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Determining optimal value of the shape parameter c in RBF for unequal distances topographical points by Cross-Validation algorithm

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Abstract. Several radial basis function based methods contain a free shape parameter which has a crucial role in the accuracy of the methods. Performance evaluation of this parameter in different functions with various data has always been a topic of study. In the present paper, we consider studying the methods which determine an optimal value for the shape parameter in interpolations of radial basis functions for data collections produced by topographical images that are not necessarily in equal distances. The Cross-Validation method is picked out of several existing algorithms proposed for determining the shape parameter.

Keywords: Radial Basis Function, Cross-Validation error, three-dimensional image.
AMS Subject Classification: 34A34, 65L05.

1 Introduction

During the last three decades, methods using radial basis functions have been used in the fields of geophysics, geology, reconstruction of image, measurement from an aerial map, solving ordinary or partial differential equations including nonlinear equations with shock waves, heat transfer, shallow water equations to simulate the tide, release without heat, transport equation and the parabolic and elliptic partial differential equations

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[4, 6, 12, 14, 16]. The radial basis functions are radially symmetric with respect to their centers and their shapes remain invariant in all dimensions. The centers themselves do not have to be specially distributed in such a way where interpolation data are difficult to obtain [6]. Moreover, the approximation accuracy provided by interpolant can be adjusted using the parameters known as shape parameters c [3, 12]. In all radial basis function topics, it is crucial to determine the best value of c, which in turn leads to the most accurate approximation of the original function [10]. For Hardy's MQ function, this value can be defined as $c = 0.815d$, where d is the distance from the nearest neighborhood [5]. There are several methods for obtaining d. One of them is based on a circle area of diameter D which contains grid points. If we divide this circle into n equal parts, each part will be as $A = \pi D^2/(4n)$. If these areas are circles of diameter $D/\sqrt{n}$, one may use them to approximate the shortest distance between two neighborhoods. Frank has used the formula $R = 1.25D/\sqrt{n}$. In fact, he used the coefficient 1.25 instead of 0.815. All these methods are based on grid points (x_i, y_i) [11, 15]. Although these values are good for initial estimations, but they do not always provide the best results [15]. Foley and Carlson have proposed a method by studying the statistical properties, such as quality of scattering and obtaining the RMS error. They did this by different functions and concluded that the value of c depends on the function which is approximated [8]. Carlson and Foley showed that the optimal value of c is independent of the number of grid points and their distribution. They have recommended using the same value of c both for the multiquadric and inverse multiquadric interpolants. Rippa in [15] showed that this is true in many cases, but in some ones, the optimal value of c for a specific function is different for various data [1]. He showed that the accuracy of computation is also significand in determining the optimal value of c. Moreover, he defined the optimal value of c as the value minimizing RMS error. He found further the general behavior of the RMS error graph is similar for their multiquadric, inverse multiquadric and Gaussian interpolants. In all the proposed methods, paying attention to the quality of data scattering and using binary precision, are significant in results, otherwise, the validity of results might fall in jeopardy. In [7], the authors provided values for the parameter c, based on estimations. Kansa obtained optimal parameter c based on the root least squares [7], but Rippa showed that in this method, no attention is paid to the Runges phenomenon [9].

Radial basis functions are an extension of transitional univariate spline functions to multivariate spline functions [8, 13]. In general form, they are defined as below

$$s(x)=\sum_{i=1}^{n}\lambda_i\varphi(r_i);\quad r_i=\|x-x_i\|,\quad x,x_i\in\mathbb{R}^d, \tag{1}$$

where $||.||$ is a norm and $\varphi:\mathbb{R}^+\longrightarrow\mathbb{R}$ is a radial basis function [13]. Some radial basis functions are presented in Table 1.

