Journal of Mathematical Modeling
Vol. 4, No. 2, 2016, pp. 161-169

JMM

An efficient nonstandard numerical method with positivity preserving property

Mohammad Mehdizadeh Khalsaraei* **and Reza Shokri Jahandizi**

Department of Mathematics, Faculty of Science, University of Maragheh
Maragheh, Iran
Emails: Muhammad.mehdizadeh@gmail.com, reza.shokri.j@gmail.com

Abstract. Classical explicit finite difference schemes are unsuitable for the solution of the famous Black-Scholes partial differential equation, since they impose severe restrictions on the time step. Furthermore, they may produce spurious oscillations in the solution. We propose a new scheme that is free of spurious oscillations and guarantees the positivity of the solution for arbitrary stepsizes. The proposed method is constructed based on a nonstandard discretization of the spatial derivatives and is applicable to Black-Scholes equation in the presence of discontinues initial conditions.

Keywords: positivity preserving, nonstandard finite differences, Black-Scholes equation.
AMS Subject Classification: 65M06, 65M12.

1 Introduction

Mathematical finance is a field of applied mathematics, concerned with financial markets. In the market of financial derivatives, options, futures and forward contracts are extremely useful tools. The simplest option gives the holder the right, but not the obligation to buy or sell an underlying asset at a specific price on or before a certain date at a fixed strike price. The simplest types of option (called European options) come in two main brands

*Corresponding author.
Received: 29 February 2016 / Revised: 4 October 2016 / Accepted: 13 October 2016.

calls and puts. European option can be exercised only at the expiry date T. For American option exercise is permitted at any time $t \leq T$. Option pricing has become increasingly important in the finance world. Black and Scholes published their germinal work on option pricing in [3]. Since double barrier options are becoming more and more popular, they are not considered to be complicated anymore. The application of the nonstandard finite difference method and investigation of its positivity preserving and smoothing properties for pricing European call option with discrete double barrier are the subject of this paper. We concentrate on a double barrier knock out option with discrete monitoring, which satisfies the Black-Scholes pricing partial differential equation:

$$-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \tag{1}$$

in which S is the asset price and $V(S,t)$ is the price of the option and endowed with initial and boundary conditions:

$$V(S,0) = \max(S-K,0)1_{[L,U]}(S),$$

$$V(S,t) \to 0 \quad \text{as} \quad S \to 0 \quad \text{or} \quad S \to \infty,$$

with updating of the initial condition at the monitoring dates t_i, $i = 1, \ldots, F$:

$$V(S,t_i) = V(S,t_i^-)1_{[L,U]}(S), \quad 0 = t_0 < t_1 < \cdots < t_F = T,$$

where $1_{[L,U]}(S)$ is the indicator function, i.e.,

$$1_{[L,U]}(S) = \begin{cases} 1, & \text{if } S \in [L,U], \\ 0, & \text{if } S \notin [L,U]. \end{cases} \tag{2}$$

Moreover K is the exercise price, T is the maturity, the parameter $r > 0$ is the interest rate and the reference volatility is $\sigma > 0$. Notice that such an option has a payoff condition equal to $\max(S-K,0)$ but the option expires worthless if the maturity of the asset price S has fallen outside the corridor $[L,U]$ at the prefixed monitoring dates. On the other hand the knock-out clause at the monitoring date introduces a discontinuity at the barriers set at $S = L$ and $S = U$ respectively. For more details see [2, 8, 15].

Explicit numerical methods based on the standard finite difference approach are consistent with the original differential equation and guarantee convergence of the discrete solution to the exact one, but they impose severe restrictions on the time step and in the presence of discontinuous payoff and low volatility, essential qualitative properties of the solution are not transferred to the numerical solution [15–18]. Spurious oscillations and negative values might be occurred in the solution. For example by using forward difference for $\frac{\partial V}{\partial t}$ and centered difference for discretization of $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$ and approximations V_j^n of V at the grid points $(j\Delta S, n\Delta t)$ for every $j = 1, \ldots, N$ and $n = 0, 1, \ldots, M$, where $N = S_{\max}/\Delta S$, $M = T/\Delta t$ and $[0, S_{\max}] \times [0, T]$ is the computational domain, an explicit finite difference method take the form:

$$-\frac{V_j^{n+1} - V_j^n}{\Delta t} + rS\frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S} + \frac{1}{2}\sigma^2 S^2 \frac{V_{j-1}^n - 2V_j^n + V_{j+1}^n}{\Delta S^2} - rV_j^n = 0. \quad (3)$$

Comparing with the analytical solution $V(T, S) = \exp(-rT)(1 - N(d_2))$, with $N(\cdot)$ standard normal cumulative distribution and

$$d_2 = \frac{\log(S/k) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},$$

we observe that method (3) has lower accuracy and often generates numerical drawbacks such as spurious oscillations and negative values in the solution applied to (1), whenever the financial parameters of the Black-Scholes model σ and r satisfy the relationship $\sigma^2 \ll r$, see Figure 1. In the case of larger time steps, we see the same behavior, see Figure 2. The analytical solution and the values of the parameters used in our simulation are taken from [15].

