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Mixed two-stage derivative estimator for sensitivity analysis

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Abstract. In mathematical modeling, determining most influential parameters on outputs is of major importance. Thus, sensitivity analysis of parameters plays an important role in model validation. We give detailed procedure of constructing a new derivative estimator for general performance measure in Gaussian systems. We will take advantage of using score function and measure-value derivative estimators in our approach. It is shown that the proposed estimator performs better than other estimators for a dense class of test functions in the sense of having smaller variance.

Keywords: derivative estimator, infinitesimal perturbation analysis, measure-valued, risk analysis, score function, stochastic activity network.
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1 Introduction

Parameter estimation is a popular estimation problem in statistics and has been carried out by using many different methods. It has many applications such as determining how much each parameter contributes to the output variability from sensitivity analysis viewpoint. As in econometric studies fluctuations are measured by the variance component, it might be of interest to consider system sensitivity with respect to the variance component. Thus, the primary purpose of this study is to consider the performance of

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scale parameter estimators comparatively and propose a new estimator at second hand.

More precisely, let X be a random variable with normal distribution $\mathcal{N}(\mu, \sigma^2)$. In sensitivity analysis the derivative estimator to measure the system sensitivity w.r.t the scale parameter is defined as

$$\frac{d}{d\sigma}\mathbb{E}[g(X)] = \frac{d}{d\sigma}\left(\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} g(x)e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx\right),$$

where $g(.)$ is a Borel measurable function with finite expectation.

There are three known procedures to derive the derivative estimator, namely the infinite perturbation analysis (IPA), the score function (SF) and the measure-valued derivatives (MVD), see for example Ho and Cao [11], Pflug [12], and Rubinstein [15] for more details. It is also worthwhile to mention that Heidergott et al. [10] defined another estimator namely coupled phantoms which interestingly performed better than IPA, SF and MV derivative estimators in the sense of having smaller variance. In this paper, we consider a different scheme by combining the methodologies behind the constructions of SF and MV derivative estimators to define a new estimator.

2 Sensitivity estimators

In this section, first we briefly review the three existing methods for the construction of derivative estimator and propose the new approach. In this regard, Ho and Cao [11] and Glasserman [5] treated IPA in the context of queueing systems. We also refer to Cao [2] for further studies. Fu and Hu [4] applied IPA and SPA estimators in inventory control and finance. The LR/SF method was introduced by Reiman and Weiss [13], Rubinstein [14], Glynn [7] and is treated in depth by Rubinstein and Shapiro [15]. Glasserman [6] also discussed both IPA (known as the pathwise method) and the LR/SF method from a financial viewpoint. The WD method was introduced by Pflug [12] and it has been put into a more general framework of "measure-valued differentiation", see for example [8].

2.1 Some existing estimators

The IPA derivative estimator is a derivative estimator for which the control parameter is present in the performance function (Glasserman, [5]).

Let Z be distributed according to the Gaussian law $\mathcal{N}(0, 1)$; then $X = \sigma Z + \mu$ is $\mathcal{N}(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$. For a differentiable function

$g : \mathbb{R} \to \mathbb{R}$, using Stein's identity (Stein, [16]) we have

$$\begin{aligned} \frac{d}{d\sigma}\mathbb{E}[g(X)] &= \frac{d}{d\sigma}\mathbb{E}[g(\sigma Z + \mu)] \\ &= \mathbb{E}[Zg'(\sigma Z + \mu)], \end{aligned}$$

where $g'(.)$ is the first derivative of g. Furthermore assume that $g(.)$ is Lipschitz continuous almost surely.

The IPA estimator is then defined by

$$D^{(IPA)}[g(X)] = \frac{(X-\mu)}{\sigma} g'(X). \tag{1}$$

For $\mu = 0$, the variance of the IPA estimator is given as

$$\mathrm{V}ar(D^{(IPA)}[g(X)]) = \mathbb{E}[Z^2(g'(\sigma Z))^2] - \mathbb{E}^2[Zg'(\sigma Z)]. \tag{2}$$

The SF derivative estimator is a distributional approach to derivative estimation, see for example Rubinstein and Shapiro [15] for more details.

