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Existence and continuous dependence for fractional neutral functional differential equations

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Abstract. In this paper, we investigate the existence, uniqueness and continuous dependence of solutions of fractional neutral functional differential equations with infinite delay and the Caputo fractional derivative order, by means of the Banach's contraction principle and the Schauder's fixed point theorem.

Keywords: Fractional differential equations, Functional differential equations, Fractional derivative and Fractional integral, Existence and continuous dependence, Fixed point theorem.
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1 Introduction

In the recent years, the fractional differential equations have attracted a considerable interest in mathematics and many applications, such as physics, mechanics, chemistry, engineering, etc. For more details, see the monographs of Kilbas et al. [18], Miller and Ross [22], Podlubny [28] and Samko et al. [30], and the papers of Delboso and Rodino [6], Diethelm et al. [8–10], Gaul et al. [13], Glockle and Nonnenmacher [14], Lakshmikantham [19], Mainardi [20], Metzler et al. [21], Momani et al. [23], Momani

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and Hadid [24], Noroozi et al. [25–27], Podlubny et al. [29], Yu and Gao [31] and the references therein.

To our knowledge, fractional delay neutral functional differential equations has not been extensively studied. Especially, the results dealing with infinite delay are comparatively scarce. Among these studies, some authors studied fractional functional differential equations [1, 3, 4, 7, 12]. For example in [4], Benchohra et al. used the Banach fixed point theorem and Leray-Schauder type nonlinear alternative to investigate the existence and uniqueness of solutions for the following problem

$$D^{\alpha}[x(t) - g(t, x_t)] = f(t, x_t), \qquad t \in [0, b], \tag{1}$$

$$x(t) = \phi(t), \qquad t \in (-\infty, 0], \tag{2}$$

where $0 < \alpha < 1$, D^{α} is the standard Riemann-Liouville fractional derivative, $\phi \in \mathcal{B}$, $\phi(0) = 0$, $\mathcal{B}$ is called a phase space $f, g : [0, b] \times \mathcal{B} \longrightarrow \mathbb{R}$ $(b > 0)$ are given functions satisfying some assumptions with $g(0, \phi) = 0$. Agarwal et al. in [1], used the Krasnoselskii's fixed point theorem to study the existence result of the following problem

$$^{c}D^{\alpha}[x(t) - g(t, x_t)] = f(t, x_t), \qquad t \in [t_0, \infty), \tag{3}$$

$$x_{t_0} = \phi \in \mathcal{C}, \tag{4}$$

where $0 < \alpha < 1$, $^{C}D^{\alpha}$ is the Caputo fractional derivative, $f, g : [t_0, \infty) \times \mathcal{C} \longrightarrow \mathbb{R}^n$ are given functions satisfying some assumptions and $\mathcal{C}$ is a space of continuous functions on $[-\tau, 0]$.

Our approach is based on the Banach contraction principle and the Schauder's fixed point theorem to get on the existence, uniqueness results and continuous dependence of solutions for the following fractional neutral functional differential equations with infinite delay

$$^{c}D^{\alpha}[x(t) - g(t, x_t)] = f(t, x_t), \qquad t \in [0, b], \tag{5}$$

$$x_0 = \phi \in \mathcal{B}, \tag{6}$$

where $0 < \alpha \leq 1$, $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f, g : [0, b] \times \mathcal{B} \longrightarrow \mathbb{R}$ $(b > 0)$ are given functions satisfying some assumptions that will be specified in Section 3, $\mathcal{B}$ the phase space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, which will be specified in Section 2 and $x_t : (-\infty, 0] \to \mathbb{R}$, such that $x_t(\theta) = x(t + \theta)$ for $\theta \leq 0$.

The paper is organized in five sections. In Section 2, we introduce some preliminaries and list the hypotheses that will be used throughout this paper. Section 3 is devoted to the study of existence and uniqueness of solutions to the problem (5)–(6). The continuous dependence of solutions to such equations in the space $C([a, b])$ is discussed in Section 4. Finally, the conclusion are given in Section 5.

2 Preliminaries

In this section, we introduce definitions and preliminary facts which are used throughout this paper. Let $C([0,b],\mathbb{R})$ be the space of all continuous real functions defined on $[0,b]$ and let $L^p([0,b])$ $(1 \leq p < \infty)$ denotes the set of Lebesgue measurable functions f on $[0,b]$ with the norm $\|f\|_p < \infty$, where $\|f\|_p = \left(\int_0^b |f(t)|^p\,dt\right)^{\frac{1}{p}}$. For any function x defined on $(-\infty,b]$ and any $t \in [0,b]$, we denote by x_t the element of $\mathcal{B}$ defined by

$$x_t(s) = x(t+s), \qquad \text{for } -\infty < s \leq 0. \tag{7}$$

We will consider the following space

$$\Omega = \{x : (-\infty,b] \to \mathbb{R} :\ x\mid_{(-\infty,0]} \in \mathcal{B},\ x\mid_{[0,b]} \in C([0,b],\mathbb{R})\},$$

where $x\mid_{[0,b]}$is the restriction of x to $[0,b]$.

Definition 1. *([18]). The fractional integral of order $\alpha > 0$ with the lower limit zero for a function $h : [0,b] \to \mathbb{R}$ is defined as*

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds, \quad t > 0,$$

provided the right-hand side is pointwise on $[0,b]$, where $\Gamma(.)$ is the gamma function.

