fixed point of T, then y is a solution of (1)−(2).

Now we prove that T has a fixed point y in $C(J,\mathbb{R})$. For that, let $y_1, y_2 \in C(J,\mathbb{R})$ and for any $x \in [0,1]$ such that

$$\begin{aligned} y_1(x) &= c + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \\ &\quad \times \left(\int_0^s K_1(s,\tau) F_1(y_1(\tau)) d\tau + \int_0^1 K_2(s,\tau) F_2(y_1(\tau)) d\tau \right) ds, \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= c + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \\ &\quad \times \left(\int_0^s K_1(s,\tau) F_1(y_2(\tau)) d\tau + \int_0^1 K_2(s,\tau) F_2(y_2(\tau)) d\tau \right) ds. \end{aligned}$$

Consequently, we get

$$\begin{aligned} &|(Ty_1)(x) - (Ty_2)(x)| \\ &\quad \le \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\begin{array}{c} \int_0^s |K_1(s,\tau)| \, |F_1(y_1(\tau)) - F_1(y_2(\tau))| \, d\tau \\ + \int_0^1 |K_2(s,\tau)| \, |F_2(y_1(\tau)) - F_2(y_2(\tau))| \, d\tau \end{array} \right) ds \\ &\quad \le \frac{K_1^* L_{F_1}}{\Gamma(\alpha+1)} |y_1(x) - y_2(x)| + \frac{K_2^* L_{F_2}}{\Gamma(\alpha+1)} |y_1(x) - y_2(x)| \\ &\quad = \left(\frac{K_1^* L_{F_1} + K_2^* L_{F_2}}{\Gamma(\alpha+1)} \right) |y_1(x) - y_2(x)| \, . \end{aligned}$$

From the inequality (24) we have

$$\|Ty_1 - Ty_2\|_\infty \leq \|y_1 - y_2\|_\infty .$$

This means that T is contraction map. By the Banach contraction principle, we can conclude that T has a unique fixed point y in $C(J,\mathbb{R})$. □

Theorem 4. *Suppose that (A1)-(A3), and (24) hold. If the series solution*

$$y(x) = \sum_{i=0}^{\infty} y_i(x),$$

and $\|y_1\|_\infty < \infty$ *obtained by the* m*-order deformation is convergent, then it converges to the exact solution of the fractional Volterra-Fredholm integro-differential equation* $(1) - (2)$.

Proof. Denote as $(C[0,1], \|.\|)$ the Banach space of all continuous functions on J, with $|y_1(x)| \leq \infty$ for all x in J.

First, we define the sequence of partial sums s_n. Let s_n and s_m be arbitrary partial sums with $n \geq m$. We are going to prove that

$$s_n = \sum_{i=0}^{n} y_i(x),$$

is a Cauchy sequence in this Banach space. To do so,

$$\begin{aligned}
\|s_n - s_m\|_\infty &= \max_{\forall x \in J} |s_n - s_m| \\
&= \max_{\forall x \in J} |\sum_{i=0}^{n} y_i(x) - \sum_{i=0}^{m} y_i(x)| \\
&= \max_{\forall x \in J} | \sum_{i=m+1}^{n} y_i(x)| \\
&= \max_{\forall x \in J} | \sum_{i=m+1}^{n} (\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [\int_0^t K_1(t,s) A_i(s) ds \\
&\quad + \int_0^1 K_2(t,s) B_i(s) ds] dt)| \\
&= \max_{\forall x \in J} |\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [\int_0^t K_1(t,s) \sum_{i=m}^{n-1} A_i(s) ds \\
&\quad + \int_0^1 K_2(t,s) \sum_{i=m}^{n-1} B_i(s) ds)] dt|.
\end{aligned}$$

From (11) and (12), we have

$$\begin{aligned}
\sum_{i=m}^{n-1} A_i &= F_1(s_{n-1}) - F_1(s_{m-1}), \\
\sum_{i=m}^{n-1} B_i &= F_2(s_{n-1}) - F_2(s_{m-1}), \\
\sum_{i=m}^{n-1} y_i &= y(s_{n-1}) - y(s_{m-1}).
\end{aligned}$$