Let $\frac{d}{d\sigma}\phi_{\mu,\sigma}(x)$ denote the derivative of the standard normal density w.r.t σ. If for an open neighborhood U of σ the following condition holds

$$\mathbb{E}\big[|g(X)| \sup_{\hat{\sigma}\in U} \big|\frac{d}{d\sigma}\log\phi_{\mu,\sigma}(X)\big|\big] < \infty, \tag{3}$$

then it follows that

$$\begin{aligned} \frac{d}{d\sigma}\mathbb{E}[g(X)] &= \frac{d}{d\sigma}\left(\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} g(x)e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx\right) \\ &= \int_{-\infty}^{\infty} g(x)\frac{d}{d\sigma}\phi_{\mu,\sigma}(x)dx. \end{aligned} \tag{4}$$

The fact that the expression on the RHS of (4) contains the derivative $\frac{d}{d\sigma}\phi_{\mu,\sigma}(x)$, imposes the problem of sampling from $\frac{d}{d\sigma}\phi_{\mu,\sigma}(x)$.

Noting that

$$\frac{d}{d\sigma}\log\phi_{\mu,\sigma}(x) = \frac{\frac{d}{d\sigma}\phi_{\mu,\sigma}(x)}{\phi_{\mu,\sigma}} \tag{5}$$

simplifies the expression in (4) to

$$\begin{aligned} \frac{d}{d\sigma}\mathbb{E}[g(X)] &= \int_{-\infty}^{\infty} g(x)\big(\frac{d}{d\sigma}\log\phi_{\mu,\sigma}(x)\big)\phi_{\mu,\sigma}(x)dx \\ &= \mathbb{E}\left[g(X)\left(\frac{X^2}{\sigma^3} - \frac{1}{\sigma}\right)\right]. \end{aligned}$$

Then the SF estimator is given by

$$D^{(SF)}[g(X)] = g(X)\left(\frac{X^2}{\sigma^3} - \frac{1}{\sigma}\right). \tag{6}$$

For $\mu = 0$, the variance of the SF estimator is given by

$$\begin{aligned} \mathrm{V}[D^{(SF)}[g(X)]] \;=\; & \frac{1}{\sigma^6}\left[\mathbb{E}[g^2(x)X^4] - \mathbb{E}^2[g(X)X^2]\right] - \frac{2}{\sigma^4}\left[\mathbb{E}[g^2(X)X^2]\right. \\ & \left.-\mathbb{E}[g(X)X^2]\mathbb{E}[g(X)]\right] + \frac{1}{\sigma^2}\left[\mathbb{E}[g^2(X)] - \mathbb{E}^2[g(X)]\right]. \end{aligned}$$

The MV derivative estimator is another distributional approach but instead considers the derivative of a measure as a finite signed measure. Pflug [12], introduced this approach for continuous and bounded functions. This technique was further generalized to $L^1(\mathbb{P})$ integrable function by Heidergott and Vazquez-Abad [9]. Differentiating from $\phi_{\mu,\sigma}(x)$ w.r.t σ gives

$$\begin{aligned} \frac{\partial}{\partial\sigma}\phi_{\mu,\sigma}(x) &= -\frac{1}{\sqrt{2\pi\sigma^2}}e^{(-\frac{(x-\mu)^2}{2\sigma^2})} + \frac{1}{\sqrt{2\pi}\sigma}\left(\frac{(x-\mu)^2}{\sigma^3}e^{(-\frac{(x-\mu)^2}{2\sigma^2})}\right) \\ &= \frac{1}{\sigma}[\mathbf{m}_{\mu,\sigma}(x) - \phi_{\mu,\sigma}(x)], \end{aligned} \tag{7}$$

where the p.d.f $\mathbf{m}_{\mu,\sigma}(.)$ is defined as the Double-Maxwell distribution with parameters (μ, σ^2), i.e. $X^+ \sim \mathcal{DM}(\mu, \sigma)$ with density function

$$\mathbf{m}_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\sigma^3}(x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