Definition 2. *([18]) The Riemann-Liouville fractional derivative of order α $(n-1 < \alpha < n)$ with the lower limit zero for a function $h \in C([0,b],\mathbb{R})$ at the point t is characterized as*

$$D_0^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}h(s)ds = D^n I_0^{n-\alpha}h(t), \;\; t > 0, D^n = \frac{d^n}{dt^n}.$$

In particular, if $0 < \alpha < 1$ and $h \in L^p([0,b])$, then $D_0^\alpha I_0^\alpha h(t) = h(t)$, for $t > 0$.

Definition 3. *([18]). Let $n-1 < \alpha < n$ and $h \in C([0,b],\mathbb{R})$. The Caputo fractional derivative with the lower limit zero for a function h is determined as*

$${}^cD_0^\alpha h(t) = D_0^\alpha\left(h(t) - \sum_{k=0}^{n-1}\frac{h^{(k)}(0)}{k!}t^k\right).$$

In particular, if $0 < \alpha < 1$, we have ${}^cD_0^\alpha h(t) = D_0^\alpha\left(h(t) - h(0)\right)$. Moreover, if ${}^cD_0^\alpha h(t) \in L^p([0,b])$, then

$$I_0^\alpha\ {}^cD_0^\alpha h(t) = h(t) - h(0).$$

Also we can write

$$ {}^cD_0^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds, \quad t > 0. $$

If $0 < \alpha < 1$, *we have*

$$ {}^cD_0^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} h'(s) ds, \quad t > 0. $$

Obviously, the Caputo derivative of a constant is equal to zero.

Lemma 1. *(Hölder's inequality). Assume that* $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $f \in L^p([a,b], \mathbb{R})$ *and* $g \in L^q([a,b], \mathbb{R})$.*, then we have Hölder's inequality for integrals states that*

$$ \int_a^b |f(t)g(t)|\, dt \le \left(\int_a^b |f(t)|^p\, dt \right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\, dt \right)^{\frac{1}{q}}. $$

Definition 4. *A function* $x \in \Omega$ *is said to be a solution of (5)–(6) if* x *satisfies the equation* ${}^cD^\alpha[x(t) - g(t, x_t)] = f(t, x_t)$, $t \in [0, b]$, *with initial condition* $x_0 = \phi$.

In this paper, we denote ${}^cD_0^\alpha$, D_0^α and I_0^α by ${}^cD^\alpha$, D^α and I^α, respectively, we also assume that the state space $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a seminormed linear of functions mapping $(-\infty, 0]$ into $\mathbb{R}$ and satisfying the following fundamental axioms which were introduced by Hale and Kato in [15] and widely discussed in [16]:

(H1) If $x : (-\infty, b] \to \mathbb{R}$, such that x is continuous on $[0, b]$ and $x_0 \in \mathcal{B}$, then for every $t \in [0, b]$ the following statements hold:

(i) $x_t \in \mathcal{B}$;

(ii) $|x(t)| \le H \|x_t\|_{\mathcal{B}}$ for some $H > 0$ which is equivalent to $|\phi(0)| \le H \|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$;

(iii) $\|x_t\|_{\mathcal{B}} \le K(t) \sup_{0 \le s \le t} |x(s)| + M(t) \|x_0\|_{\mathcal{B}}$, where $K, M : [0, +\infty) \to [0, +\infty)$ with K continuous and M locally bounded, such that K, M are independent of $x(.)$. Denote $K_b = \sup\{K(t) : t \in [0, b]\}$ and $M_b = \sup\{M(t) : t \in [0, b]\}$.

(H2) For the function $x(.)$ in (H1), the function $t \to x_t$ is continuous from $[0, b]$ into $\mathcal{B}$.

(H3) The space $\mathcal{B}$ is complete.

3 Main results

In this section, we give an existence and uniqueness results of (5)–(6) and prove it by the Banach contraction principle and the Schauder's fixed point theorem. Before starting and proving the main results, we introduce the following hypotheses:

(A1) There exists a positive constant L_f such that

$$|f(t,u) - f(t,v)| \leq L_f \|u - v\|_{\mathcal{B}}, \quad t \in [0,b], \ u, v \in \mathcal{B}.$$

(A2) There exists a positive constant L_g such that

$$|g(t,u) - g(t,v)| \leq L_g \|u - v\|_{\mathcal{B}}, \quad t \in [0,b], \ u, v \in \mathcal{B}.$$

(A3) $f : [0,b] \times \mathcal{B} \to \mathbb{R}$ is continuous.

(A4) There exist an $\eta \in L^p([0,b], \mathbb{R})$ with $p > \frac{1}{\alpha}$ and a continuously non-decreasing function $\Psi : [0, +\infty) \to [0, +\infty)$ such that

$$|f(t,u)| \leq \eta(t)\Psi(\|u\|_{\mathcal{B}}), \quad t \in [0,b], \ u \in \mathcal{B}.$$

(A5) The function g is continuous and completely continuous and for any bounded set in Ω, the set $\{t \to g(t, x_t) : x \in \mathcal{B}\}$ is equicontinuous in $C([0,b], \mathbb{R})$ and there exist constants $0 \leq c_1 < 1, c_2 > 0$ such that

$$|g(t,u)| \leq c_1 \|u\|_{\mathcal{B}} + c_2, \quad t \in [0,b], \ u \in \mathcal{B}.$$

Firstly, we prove the uniqueness result by means of the Banach contraction principle theorem.