So,

$$\begin{aligned}
\|s_n - s_m\|_\infty &= \max_{\forall x\in J}(|\int_0^t K_1(t,s)(F_1(s_{n-1}) - F_1(s_{m-1}))ds \\
&\quad + \int_0^1 K_2(t,s)(F_2(s_{n-1}) - F_2(s_{m-1}))ds]dt|), \\
&\leq \max_{\forall x\in J}(\frac{1}{\Gamma(\alpha)}[\int_0^t |K_1(t,s)||(F_1(s_{n-1}) - F_1(s_{m-1}))|ds \\
&\quad + \int_0^1 |K_2(t,s)||(F_2(s_{n-1}) - F_2(s_{m-1}))|ds]dt), \\
&\leq \frac{1}{\Gamma(\alpha+1)}[K_1^* L_{F_1}\|s_{n-1} - s_{m-1}\|_\infty + K_2^* L_{F_2}\|s_{n-1} - s_{m-1}\|_\infty], \\
&= \left(\frac{K_1^* L_{F_1} + K_2^* L_{F_2}}{\Gamma(\alpha+1)}\right)\|s_{n-1} - s_{m-1}\|_\infty, \\
&= \delta\|s_{n-1} - s_{m-1}\|_\infty,
\end{aligned}$$

where

$$\delta = \left(\frac{K_1^* L_{F_1} + K_2^* L_{F_2}}{\Gamma(\alpha+1)}\right).$$

Let $n = m+1$, then

$$\begin{aligned}
\|s_n - s_m\|_\infty &\leq \delta\|s_m - s_{m-1}\|_\infty \\
&\leq \delta^2\|s_{m-1} - s_{m-2}\|_\infty \\
&\vdots \\
&\leq \delta^m\|s_1 - s_0\|_\infty,
\end{aligned}$$

so,

$$\begin{aligned}
\|s_n - s_m\|_\infty &\leq \|s_{m+1} - s_m\|_\infty + \|s_{m+2} - s_{m+1}\|_\infty \\
&\quad + \cdots + \|s_n - s_{n-1}\|_\infty \\
&\leq [\delta^m + \delta^{m+1} + \cdots + \delta^{n-1}]\|s_1 - s_0\|_\infty \\
&\leq \delta^m[1 + \delta + \delta^2 + \cdots + \delta^{n-m-1}]\|s_1 - s_0\|_\infty \\
&\leq \delta^m(\frac{1-\delta^{n-m}}{1-\delta})\|y_1\|_\infty.
\end{aligned}$$

Since $0 < \delta < 1$, we have $(1 - \delta^{n-m}) < 1$, and then

$$\|s_n - s_m\|_\infty \leq \frac{\delta^m}{1-\delta}\|y_1\|_\infty.$$

But $\|y_1(x)\|_\infty < \infty$, so, as $m \longrightarrow \infty$, then

$$\|s_n - s_m\|_\infty \longrightarrow 0.$$

We conclude that s_n is a Cauchy sequence in $C[0,1]$, therefore

$$y = \lim_{n\to\infty} y_n.$$

Then, the series is convergence and the proof is complete. □

6 Illustrative example

In this section, we present the analytical techniques based on ADM and MLADM to solve fractional Volterra-Fredholm integro-differential equation.

Example 1. Consider the following fractional Volterra-Fredholm integro-differential equation.

$${}^cD^{0.75}[y(t)] + \frac{t^2e^t}{5}y(t) = \frac{6t^{2.25}}{\Gamma(3.25)} + \int_0^t e^t s y(s)ds + \int_0^1 (4 - s^{-3})y(s)ds,$$

with the initial condition

$$y(0) = 0, \tag{25}$$

and the the exact solution is $y(t) = t^3$.

Firstly, we apply the Adomian decomposition method. Applying the operator J^α to both sides of Eq. (25) gives

$$\begin{aligned} y(t) \;&=\; \sum_{k=0}^{m-1} \frac{dy(0)}{dt^k}\frac{t^k}{k!} + \frac{6}{\Gamma(3.25)}J^\alpha(t^{2.25}) - \frac{1}{5}J^\alpha(t^2e^ty(t)) \\ &\quad + J^\alpha\left(\int_0^t e^t s y(s)ds + \int_0^1 (4 - s^{-3})y(s)ds\right). \end{aligned}$$