It follows from (7) that

$$\begin{aligned} \frac{d}{d\sigma}\mathbb{E}[g(X)] \;&=\; \frac{1}{\sigma}\left(\int_{-\infty}^{\infty} g(x)\mathbf{m}_{\mu,\sigma}(x)dx - \int_{-\infty}^{\infty} g(x)\phi_{\mu,\sigma}(x)dx\right) \\ &=\; \frac{1}{\sigma}(\mathbb{E}[g(X^+)] - \mathbb{E}[g(X)]), \end{aligned}$$

which is satisfied whenever $\sup_{\sigma\in U}\mathbb{E}[|g(X^+)|] < \infty$. Thus the MV derivative estimator is defined as

$$D^{(MVD)}[g(X)] = \frac{1}{\sigma}(g(X^+) - g(X)). \tag{8}$$

The transformation $X^+ = \mu + \sigma Z^+$ holds for the Double-Maxwell distribution where $Z^+ \sim \mathcal{DM}(0,1)$. The variance of the MVD estimator, assuming X and X^+ are independent is given as

$$\mathrm{V}ar(D^{(MVD)}) = \frac{1}{\sigma^2}\left[\mathbb{E}[g^2(X^+)] - \mathbb{E}^2[g(X^+)] + \mathbb{E}[g^2(X)] - \mathbb{E}^2[g(X)]\right]. \tag{9}$$

2.2 New estimator

In this part, a new derivative estimator will be introduced using a relevant combination of both SF and MV derivative estimators construction methodologies. Let $X \sim \mathcal{N}(\mu, \sigma^2)$, by definition

$$
\begin{aligned}
\frac{\partial}{\partial \sigma}\mathbb{E}[g(X)] &= \frac{\partial}{\partial \sigma}\int g(x)\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}}dx \\
&= \int g(x)\frac{\partial}{\partial \sigma}\left(\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}}\right)dx \\
&= \int g(x)\left(\frac{\partial}{\partial \sigma}\sigma^2 \times \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}} \times \frac{1}{\sigma^2}\right)dx.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\frac{\partial}{\partial \sigma}\left(\sigma^2 \times \frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}} \times \frac{1}{\sigma^2}\right) \\
&\quad = 2\sigma\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}} \times \frac{1}{\sigma^2} + \sigma^2\frac{\partial}{\partial \sigma}\left(\frac{1}{\sqrt{2\pi\sigma^3}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}\right) \\
&\quad = 2\sigma\frac{1}{\sqrt{2\pi\sigma^3}}e^{\frac{-(x-\mu)^2}{2\sigma^2}} + \sigma^2\left(\frac{-3}{\sigma} + \frac{(x-\mu)^2}{\sigma^3}\right)\frac{1}{\sqrt{2\pi\sigma^3}}e^{\frac{-(x-\mu)^2}{2\sigma^2}} \\
&\quad = \left(\frac{-\sigma}{(x-\mu)^2} + \frac{1}{\sigma}\right)\frac{(x-\mu)^2}{\sqrt{2\pi\sigma^3}}e^{\frac{-(x-\mu)^2}{2\sigma^2}} \\
&\quad = \left(\frac{-\sigma}{(x-\mu)^2} + \frac{1}{\sigma}\right)\mathbf{m}_{\mu,\sigma},
\end{aligned} \tag{10}
$$

where $\frac{\partial}{\partial \sigma}\frac{1}{\sqrt{2\pi\sigma^3}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ is calculated as