Theorem 1. *Assume that (A1) and (A2) hold. If*

$$K_b \left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)} L_f \right) < 1, \tag{8}$$

then there exists a unique solution to (5)-(6) on $(-\infty, b]$.

Proof. In view of Definition 3, the function x is a solution to (5)-(6) iff x satisfies

$$x(t) = \begin{cases} \phi(0) - g(0,\phi) + g(t,x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x_s) ds, & t \in [0,b], \\ \phi(t), & t \in (-\infty, 0]. \end{cases}$$

Now, we transform the problem (5)-(6) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$(Nx)(t) = \begin{cases} \phi(0) - g(0,\phi) + g(t,x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x_s)ds, & t \in [0,b] \\ \phi(t), & t \in (-\infty,0]. \end{cases}$$

For $\phi \in \mathcal{B}$, let $w : (-\infty,b] \to \mathbb{R}$ be the function defined by

$$w(t) = \begin{cases} \phi(0), & t \in [0,b], \\ \phi(t), & t \in (-\infty,0]. \end{cases} \tag{9}$$

Then, we get $w_0 = \phi$. For each function $z \in C([0,b],\mathbb{R})$, let $\overline{z} : (-\infty,b] \to \mathbb{R}$ be the extension of z to $(-\infty,b]$ such that

$$\overline{z}(t) = \begin{cases} z(t), & t \in [0,b], \\ 0, & t \in (-\infty,0]. \end{cases}$$

If $x(.)$ satisfies the integral equation

$$x(t) = \phi(0) - g(0,\phi) + g(t,x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x_s)ds, \quad t \in [0,b],$$

then we can decompose $x(.)$ as follows $x(t) = w(t) + \overline{z}(t)$, $t \in (-\infty,b]$, which implies $x_t = w_t + \overline{z}_t$, for every $t \in [0,b]$ and the function $z(.)$ satisfies

$$z(t) = -g(0,\phi) + g(t,w_t+\overline{z}_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,w_s+\overline{z}_s)ds, \quad t \in [0,b], \tag{10}$$

with $\overline{z}_0 = 0$. Set $\Omega_0 = \{z \in \Omega,\ z_0 = 0\}$. For $z \in \Omega_0$ let $\|.\|_{\Omega_0}$ be seminorm in Ω_0 defined by

$$\|z\|_{\Omega_0} = \|z_0\|_{\mathcal{B}} + \|z\|_C = \sup\{|z(t)| : t \in [0,b]\}. \tag{11}$$

Then $(\Omega_0, \|z\|_{\Omega_0})$ is a Banach space, which was proved by Arion [2]. Let the operator $T : \Omega_0 \to \Omega_0$ be defined by

$$(Tz)(t) = -g(0,\phi) + g(t,w_t+\overline{z}_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,w_s+\overline{z}_s)ds, \quad t \in [0,b] \tag{12}$$

and $(Tz)(t) = 0$, for $t \in (-\infty,0]$. Then, we get $(Tz)_0 = 0$.

Obviously, the operator N has a fixed point equivalent to T that has a fixed point too. So we turn to prove that T has a fixed point. If $z \in \Omega_0$ is a fixed point of T, then $x = w + \overline{z}$ is the unique solution to (5)-(6).

Now, we will show that the operator $T:\Omega_0\to\Omega_0$ is a contraction map. Indeed, consider $z,z^*\in\Omega_0$. For each $t\in[0,b]$, we have

$$\begin{aligned}
& |(Tz)(t)-(Tz^*)(t)| \\
\leq\ & \left|g(t,w_t+\overline{z}_t)-g(t,w_t+\overline{z^*}_t)\right| \\
& +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,w_s+\overline{z}_s)-f(s,w_s+\overline{z^*}_s)\right|ds \\
\leq\ & L_g\left\|z_t-z_t^*\right\|_{\mathcal{B}}+\frac{1}{\Gamma(\alpha)}L_f\left\|z_t-z_t^*\right\|_{\mathcal{B}}\int_0^t (t-s)^{\alpha-1}ds.
\end{aligned}$$

Since

$$\begin{aligned}
\left\|z_t-z_t^*\right\|_{\mathcal{B}} &\leq K(t)\sup_{0\leq\tau\leq t}\{|z(\tau)-z^*(\tau)|\}+M(t)\left\|z_0-z_0^*\right\|_{\mathcal{B}} \\
&\leq K_b\sup_{0\leq\tau\leq t}\{|z(\tau)-z^*(\tau)|\} \\
&= K_b\left\|z-z^*\right\|_C \qquad (13)\\
&= K_b\left\|z-z^*\right\|_{\Omega_0}, \qquad (14)
\end{aligned}$$

we get

$$\begin{aligned}
& |(Tz)(t)-(Tz^*)(t)| \\
\leq\ & L_gK_b\left\|z-z^*\right\|_{\Omega_0}+\frac{1}{\Gamma(\alpha)}L_fK_b\left\|z-z^*\right\|_{\Omega_0}\int_0^t (t-s)^{\alpha-1}ds \\
\leq\ & K_b\left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)\left\|z-z^*\right\|_{\Omega_0}.
\end{aligned}$$

Consequently,

$$\left\|Tz-Tz^*\right\|_{\Omega_0}\leq K_b\left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)\left\|z-z^*\right\|_{\Omega_0}.$$

By the inequality (8), we conclude that

$$\left\|Tz-Tz^*\right\|_{\Omega_0}\leq\left\|z-z^*\right\|_{\Omega_0}.$$

This proves that T is a contraction map. As a consequence of the Banach contraction principle, we can deduce that T has a unique fixed point which is just the unique solution to the integral equation (10) on $[0,b]$.