Then,

$$
\begin{aligned}
y_0(t) &= \sum_{k=0}^{m-1} \frac{dy(0)}{dt^k}\frac{t^k}{k!} + \frac{6}{\Gamma(3.25)} J^\alpha t^{2.25} \\
&= 0 + \frac{6}{\Gamma(3.25)} \frac{\Gamma(9/4+1)}{\Gamma(9/4+3/4+1)} t^{(9/4+3/4)} \\
&= t^3, \\
y_1(t) &= -\frac{1}{5} J^\alpha (t^2 e^t y_0(t)) + J^\alpha \Big(\int_0^t e^t s A_0(s) ds + \int_0^1 (4 - s^{-3}) B_0(s) ds \Big) \\
&= -\frac{1}{5} J^\alpha (t^2 e^t y_0(t)) + J^\alpha \Big(\int_0^t e^t s^4 ds + \int_0^1 (4 - s^{-3}) s^3 ds \Big) \\
&= -\frac{1}{5} J^\alpha (t^2 e^t y_0(t)) + J^\alpha \Big(\frac{1}{5} e^t t^5 + 0 \Big) \\
&= -\frac{1}{5} J^\alpha (t^2 e^t y_0(t)) + \frac{1}{5} J^\alpha (e^t t^2 y_0(t)) \\
&\vdots \\
y_n(t) &= 0. \qquad (26)
\end{aligned}
$$

Therefore, the obtained solution is $y(t) = t^3$.

Secondly, we employ the modified Laplace Adomian decomposition method. We apply the Laplace transform to both sides of (25)

$$
\begin{aligned}
\mathcal{L}\left[{}^c D^{0.75} y(t)\right] &= \mathcal{L}\left[(-\frac{t^2 e^t}{5}) y(t)\right] + \mathcal{L}\left[\frac{6t^{2.25}}{\Gamma(3.25)}\right] \\
&\quad + \mathcal{L}\left[\int_0^t e^t s y(s) ds + \int_0^1 (4 - s^{-3}) y(s) ds\right].
\end{aligned}
$$

Using the property of Laplace transform and the initial conditions (25), we get

$$
\begin{aligned}
s^{\frac{3}{4}} \mathcal{L}[y(t)] &= \mathcal{L}\left[(-\frac{t^2 e^t}{5}) y(t)\right] + \mathcal{L}\left[\frac{6t^{2.25}}{\Gamma(3.25)}\right] \\
&\quad + \mathcal{L}\left[\int_0^t e^t s y(s) ds + \int_0^1 (4 - s^{-3}) y(s) ds\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}[y(t)] &= \frac{1}{s^{\frac{3}{4}}} \Big(\mathcal{L}\left[(-\frac{t^2 e^t}{5}) y(t)\right] + \mathcal{L}\left[\frac{6t^{2.25}}{\Gamma(3.25)}\right] \\
&\quad + \mathcal{L}\left[\int_0^t e^t s y(s) ds + \int_0^1 (4 - s^{-3}) y(s) ds\right] \Big).
\end{aligned}
$$

Substituting (11) and (12) into above equation, we have

$$\begin{aligned}\mathcal{L}\left[\sum_{n=0}^{\infty} y_n(t)\right] &= \frac{1}{s^{\frac{3}{4}}}(\mathcal{L}\left[(-\frac{t^2e^t}{5})\sum_{n=0}^{\infty} y_n(t)\right] + \mathcal{L}\left[\frac{6t^{2.25}}{\Gamma(3.25)}\right] \\ &+\mathcal{L}\left[\int_0^t e^t s\sum_{n=0}^{\infty} A_n ds + \int_0^1 (4-s^{-3})\sum_{n=0}^{\infty} B_n ds\right]).\end{aligned}$$

By matching both sides of above equation, we have the following relations

$$\begin{aligned}\mathcal{L}\left[y_0(t)\right] &= \frac{1}{s^{\frac{3}{4}}}\mathcal{L}\left[\frac{6t^{2.25}}{\Gamma(3.25)}\right] \\ \mathcal{L}\left[y_1(t)\right] &= \frac{1}{s^{\frac{3}{4}}}\left(\mathcal{L}\left[(-\frac{t^2e^t}{5})y_0(t)\right] + \mathcal{L}\left[\int_0^t e^t sA_0 ds + \int_0^1 (4-s^{-3})B_0 ds\right]\right). \\ &\vdots \\ \mathcal{L}\left[y_{n+1}(t)\right] &= \frac{1}{s^{\frac{3}{4}}}\left(\mathcal{L}\left[(-\frac{t^2e^t}{5})y_n(t)\right] + \mathcal{L}\left[\int_0^t e^t sA_n ds + \int_0^1 (4-s^{-3})B_n ds\right]\right).\end{aligned}$$

By applying the inverse Laplace transform to above equations we get

$$\begin{aligned}y_0(t) &= t^3, \\ y_0(t) &= \mathcal{L}^{-1}(\frac{1}{s^{\frac{3}{4}}}(\mathcal{L}\left[(-\frac{t^2e^t}{5})y_0(t)\right] \\ &+\frac{1}{s^{\frac{3}{4}}}\mathcal{L}\left[\int_0^t e^t s^4 ds + \int_0^1 (4-s^{-3})s^3 ds\right])) = 0. \\ &\vdots \\ y_n(t) &= 0.\end{aligned}$$

Therefore, the obtained solution is $y(t) = t^3$.