$$
\begin{aligned}
\frac{\partial}{\partial \sigma}\left(\frac{1}{\sqrt{2\pi\sigma^3}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}\right) &= \frac{\partial}{\partial \sigma}\ln\left(\frac{1}{\sqrt{2\pi\sigma^3}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}\right) \times \frac{1}{\sqrt{2\pi\sigma^3}}e^{\frac{-(x-\mu)^2}{2\sigma}} \\
&= \frac{\partial}{\partial \sigma}\left(-3\ln\sigma - \frac{(x-\mu)^2}{2\sigma^2}\right)\frac{1}{\sqrt{2\pi\sigma^3}}e^{\frac{-(x-\mu)^2}{2\sigma^2}} \\
&= \left(\frac{-3}{\sigma} + \frac{(x-\mu)^2}{\sigma^3}\right)\frac{1}{\sqrt{2\pi\sigma^3}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}.
\end{aligned} \tag{11}
$$

Then we define the mixed two-staged (MTS) derivative estimator as

$$
D^{(MTS)}[g(X)] = g(X^+)\frac{1}{\sigma}\left[1 - \frac{1}{Z^{+2}}\right]. \tag{12}
$$

Consider that in deriving (12), we proceeded in the same fashion as in MV method, however expression (11) was followed according to the SF.

In conclusion we mixed the two methods to get a new construction procedure. Assuming $\mu = 0$, after some algebra, the variance of MTS derivative estimator is given by

$$\mathrm{V}ar(D^{(MTS)}) = \frac{1}{\sigma^2}\mathbb{E}\left((1-\frac{1}{Z^{+2}})^2 g^2(\sigma Z^{+2})\right) - \mathbb{E}\left((1-\frac{1}{Z^{+2}})g(\sigma Z^{+2})\right).$$

3 Comparison study

From the general expressions obtained in the previous section for the sensitivity estimators, we now compare their performance using the derived variances for polynomial costs. The variances of the various proposed derivative estimators are provided for odd and even p in Tables 1 & 2 respectively.

Table 1: Variance of the estimators for add p.

Estimator	Variance
IPA	$\sigma^{2p-2}p^2(2p-1)!!$
SF	$\sigma^{2p-2}[(2p+1)^2+1](2p-1)!!$
MVD	$\sigma^{2p-2}(2p+2)(2p-1)!!$
MTS	$\sigma^{2p-2}(2p-3)!!(4p^2-4p+2)$

Table 2: Variance of the estimators for even p.

Estimator	Variance
IPA	$\sigma^{2p-2}[p^2(2p-1)!!-p^2[(p-1)!!]^2]$
SF	$\sigma^{2p-2}[[(2p+1)^2+1](2p-1)!!-P^2[(P-1)!!]^2]$
MVD	$\sigma^{2p-2}[[(2p+2)(2p-1)!!-[(P+1)^2+1][(P-1)!!]^2]$
MTS	$\sigma^{2P-2}((2P-3)!!(4P^2-4P+2)-(P-1)!!^2P^2)$

To illustrate the behavior of of different estimators numerically, Table 3 gives their exact variances based on σ for p from 1 to 6.

The precise ordering statement between the exhibited estimators, is provided in the following result.

Proposition 1. *Let $X \sim \mathcal{N}(0, \sigma^2)$. For the cost function $g(x) = x^p$, with $p \in \mathbb{N}$, the following inequalities hold:*

$$\begin{aligned} &Var(D^{IPA}) < Var(D^{SM}) < Var(D^{MVD}) < Var(D^{SF}), && p = 0, 1, \\ &Var(D^{MTS}) \le Var(D^{MVD}) \le Var(D^{IPA}) \le Var(D^{SF}), && p \ge 2. \end{aligned}$$

Proof. Let p be odd. For $p > 1$ we have

$$\sigma^{2-2p}[Var(D^{MVD}) - Var(D^{SM})] = (2p-3)!!(6p-4) \geq 0.$$

Thus $Var(D^{MTS}) \leq Var(D^{MVD})$. The proof of the rest is similar and therefore omitted. □

Table 3: Variance of the estimators for simple cost function.