Now set $x=w+\overline{z}$, then x is the unique solution to the fractional differential equation (5)-(6) on $(-\infty,b]$. □

Next, the following result is based on Schauder's fixed point theorem.

Theorem 2. *Assume that (A3), (A4) and (A5) hold. Then there exists at least a solution to (5)-(6) on $(-\infty, b]$ provided that*

$$K_b\left(c_1+\frac{b^{\alpha+1}\left\|\eta\right\|_p}{\Gamma(\alpha+1)}\lim_{\zeta\to+\infty}\sup\frac{\Psi\left(\zeta\right)}{\zeta}\right)<1. \tag{15}$$

Proof. As in proof of Theorem 1, we define the operator $T:\Omega_0\to\Omega_0$ and we also show that the operator T has a fixed point by using the Schauder's fixed point theorem. This fixed point is the solution to the problem (5)-(6). For this purpose, we proceed in several steps.

Step 1: T is continuous.

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in Ω_0 such that $z_n\to z$ in Ω_0, as well $(z_n)_t\to z_t$ in Ω_0 as $n\to\infty$. By (12), then for every $t\in[0,b]$ and for each $z_n, z\in\Omega_0$, we have

$$\begin{aligned}
& \left|(Tz_n)(t)-(Tz)(t)\right| \\
\leq\ & \left|g(t,w_t+(\overline{z_n})_t)-g(t,w_t+\overline{z}_t)\right| \\
& +\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|f(s,w_s+(\overline{z_n})_s)-f(s,w_s+\overline{z}_s)\right|ds.
\end{aligned}$$

From the continuity of f, the complete continuity of g and the Lebesgue dominated convergence theorem, we get $\left\|Tz_n-Tz\right\|_{\Omega_0}\to 0$ as $n\to\infty$. So T is continuous.

Step 2: T maps bounded sets into bounded sets in Ω_0.

Consider $\mathbb{B}_r=\{z\in\Omega_0:\left\|z\right\|_{\Omega_0}\leq r\}$. For any $r>0$, it can be shown that there exists a positive constant ℓ such that for all $z\in\mathbb{B}_r$, $\left\|Tz\right\|_{\Omega_0}\leq\ell$. Let $z\in\mathbb{B}_r$, for each $t\in[0,b]$, we have

$$\begin{aligned}
& \left|(Tz)(t)\right| \\
\leq\ & \left|g(0,\phi)\right|+\left|g(t,w_t+\overline{z}_t)\right|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|f(s,w_s+\overline{z}_s)\right|ds \\
\leq\ & (c_1\left\|\phi\right\|_{\mathcal{B}}+c_2)+(c_1\left\|w_t+\overline{z}_t\right\|_{\mathcal{B}}+c_2) \\
& +\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\eta(s)\Psi\left(\left\|w_s+\overline{z}_s\right\|_{\mathcal{B}}\right)ds \\
=\ & c_1(\left\|\phi\right\|_{\mathcal{B}}+\left\|w_t+\overline{z}_t\right\|_{\mathcal{B}})+2c_2 \\
& +\frac{1}{\Gamma(\alpha)}\Psi\left(\left\|w_t+\overline{z}_t\right\|_{\mathcal{B}}\right)\int_0^t(t-s)^{\alpha-1}\eta(s)ds.
\end{aligned} \tag{16}$$

Since

$$
\begin{aligned}
\|w_t+\overline{z}_t\|_{\mathcal{B}} &\leq \|w_t\|_{\mathcal{B}}+\|\overline{z}_t\|_{\mathcal{B}} \\
&\leq K(t)\sup_{0\leq\tau\leq t}|w(\tau)|+M(t)\|w_0\|_{\mathcal{B}}+K(t)\sup_{0\leq\tau\leq t}|\overline{z}(\tau)|+M(t)\|\overline{z}_0\|_{\mathcal{B}} \\
&\leq K_bH\|\phi\|_{\mathcal{B}}+M_b\|\phi\|_{\mathcal{B}}+K_b\sup_{0\leq\tau\leq t}|z(\tau)| \\
&= (K_bH+M_b)\|\phi\|_{\mathcal{B}}+K_b\|z\|_{\mathbf{\Omega_0}} \\
&\leq (K_bH+M_b)\|\phi\|_{\mathcal{B}}+K_br \qquad (17)\\
&: = r_0 \qquad (18)
\end{aligned}
$$