Table 1: Some radial basis functions

Function name	**Function form**
Maltiquadric (MQ)	$\sqrt{r^2+c^2}$
Inverse Maltiquadric (IMQ)	$1/\sqrt{r^2+c^2}$
Gaussian (GA)	$\exp(-(cr)^2)$
Thin Plate Splines (TSP)	$r^2\log r$
Triharmonic	$\|r\|^3$

Let f be a known function in the grid points x_i, then, one can consider an interpolation function for f so that the following interpolational conditions are satisfied:

$$s(x_i)=f(x_i),\quad i=1,2,\ldots,n. \tag{2}$$

By choosing a radial basis function and applying the interpolational conditions, we have:

$$f(x_j)=\sum_{i=1}^{n}\lambda_i\varphi\big(\|x_j-x_i\|\big),\quad j=1,2,\ldots,n, \tag{3}$$

where $\varphi(r)$ is a radial basis function and the unknown coefficients λ_i $i=1,2,\ldots,n$, should be determined. Equation (3) is equivalent to solve the following system of linear equations:

$$A\lambda=b, \tag{4}$$

where $A_{ji}=\varphi\big(\|x_j-x_i\|\big)$ and $b_j=f(x_j)$.

In this article, we review the determination of the optimum c for dissimilar interval data. Here, we proposed an algorithm which is capable of providing the best value of c for distinct distance points. A similar method is achieved by Rippa in [10]. Although this method has high computational costs, it is able to suggest the optimal amount of c with a high precision. In the second part of this article, we have defined Cross-Validation error and expressed an algorithm for it. In the third section, some numerical results on different data are given. Finally, the conclusions are given.

2 Cross-Validation error and process of choosing c

Cross-Validation concept can be explained as follows. Let us consider a data set such as $p = \{x_1, x_2, \ldots, x_n\}$. Now, if a datum like x_i is omitted from the set p, a new p_{-i} set is obtained which is defined as follows:

$$p_{-i} = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\},$$

construct the Meta model with p_{-i} and use x_i as validate point. Also the absolute error at x_i is denoted by

$$e_j = |f(x_j) - \hat{f}_{-i}(x_j)|, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n, \quad j \neq i \tag{5}$$

where $\hat{f}_{-i}$ is the function corresponding the points in p_{-i}. Let $\lambda = x - x_i$, then by (5) we have to solve a linear system $A\lambda = e$, where the coefficient matrix A is defined in Section 1.

After N times computation of e_i such that $N < n$, the Cross-Validation RMS error can be obtained as following:

$$\mathbf{RMSE}_{Cross} = \sqrt{\frac{\sum_{i=1}^{N} e_i^2}{N}}. \tag{6}$$

When there are enough sample points, the Cross-Validation error gets closer to the prediction error. Since all $\hat{f}_{-i}(x_j)$ are functions of c, the result obtained from RMSE [1] in the Cross-Validation method can be denoted as RMS_{Cross}. It is the cost function which we need to find an appropriate c.

The following algorithm has been used to determine the shape parameter in order to demonstrate its validity for our data that are not in equal distances. In Step 1, the distance between sample points influences the location of the critical value, so we suggest the initial value of the average d_i, that is the minimum distance between x_i and other sample points.

$$c_{primary} = \mathrm{Average}(d_i).$$

[1] Root mean square error

Algorithm: The process of choosing c
Step 1: Choose an initial value for c.
Step 2: Take c as variable, and use (5) as the cost function.
Step 3: Look for the trend of error, and determine whether to increase or decrease c.
Step 4: Stop until the error is no smaller than ε(tolerance), and take the current c as the solution of the cost function.

3 Numerical results

A major problem with civil engineers always is to measure distant points on which little data is available, such as the points which are located across a valley and are so hard to reach due to the impassability of the area, and are not necessarily scattered regularly. Due to the radial characteristic of the radial basis functions, we can define a more appropriate procedure for such points.

The first data belong to the map of a valley located near the city of Kerman. Using AutoCAD, 81 grid points have been taken out of this map. Figure 1 shows a topographic map of the area.

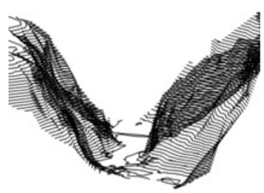

Figure 1: Topographic map of the valley.