MTS	MVD	SF	IPA	p
2	4	10	1	1
$6\sigma^2$	$8\sigma^2$	$74\sigma^2$	$8\sigma^2$	2
$78\sigma^4$	$120\sigma^4$	$750\sigma^4$	$135\sigma^4$	3
$606\sigma^6$	$816\sigma^6$	$8466\sigma^6$	$1536\sigma^6$	4
$8610\sigma^8$	$14940\sigma^8$	$151890\sigma^8$	$31125\sigma^8$	5
$143790\sigma^{10}$	$180480\sigma^{10}$	$2320050\sigma^{10}$	$484920\sigma^{10}$	6

From Proposition 1, it can be immediately deduced that the new estimator, namely mixed two-stage derivative estimator performs better than the other three types, for the polynomial cost function whenever $p \geq 2$.

4 Multivariate case

Consider the Gaussian system, whose elements are driven by the variables $(X_1, \ldots, X_n)$ under the law $X_i \sim \mathcal{N}(0, \sigma^2)$. We are interested in estimating the sensitivity $\frac{d}{d\sigma}\mathbb{E}[g(X_1, \ldots, X_n)]$. Using the product rule, similar to the proposal of Heidergott et al. [10], the IPA estimator is defined by

$$D^{(IPA)} = \sum_{i=1}^{n} Z_i \frac{\partial}{\partial x_i} g(\sigma Z_1, \ldots, \sigma Z_n). \tag{13}$$

For the SF estimator, assuming the integrability condition

$$\mathbb{E}\big[|g(X)| \sup_{\hat{\sigma}\in U} \big| \sum_{k=1}^{n} \frac{d}{d\sigma} \log \phi_{0,\sigma}(X_k) \big|\big] < \infty, \tag{14}$$

we have that

$$D^{(SF)} = g(X) \sum_{k=1}^{n} \big(\frac{X_k^2}{\sigma^3} - \frac{1}{\sigma}\big). \tag{15}$$

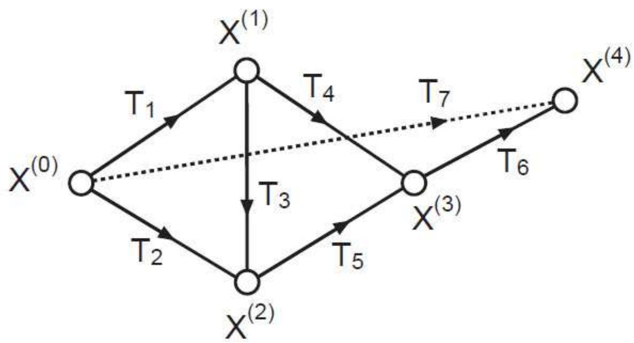

Figure 1: Diagram of the stochastic activity network (SAN).

To obtain the MV derivative estimator, Let X_n^+, $n = 1, \dots, n$ be iid with double-sided Maxwell density. For each $k = 1, \dots, n$, define

$$X_n^+(k) = \begin{cases} X_n & n \neq k, \\ X_k^+ & n = k, \end{cases}$$

then the multidimensional phantom MV estimator is as follows

$$D^{(WD)} = \frac{1}{\sigma} \sum_{k=1}^{n} [g(X_1^+(k), \dots, g(X_n^+(k)) - g(X_1, \dots, X_n)]. \tag{16}$$

Finally, the mulltidimensional MTS derivative estimator is defined as follows

$$D^{(MTS)} = \frac{1}{\sigma} \sum_{k=1}^{n} \left[g(X_1^+(k), \dots, X_n^+(k)) \left(1 - \frac{1}{Z^{+2}}\right)\right]. \tag{17}$$