then by Lemma 1, (16) leads to

$$
\begin{aligned}
|(Tz)(t)| &\leq c_1(\|\phi\|_{\mathcal{B}}+r_0)+2c_2+\frac{1}{\Gamma(\alpha)}\Psi(r_0)\left(\int_0^t(t-s)^{(\alpha-1)q}ds\right)^{\frac{1}{q}}\|\eta\|_p \\
&\leq c_1(\|\phi\|_{\mathcal{B}}+r_0)+2c_2+\frac{b^{\alpha+1}\|\eta\|_p}{\Gamma(\alpha+1)}\Psi(r_0) \\
&:= \ell,
\end{aligned}
$$

where $q>1$, $\frac{1}{p}<\alpha$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\eta\|_p=\left(\int_0^t|\eta(s)|^p\,ds\right)^{\frac{1}{p}}$. Therefore, $\|Tz\|_{\Omega_0}\leq\ell$, for every $z\in\mathbb{B}_r$. This means that $T\mathbb{B}_r\subset\mathbb{B}_\ell$. i.e. T maps bounded sets into bounded sets in Ω_0.

Step 3: T maps bounded sets into equicontinuous sets in Ω_0.

Let $z\in\mathbb{B}_r$ such that $\mathbb{B}_r$ be a bounded set of Ω_0 as in Step 2 and let $t_1,t_2\in[0,b]$ with $t_1<t_2$, we have

$$
\begin{aligned}
& |(Tz)(t_2)-(Tz)(t_1)| \\
\leq\ & |g(t_2,w_{t_2}+\overline{z}_{t_2})-g(t_1,w_{t_1}+\overline{z}_{t_1})| \\
& +\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,w_s+\overline{z}_s)ds-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,w_s+\overline{z}_s)ds\right| \\
\leq\ & |g(t_2,w_{t_2}+\overline{z}_{t_2})-g(t_1,w_{t_1}+\overline{z}_{t_1})| \\
& +\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right||f(s,w_s+\overline{z}_s)|\,ds \\
& +\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,w_s+\overline{z}_s)|\,ds.
\end{aligned}
$$

By the complete continuity of g we have

$$|g(t_2,w_{t_2}+\overline{z}_{t_2})-g(t_1,w_{t_1}+\overline{z}_{t_1})|\to 0,$$

as $t_1 \to t_2$. Hence, by (18) and Lemma 1, we obtain

$$
\begin{aligned}
& |(Tz)(t_2) - (Tz)(t_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) \eta(s) \Psi\left(\|w_s + \overline{z}_s\|_{\mathcal{B}}\right) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \eta(s) \Psi\left(\|w_s + \overline{z}_s\|_{\mathcal{B}}\right) ds \\
\leq\ & \frac{\Psi(r_0)\,\|\eta\|_p}{\Gamma(\alpha)} \left(\int_0^{t_1} ((t_1 - s)^{(\alpha-1)} - (t_2 - s)^{(\alpha-1)})^q ds \right)^{\frac{1}{q}} \\
& + \frac{\Psi(r_0)\,\|\eta\|_p}{\Gamma(\alpha)} \left(\int_{t_1}^{t_2} (t_2 - s)^{(\alpha-1)q} ds \right)^{\frac{1}{q}} \\
\leq\ & \frac{\Psi(r_0)\,\|\eta\|_p}{r_2 \Gamma(\alpha)} ((t_1^{r_1} - t_2^{r_1}) + 2(t_2 - t_1)^{r_1}) \\
\leq\ & \frac{2\Psi(r_0)\,\|\eta\|_p}{r_2 \Gamma(\alpha)} (t_2 - t_1)^{r_1},
\end{aligned}
$$

where r_0 is defined as in Step 2, $r_1 = \frac{(\alpha-1)q+1}{q}$ and $r_2 = ((\alpha-1)q+1)^{\frac{1}{q}} > 0$.

It follows that $|(Tz)(t_2) - (Tz)(t_1)| \to 0$, as $t_2 - t_1 \to 0$ and the convergence is independent of z in $\mathbb{B}_r$. This implies that the set $\{T\mathbb{B}_r\}$ is equicontinuous. The equicontinuity for the cases $t_1 < t_2 \leq 0$, and $t_1 \leq 0 \leq t_2$ obvious.

As a consequence of Steps 1–3, and along with the Arzela–Ascoli theorem, we can conclude that $T : \Omega_0 \to \Omega_0$ is completely continuous.

Finally, we need to verify that there exists a closed convex bounded subset $\mathbb{B}_\epsilon = \{z \in \Omega_0 : \|z\|_{\Omega_0} \leq \epsilon\} \subseteq \Omega_0$ such that $T\mathbb{B}_\epsilon \subseteq \mathbb{B}_\epsilon$. For each positive integer ϵ, clearly $\mathbb{B}_\epsilon$ is closed, convex and bounded subset of Ω_0. We claim that there exists a positive integer ϵ such that $T\mathbb{B}_\epsilon \subseteq \mathbb{B}_\epsilon$. If this property is not true, then for every positive integer ϵ, there exists $z_\epsilon \in \mathbb{B}_\epsilon$ such that $(Tz_\epsilon) \notin \mathbb{B}_\epsilon$, i.e. $\|Tz_\epsilon(t)\|_{\Omega_0} > \epsilon$ for some $t(\epsilon) \in [0, b]$, where $t(\epsilon)$ denotes t depending on ϵ. But by using the previous hypotheses, we obtian