7 Conclusion

This paper successfully applied the Adomian decomposition method and the modified Laplace Adomian decomposition method to find the approximate solution of nonlinear fractional Volterra-Fredholm integro-differential equation. The reliability of the methods and reduction in the size of the computational work give these methods a wider applicability. The methods are very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and fractional Volterra-Fredholm integro-differential equations. They provide more realistic series solutions that converge very rapidly in real physical problems. Finally, the behavior of the solution can be formally determined by analytical approximate. The proposed methods can be applied to other nonlinear fractional differential equations, systems of differential and integral equation.

References

[1] K. Abbaoui and Y. Cherruault, *Convergence of Adomian's method applied to nonlinear equations,* Math. Comput. Modelling **20** (1994) 69–73.

[2] M. AL-Smadi and G. Gumah, *On the homotopy analysis method for fractional SEIR epidemic model,* Research J. Appl. Sci. Engrg. Technol. **(7)** (2014) 3809–3820.

[3] M. Araghi and S. Behzadi, *Solving nonlinear Volterra-Fredholm integro-differential equations using the modified Adomian decomposition method,* Comput. Methods Appl. Math. **9** (2009) 321–331.

[4] M. Bani Issa, A. Hamoud, K. Ghadle and Giniswamy, *Hybrid method for solving nonlinear Volterra-Fredholm integro-differential equations*, J. Math. Comput. Sci. **7** (2017) 625–641.

[5] V. Daftardar-Gejji and H. Jafari, *Solving a multi-order fractional differential equation using Adomian decomposition,* Appl. Math. Comput. **189** (2007) 541–548.

[6] K. Ghadle and A. Hamoud, *Study of the approximate solution of fuzzy Volterra-Fredholm integral equations by using (ADM),* Elixir Appl. Math. **98** (2016) 42567–42573.

[7] A. Hamoud and K. Ghadle, *On the numerical solution of nonlinear Volterra-Fredholm integral equations by variational iteration method,* Int. J. Adv. Sci. Tech. Research **3** (2016) 45–51.

[8] A. Hamoud and K. Ghadle, *The reliable modified of Laplace Adomian decomposition method to solve nonlinear interval Volterra-Fredholm integral equations,* Korean J. Math. **25** (2017) 323–334.

[9] A. Hamoud and K. Ghadle, *The combined modified Laplace with Adomian decomposition method for solving the nonlinear Volterra-Fredholm integro-differential equations,* J. Korean Soc. Ind. Appl. Math. **21** (2017) 17–28.

[10] A. Hamoud and K. Ghadle, *Modified Adomian decomposition method for solving fuzzy Volterra-Fredholm integral equations,* J. Indian Math. Soc. **85** (1-2) (2018), 52–69.

[11] A. Hamoud and K. Ghadle, *A study of some reliable methods for solving fuzzy Volterra-Fredholm integral equations*, Acta Univ. Apul. **53** (2018) 65–92.

[12] A. Hamoud and K. Ghadle, *Recent advances on reliable methods for solving Volterra-Fredholm integral and integro-differential equations*, Asian J. Math. Comput. Res. **24** (2018) 128-157.

[13] X. Ma and C. Huang,*Numerical solution of fractional integro-differential equations by a hybrid collocation method*, Appl. Math. Comput. **219** (2013) 6750–6760.

[14] R. Mittal and R. Nigam, *Solution of fractional integro-differential equations by Adomian decomposition method*, Int. J. Appl. Math. Mech. **4** (2008) 87–94.

[15] X. Yang, *Local fractional integral equations and their applications*, Advs. Comput. Sci. Appl. **1** (2012) 234–239.

[16] A. Wazwaz, *A reliable modification of Adomian decomposition method*, Appl. Math. Comput. **102** (1999) 77–86.

[17] C. Yang and J. Hou, *Numerical solution of integro-differential equations of fractional order by Laplace decomposition method*, Wseas Trans. Math. **12** (2013) 1173–1183.

[18] X. Zhang, B. Tang, and Y. He, *Homotopy analysis method for higher-order fractional integro-differential equations*, Comput. Math. Appl. **62** (2011) 3194–3203.