4.1 Application to stochastic activity networks

A simple example, which is inspired by Fu [3] and also nicely used by Heidergott et al. [10], is depicted in Figure 1. The stochastic activity network under consideration is consisting of seven arcs $T = (T_1 \dots, T_7)$ connected to four nodes $X = (X^{(0)}, \dots, X^{(4)})$. T represents the completion time for each activity. The solid arcs, T_j, are iid $\mathcal{N}(t_i, \sigma^2)$. The dashed arc, T_7, is deterministic with value $t_7 > 0$. Similar to Heidergott et al. [10] we are interested in the sensitivities of the first two moments of completion times $\mathbb{E}[\tau]$ and $\mathbb{E}[\tau^2]$ with respect to σ. For each path, the completion time is additive, so that $\tau(\pi) = \sum_{n\in\pi} T_n$, where the paths are labeled by

$$\pi_1 = (1,4,6), \qquad \pi_2 = (1,3,5,6), \qquad \pi_3 = (2,5,6), \qquad \pi_4 = (7).$$

Specifically, the completion time is given by

$$\tau = \max(\max(T_1 + T_4, T_1 + T_3 + T_5, T_2 + T_5) + T_6, T_7). \tag{18}$$

Then we have

$$D^{(IPA)}(\tau) = Z_6 + \begin{cases} Z_1 + Z_4, & \pi^* = \pi_1, \\ Z_1 + Z_3 + Z_5, & \pi^* = \pi_2, \\ Z_2 + Z_5, & \pi^* = \pi_3, \\ 0, & \pi^* = \pi_4, \end{cases}$$

$$D^{(IPA)}(\tau^2) = 2\tau D^{(IPA)}(\tau).$$

The score function estimator is given by

$$D^{(SF)}(\tau) = \tau \sum_{i=1}^{6} (\frac{X_i^2}{\sigma^3} - \frac{1}{\sigma}) = \tau \sum_{i=1}^{6} \frac{Z_i^2 - 1}{\sigma},$$

$$D^{(SF)}(\tau^2) = \tau^2 \sum_{i=1}^{6} (\frac{X_i^2}{\sigma^3} - \frac{1}{\sigma}).$$

Also we have

$$\begin{aligned} D^{(MV)}(\tau) = \tfrac{1}{\sigma} \textstyle\sum_{i=1}^{6} \Big(& \max\big(\max(T_1^{(k)} + T_4^{(k)}, T_1^{(k)} + T_3^{(k)} + T_5^{(k)}) + T_6^{(k)}, \\ & T_7\big) - \max\big(\max(T_1^{+} + T_4^{+}, T_1^{+} + T_3^{+} + T_5^{+}, T_2^{+} + T_5^{+}) + T_6^{+}, \\ & T_7)\Big), \end{aligned}$$

$$\begin{aligned} D^{(MV)}(\tau^2) = \tfrac{1}{\sigma} \textstyle\sum_{i=1}^{6} \Big(& \Big[\max\big(\max(T_1^{(k)} + T_4^{(k)}, T_1^{(k)} + T_3^{(k)} + T_5^{(k)}) + T_6^{(k)}, \\ & T_7\big)\Big]^2 - \Big[\max\big(\max(T_1^{+} + T_4^{+}, T_1^{+} + T_3^{+} + T_5^{+}, T_2^{+} + T_5^{+}) \\ & + T_6^{+}, T_7\big)\Big]^2\Big). \end{aligned}$$

Moreover, the MTS estimator has the form

$$\begin{aligned} D^{(MTS)}(\tau) = \tfrac{1}{\sigma} \textstyle\sum_{i=1}^{6} & \max\big(\max(T_1^{(k)} + T_4^{(k)}, T_1^{(k)} + T_3^{(k)} + T_5^{(k)}) \\ & + T_6^{(k)}, T_7\big)\Big(1 - \tfrac{1}{Z_i^2}\Big), \end{aligned}$$

$$D^{(MTS)}(\tau^2) = \frac{1}{\sigma}\sum_{i=1}^{6}\Big[\max\big(\max(T_1^{(k)}+T_4^{(k)}, T_1^{(k)}+T_3^{(k)}+T_5^{(k)}) +T_6^{(k)}, T_7\big)\Big]^2\Big(1-\frac{1}{Z_i^2}\Big).$$