$$
\begin{aligned}
\epsilon\ <\ & \|Tz_\epsilon\|_{\Omega_0} = \|(Tz_\epsilon)_0\|_{\mathcal{B}} + \sup_{0 \leq t \leq b} |(Tz_\epsilon)(t)| \\
\leq\ & \sup_{0 \leq t \leq b} \left\{ \begin{array}{c} |g(0, \phi)| + |g(t, w_t + (\overline{z_\epsilon})_t)| \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s, w_s + (\overline{z_\epsilon})_s) \right| ds \end{array} \right\} \\
\leq\ & \sup_{0 \leq t \leq b} \left\{ \begin{array}{c} (c_1 \|\phi\|_{\mathcal{B}} + c_2) + (c_1 \|w_t + (\overline{z_\epsilon})_t\|_{\mathcal{B}} + c_2) \\ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \Psi\left(\|w_s + (\overline{z_\epsilon})_s\|_{\mathcal{B}}\right) ds \end{array} \right\}.
\end{aligned}
$$

According to the inequality (17), we can conclude that

$$\|w_t + (\overline{z_\epsilon})_t\|_{\mathcal{B}} \leq (K_b H + M_b)\|\phi\|_{\mathcal{B}} + K_b\epsilon := \zeta$$

and by Lemma 1, we have

$$\zeta < (K_b H + M_b)\|\phi\|_{\mathcal{B}} + K_b\left(c_1\left(\|\phi\|_{\mathcal{B}} + \zeta\right) + 2c_2 + \frac{b^{\alpha+1}\|\eta\|_p}{\Gamma(\alpha+1)}\Psi(\zeta)\right).$$

Dividing both sides by ζ and taking the upper limit as $\zeta \to +\infty$, we obtain

$$1 < K_b\left(c_1 + \frac{b^{\alpha+1}\|\eta\|_p}{\Gamma(\alpha+1)}\lim_{\zeta\to+\infty}\sup\frac{\Psi(\zeta)}{\zeta}\right),$$

which contradicts our assumption (15). Thus, for some positive integer ϵ, we must have $T\mathbb{B}_\epsilon \subseteq \mathbb{B}_\epsilon$.

An application of Schauder's fixed point theorem shows that there exists at least a fixed point z of T in Ω_0. Therefore, $x = w + \overline{z}$ is the solution to (5)-(6) on $(-\infty, b]$, and the proof is completed. $\square$

4 Continuous dependence

In this section, we discuss the influence of perturbed data on the solution.

Definition 5. *The functions $x(\phi,.), x(\psi,.) \in C([0,b])$ are solutions of the problems (5)-(6) and*

$$^{c}D^{\alpha}[x(t) - g(t,x_t)] = f(t,x_t), \quad t \in [0,b], \tag{19}$$

$$x_0 = \psi \in \mathcal{B}, \tag{20}$$

respectively on $(-\infty, b]$ if $x(\phi,.) = w_\phi + \overline{z_1}$ and $x(\psi,.) = w_\psi + \overline{z_2}$, where $w_\phi(t) = \phi(0)$, $w_\psi(t) = \psi(0)$, $\overline{z_1}(t) = z_1(t)$, $\overline{z_2}(t) = z_2(t)$ and

$$\begin{aligned} z_1(t) &= -g(0,\phi) + g(t,(w_\phi)_t + (\overline{z_1})_t) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,(w_\phi)_s + (\overline{z_1})_s)ds, \end{aligned} \tag{21}$$

$$\begin{aligned} z_2(t) &= -g(0,\psi) + g(t,(w_\psi)_t + (\overline{z_2})_t) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,(w_\psi)_s + (\overline{z_2})_s)ds, \end{aligned} \tag{22}$$

for each $t \in [0,b]$ and for every $z_1, z_2 \in C([0,b])$.

Definition 6. *([11]). The solution $x \in C([0,b])$ of the problem (5)-(6) is continuously dependent on initial data if for every $\phi, \psi \in \mathcal{B}$,*

$$\|x(\phi,.) - x(\psi,.)\|_C \leq O(\|\phi - \psi\|_{\mathcal{B}}). \tag{23}$$

Definition 7. *The functions $x(g,f,.), x(\widetilde{g},\widetilde{f},.) \in C([0,b])$ are solutions of the problems (5)-(6) and*

$${}^cD^\alpha[x(t) - \widetilde{g}(t,x_t)] = \widetilde{f}(t,x_t), \quad t \in [0,b], \tag{24}$$

$$x_0 = \phi \in \mathcal{B}, \tag{25}$$

respectively on $(-\infty, b]$ if $x(g,f,.) = w + \overline{z_1}$ and $x(\widetilde{g},\widetilde{f},.) = w + \overline{z_2}$, where $w(t) = \phi(0)$, $\overline{z_1}(t) = z_1(t)$, $\overline{z_2}(t) = z_2(t)$ and

$$\begin{aligned} z_1(t) &= -g(0,\phi) + g(t, w_t + (\overline{z_1})_t) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, w_s + (\overline{z_1})_s) ds, \end{aligned} \tag{26}$$

$$\begin{aligned} z_2(t) &= -\widetilde{g}(0,\phi) + \widetilde{g}(t, w_t + (\overline{z_2})_t) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \widetilde{f}(s, w_s + (\overline{z_2})_s) ds, \end{aligned} \tag{27}$$

for each $t \in [0,b]$ and for every $z_1, z_2 \in C([0,b])$.