Figure 2(a) is constructed from Figure 1 by Matlab. The results of interpolation of radial basis functions for the valley is presented in Figure 2. The data for the second examples is taken from Figure 3 which is a set

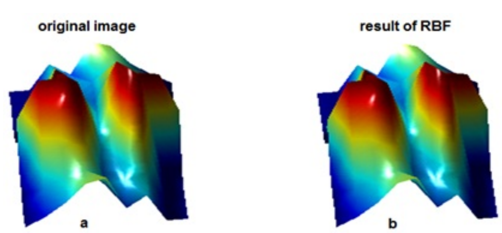

Figure 2: Figure (a) is a result of Figure 1 by Matlab and Figure (b) is an interpolation of radial basis function.

of 3721 points [2].

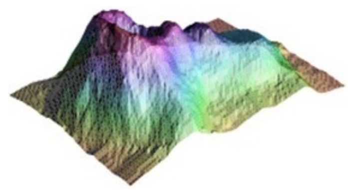

Figure 3: A surf of volcano.

By applying our proposed algorithm to this example, we have the suitable interpolation of radial basis functions of these data. The result of interpolation of radial basis functions for the volcano is presented in Figure 4.

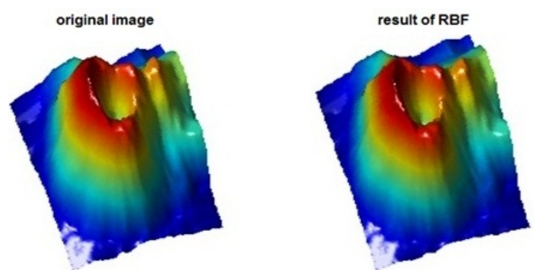

Figure 4: The result of Figure 3 by MATLAB (left) and interpolation of radial basis function (right).

Also, two radial basis functions and parameter values c having the lowest Cross-Validation error are presented in Table 2.

Table 2: Numerical result of Figures 1 and 3.

Radial Basis Function	c **value**	**RMS error**	**Data**
$e^{-(cr)^2}$	1.493	$1.4e-7$	Valley
$e^{-(cr)^2}$	0.2336	$3.7e-7$	Volcano
$(c^2+r^2)^\beta,\ \ \beta=-3.5$	0.1852	$1.2e-5$	Valley
$(c^2+r^2)^\beta,\ \ \beta=-3.5$	1.0751	$2.5e-5$	Volcano

Finally, we have used this algorithm to design three-dimensional portraits of models or animations. We used pixels of an image instead of the heights used in geographic regions and presented three-dimensional images of a two-dimensional image by use of classical RBF interpolation where the parameter c is chosen by the process introduced in Section 2.

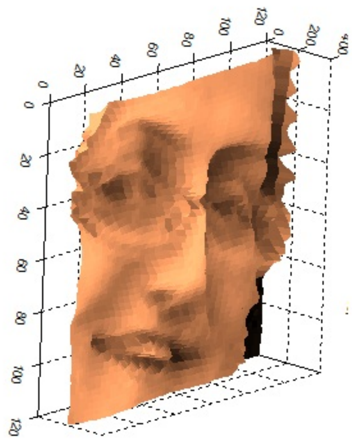

Figure 5: Two and three-dimensional images, left and right respectively.

The *imread* command in MATLAB software was used to obtain the pixel data. A total number of 2916 pixels were used in this image. The results approve the accuracy of this algorithm, as seen in Figure 5 and table 3.

Table 3: Numerical result of Figure 5.

Radial Basis Function	c value	RMS Error
$e^{-(cr)^2}$	2.43	$2.8e-7$
$(c^2+r^2)^\beta,\ \ \beta=-3.5$	0.57	$1.3e-5$

4 Conclusion

We have utilized Cross-Validation algorithm to pick out the best shape parameter c in RBF using two types of data. In the first type, the distance between geographical points was different, but in the second type it was equal. The pixels in the image are equal in both forms and the results are favorable.

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