We performed $N = 10^5$ independent simulations with $\sigma = 1$, and $(t_1,\dots,t_6) = (6,8,5,8,4,7)$ to calculate the estimates for $\mathbb{E}\frac{d\tau}{d\sigma}$ and$\mathbb{E}\frac{d\tau^2}{d\sigma}$. The obtained results are summarized in Tables 4-6. In these tables, $\mathbb{E}(d\tau_1^j/d\sigma) = D^{(M)}(\tau_1^j)$, $j = 1,2$ for $M = \text{IPA}, \text{SF}, \text{WD}, \text{MTS}$. In all cases (different T_7 values) the variance of SF is considerably larger than the variance of the other three estimators. As T_7 gets larger the variances are increased. Although overall, the WD estimation method presents the advantages over others, but the new estimator is better than two others and in general easier to compute. Note that the WD estimation method needs simulating from double-sided Maxwell random variables. Finally, as it is evident from Table 3, as σ gets larger the new estimator is preferred among others.

Table 4: Results for $T_7 = 21$.

Estimator	$D^{(M)}(\tau_1)$	$\mathbb{V}ar\left(D^{(M)}(\tau_1)\right)$	$D^{(M)}(\tau_1^2)$	$Var\frac{d\tau_1^2}{d\sigma}$
IPA	$1.0072028e+000$	$1.7688103e+000$	$4.9084149e+001$	$4.4482945e+003$
SF	$1.3247921e+000$	$6.3264932e+003$	$5.6897842e+001$	$3.4168694e+006$
WD	$1.0042531e+000$	$2.1301063e+000$	$4.8918532e+001$	$4.8284316e+003$
MTS	$3.8228814e+0001$	$.1996669e+005$	$3.8306191e+000$	$1.1997262e+005$
$\mathbb{E}(\tau_1) = 22.55$ $\quad Var(\tau_1) = 2.19$ $\quad p = 0.2359$				

Table 5: Results for $T_7 = 23$.

Estimator	$D^{(M)}(\tau_1)$	$\mathbb{V}ar\left(D^{(M)}(\tau_1)\right)$	$D^{(M)}(\tau_1^2)$	$\mathbb{V}ar\left(D^{(M)}(\tau_1^2)\right)$
IPA	$8.4654301e-001$	$1.7444848e+000$	$4.1768305e+001$	$4.4746775e+003$
SF	$1.1637515e+000$	$6.7803099e+003$	$4.9565389e+001$	$3.8635188e+006$
WD	$8.4212505e-001$	$1.5042224e+000$	$4.1529196e+001$	$3.7018445e+003$
MTS	$3.7736040e+000$	$1.3039099e+005$	$3.7820487e+000$	$1.3039806e+005$
$\mathbb{E}(\tau_1) = 23.43$ $\quad Var(\tau_1) = 0.72$ $\quad p = 0.6555$				

Table 6: Results for $T_7 = 25$.

Estimator	$D^{(M)}(\tau_1)$	$\mathbb{V}ar\left(D^{(M)}(\tau_1)\right)$	$D^{(M)}(\tau_1^2)$	$\mathbb{V}ar\left(D^{(M)}(\tau_1^2)\right)$
IPA	$2.8802098e-001$	$1.1015529e+000$	$1.4983166e+001$	$3.0284490e+003$
SF	$6.1557906e-001$	$7.6545519e+003$	$2.3286288e+001$	$4.8775811e+006$
WD	$2.8378617e-001$	$5.7967241e-001$	$1.4759672e+001$	$1.5904695e+003$
MTS	$3.4100394e+000$	$1.5117733e+005$	$3.4200166e+000$	$1.5118722e+005$
$\mathbb{E}(\tau_1) = 25.06$ $\quad Var(\tau_1) = 0.09$ $\quad p = 0.9271$				

5 Conclusion

In this paper, we briefly reviewed construction methodologies behind three estimators measuring system sensitivity w.r.t scale parameter. A new estimator, namely mixed two-stage derivative estimator constructed and demonstrated that performs better than the other three types, for the polynomial cost function whenever $p \geq 2$. Extension to multivariate case also considered.

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