Definition 8. *([11]). The solution $x \in C([0,b])$ of the problem (5)-(6) is continuously dependent on the parameters f and g if for every $f, \widetilde{f}, g, \widetilde{g} \in C([0,b] \times \mathcal{B}, \mathbb{R})$,*

$$\left\| x(g,f,.) - x(\widetilde{g},\widetilde{f},.) \right\|_C \leq O(\sup|g - \widetilde{g}|) + O(\sup\left|f - \widetilde{f}\right|).$$

Firstly, we have the following theorem regarding the continuous dependence of solution on the initial values.

Theorem 3. *Suppose that the assumptions of Theorem 1 hold. Then there exists a constant κ such that*

$$\|x(\phi,.) - x(\psi,.)\|_C \leq \kappa \|\phi - \psi\|_{\mathcal{B}}, \quad \forall \phi, \psi \in \mathcal{B}.$$

Proof. By Theorem 1, we know that for every $\phi, \psi \in \mathcal{B}$ the problems (5)-(6) and (19)-(20) have solutions $x(\phi,.)$ and $x(\psi,.)$, respectively on $(-\infty, b]$. Further, there are $z_1, z_1 \in C([0,b])$ such that $x(\phi,.) = w_\phi + \overline{z_1}$, $x(\psi,.) = w_\psi + \overline{z_2}$ and satisfying (21) and (22) for $t \in [0,b]$.

Now, $x(\phi,t) = \phi(0) + z_1(t)$ and $x(\psi,t) = \psi(0) + z_2(t)$ for $t \in [0,b]$. Hence, we have

$$
\begin{aligned}
& |x(\phi,t) - x(\psi,t)| \\
\leq\; & |\phi(0) - \psi(0)| + |z_1(t) - z_2(t)| \\
\leq\; & |\phi(0) - \psi(0)| + |g(0,\phi) - g(0,\psi)| + |g(t,(w_\phi)_t + (\overline{z_1})_t) - g(t,(w_\psi)_t + (\overline{z_2})_t)| \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s,(w_\phi)_s + (\overline{z_1})_s) - f(s,(w_\psi)_s + (\overline{z_2})_s) \right| ds \\
\leq\; & H \|\phi - \psi\|_{\mathcal{B}} + L_g \|\phi - \psi\|_{\mathcal{B}} + L_g \left(\|\phi_t - \psi_t\|_{\mathcal{B}} + \|(\overline{z_1})_t - (\overline{z_2})_t\|_{\mathcal{B}} \right) \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L_f \left(\|\phi_s - \psi_s\|_{\mathcal{B}} + \|(\overline{z_1})_s - (\overline{z_2})_s\|_{\mathcal{B}} \right) ds.
\end{aligned}
$$

Since

$$
\begin{aligned}
\|(\overline{z_1})_t - (\overline{z_2})_t\|_{\mathcal{B}} &\leq K(t) \sup_{0 \leq \tau \leq t} |\overline{z_1}(\tau) - \overline{z_2}(\tau)| + M(t) \|(\overline{z_1})_0 - (\overline{z_2})_0\|_{\mathcal{B}} \\
&\leq K_b \sup_{0 \leq \tau \leq t} |z_1(\tau) - z_2(\tau)| \\
&= K_b \|z_1 - z_2\|_C
\end{aligned}
$$

and

$$
\begin{aligned}
\|\phi_t - \psi_t\|_{\mathcal{B}} &\leq K(t) \sup_{0 \leq \tau \leq t} |\phi(\tau) - \psi(\tau)| + M(t) \|\phi_0 - \psi_0\|_{\mathcal{B}} \\
&\leq K_b |\phi(0) - \psi(0)| + M_b \|\phi - \psi\|_{\mathcal{B}} \\
&\leq (K_b H + M_b) \|\phi - \psi\|_{\mathcal{B}},
\end{aligned}
$$

we get

$$
\begin{aligned}
|x(\phi,t) - x(\psi,t)| &\leq \left(H + L_g + \left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)} L_f \right) (K_b H + M_b) \right) \|\phi - \psi\|_{\mathcal{B}} \\
&\quad + \left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)} L_f \right) K_b \|z_1 - z_2\|_C \\
&\leq \left(H + L_g + \left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)} L_f \right) (K_b H + M_b) \right) \|\phi - \psi\|_{\mathcal{B}} \\
&\quad + \left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)} L_f \right) K_b \left(\|x(\phi,.) - x(\psi,.)\|_C + H \|\phi - \psi\|_{\mathcal{B}} \right).
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\|x(\phi,.) - x(\psi,.)\|_C &\leq \left(H(1+K_b) + L_g + \left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)} L_f \right) (K_b H + M_b) \right) \|\phi - \psi\|_{\mathcal{B}} \\
&\quad + \left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)} L_f \right) K_b \|x(\phi,.) - x(\psi,.)\|_C .
\end{aligned}
$$

By inequality (8), we conclude that

$$\|x(\phi,.)-x(\psi,.)\|_C \le \kappa \|\phi-\psi\|_{\mathcal{B}},$$

where $\kappa = \frac{\left(H(1+K_b)+L_g+\left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)(K_bH+M_b)\right)}{1-\left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b}$.

This means that the solution of the problem (5)-(6) is continuously dependent on the random data ϕ. □

THe following theorem is devoted to the study of the continuous dependence of solution on the given functions f and g.

Theorem 4. *Let f, $\widetilde{f}$, g, and $\widetilde{g}$ fulfill hypotheses (A1), (A2) and (A3). Then there exist two constants K_1 and K_2 such that*

$$\begin{aligned} &\left\|x(g,f,.)-x(\widetilde{g},\widetilde{f},.)\right\|_C \\ \le\ & K_1 \sup_{(t,u)\in[0,b]\times\mathcal{B}} |g(t,u)-\widetilde{g}(t,u)| + K_2 \sup_{(t,u)\in[0,b]\times\mathcal{B}} \left|f(t,u)-\widetilde{f}(t,u)\right|, \end{aligned}$$

provided that $\left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b<1$.

Proof. The existence and uniqueness results can be confirmed by Theorem 1 and Theorem 2. Let $t\in[0,b]$ and let $z_1, z_1 \in C([0,b])$ such that $x(g,f,t) = w(t)+\overline{z_1}(t)$, $x(\widetilde{g},\widetilde{f},t) = w(t)+\overline{z_2}(t)$ and satisfying (26) and (27). Hence, we have

$$\begin{aligned} &\left|x(g,f,t)-x(\widetilde{g},\widetilde{f},t)\right| \\ =\ & |z_1(t)-z_2(t)| \\ \le\ & |g(0,\phi)-\widetilde{g}(0,\phi)| + |g(t,w_t+(\overline{z_1})_t)-g(t,w_t+(\overline{z_2})_t)| \\ & + |g(t,w_t+(\overline{z_2})_t)-\widetilde{g}(t,w_t+(\overline{z_2})_t)| \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} |f(s,w_s+(\overline{z_1})_s)-f(s,w_s+(\overline{z_2})_s)|\,ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \left|f(s,w_s+(\overline{z_2})_s)-\widetilde{f}(s,w_s+(\overline{z_2})_s)\right|ds \\ \le\ & L_g\|(\overline{z_1})_t-(\overline{z_2})_t\|_{\mathcal{B}} + 2\sup_{(t,u)\in[0,b]\times\mathcal{B}} |g(t,u)-\widetilde{g}(t,u)| \\ & + \frac{t^\alpha}{\Gamma(\alpha+1)}L_f\|(\overline{z_1})_t-(\overline{z_2})_t\|_{\mathcal{B}} + \frac{t^\alpha}{\Gamma(\alpha+1)}\sup_{(t,u)\in[0,b]\times\mathcal{B}}\left|f(t,u)-\widetilde{f}(t,u)\right| \\ \le\ & \left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b\|z_1-z_2\|_C + 2\sup_{(t,u)\in[0,b]\times\mathcal{B}}|g(t,u)-\widetilde{g}(t,u)| \\ & + \frac{b^\alpha}{\Gamma(\alpha+1)}\sup_{(t,u)\in[0,b]\times\mathcal{B}}\left|f(t,u)-\widetilde{f}(t,u)\right|. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\|x(g,f,.)-x(\widetilde{g},\widetilde{f},.)\right\|_C \\
\leq\ & \left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b\left\|x(g,f,.)-x(\widetilde{g},\widetilde{f},.)\right\|_C+2\sup_{(t,u)\in[0,b]\times\mathcal{B}}|g(t,u)-\widetilde{g}(t,u)| \\
&+\frac{b^\alpha}{\Gamma(\alpha+1)}\sup_{(t,u)\in[0,b]\times\mathcal{B}}\left|f(t,u)-\widetilde{f}(t,u)\right|.
\end{aligned}$$

Since $\left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b<1$, we have

$$\begin{aligned}
&\left\|x(g,f,.)-x(\widetilde{g},\widetilde{f},.)\right\|_C \\
\leq\ & \frac{2}{1-\left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b}\sup_{(t,u)\in[0,b]\times\mathcal{B}}|g(t,u)-\widetilde{g}(t,u)| \\
&+\frac{b^\alpha}{\Gamma(\alpha+1)\left(1-\left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b\right)}\sup_{(t,u)\in[0,b]\times\mathcal{B}}\left|f(t,u)-\widetilde{f}(t,u)\right|.
\end{aligned}$$

Let $K_1=\frac{2}{1-\left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b}$ and $K_2=\frac{b^\alpha}{\Gamma(\alpha+1)(1-\left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b)}$. Then, we obtain

$$\begin{aligned}
&\left\|x(g,f,.)-x(\widetilde{g},\widetilde{f},.)\right\|_C \\
\leq\ & K_1\sup_{(t,u)\in[0,b]\times\mathcal{B}}|g(t,u)-\widetilde{g}(t,u)|+K_2\sup_{(t,u)\in[0,b]\times\mathcal{B}}\left|f(t,u)-\widetilde{f}(t,u)\right|.
\end{aligned}$$

This confirms that the solution to the problem (5)-(6) is continuously dependent on the given functions f and g. □

5 Conclusion

In this paper, the Banach contraction principle and the Schauder's fixed point theorem are used to prove the existence and uniqueness results for fractional neutral functional differential equations (5)-(6) with infinite delay and Caputo fractional derivative. Also the influence of perturbed data have been discussed.

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