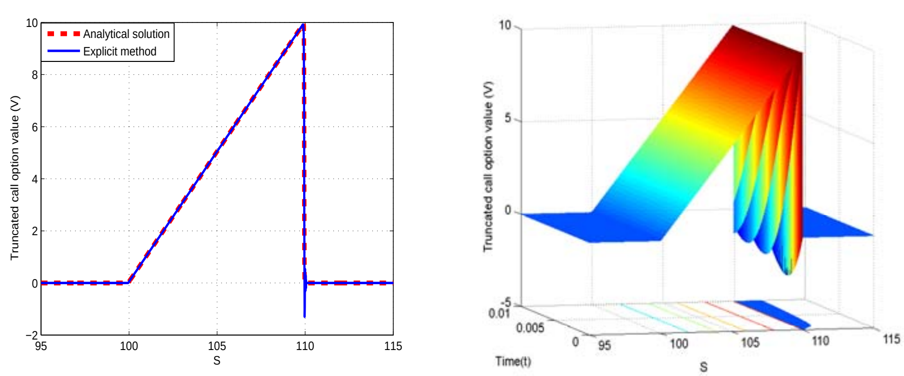

Figure 1: Truncated call option value for the explicit method with $\Delta S = 0.01$, $\Delta t = 10^{-6}$. parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 0.01$, $S_{max} = 120$.

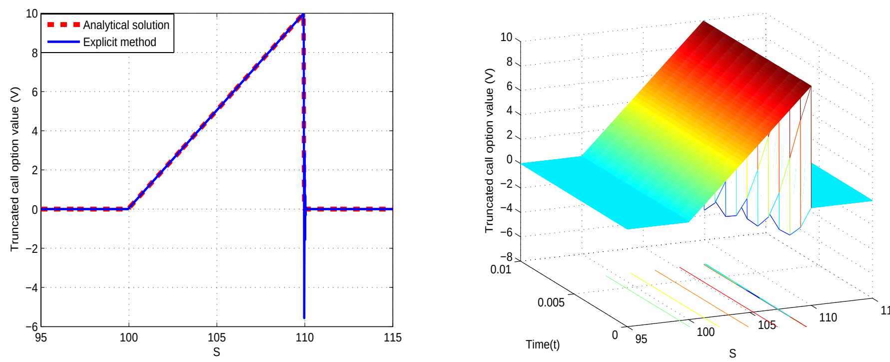

Figure 2: Truncated call option value for the explicit method with $\Delta S = 0.01$, $\Delta t = 10^{-3}$. parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 0.01$, $S_{max} = 120$.

2 Scheme construction

In this section by using the strategy of nonstandard discretization methods (using values at different time levels for discretization of $\frac{\partial V}{\partial S}$, $\frac{\partial^2 V}{\partial S^2}$ and approximation V_j^{n+1} of V, see [4–10, 12–14]) we propose our new scheme as:

$$-\frac{V_j^{n+1}-V_j^n}{\Delta t}+rS_j\frac{V_{j+1}^n-V_j^{n+1}}{\Delta S}+\frac{1}{2}\sigma^2S_j^2\frac{V_{j-1}^n-2V_j^{n+1}+V_{j+1}^n}{\Delta S^2}-rV_j^{n+1}=0, \tag{4}$$

The explicit form of (4) is:

$$V_j^{n+1}=\frac{\frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2V_{j-1}^n+\frac{1}{\Delta t}V_j^n+\left(\frac{rS_j}{\Delta S}+\frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2\right)V_{j+1}^n}{\frac{1}{\Delta t}+\frac{rS_j}{\Delta S}+(\frac{\sigma S_j}{\Delta S})^2+r}. \tag{5}$$

Similar to the method in [8,11,17], the new proposed scheme performs well for larger time steps but the main advantage of it is unconditional positivity and stability (these results will be discussed in following).

Proposition 1. *Assuming V_{j-1}^n, V_j^n and V_{j+1}^n are nonnegative real numbers, then (5) provides a nonnegative approximation V_j^{n+1} to the solution of (1).*

Theorem 1. *The new proposed scheme is unconditionally stable and convergent with local truncation error $\mathcal{O}(\Delta t, \Delta S^2)$.*

Proof. Using the Fourier stability method [1] put

$$V_j^n=e^{\alpha n\Delta t}e^{i\beta j\Delta S}, \tag{6}$$

where $i = \sqrt{(-1)}$, and β is an arbitrary real parameter. By substituting (6) into (5) we obtain

$$e^{\alpha(n+1)\Delta t}e^{i\beta j\Delta S} = \frac{\frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2 e^{\alpha n\Delta t}e^{i\beta(j-1)\Delta S} + \frac{1}{\Delta t}e^{\alpha n\Delta t}e^{i\beta j\Delta S} + \big(\frac{rS_j}{\Delta S} + \frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2\big)e^{\alpha n\Delta t}e^{i\beta(j+1)\Delta S}}{\frac{1}{\Delta t} + \frac{rS_j}{\Delta S} + (\frac{\sigma S_j}{\Delta S})^2 + r}, \tag{7}$$

and division by $e^{\alpha n\Delta t}e^{i\beta j\Delta S}$ leads to

$$e^{\alpha\Delta t} = \frac{\frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2 e^{-i\beta\Delta S} + \frac{1}{\Delta t} + \big(\frac{rS_j}{\Delta S} + \frac{1}{2}(\frac{\sigma S_j}{\Delta S})^2\big)e^{i\beta\Delta S}}{\frac{1}{\Delta t} + \frac{rS_j}{\Delta S} + (\frac{\sigma S_j}{\Delta S})^2 + r}. \tag{8}$$

By taking the real part, it is seen that the absolute value of the amplification factor $e^{\alpha\Delta t} \leq 1$. Therefore the scheme is stable and convergent with local truncation error:

$$T_j^n = -\frac{V(S_j,t_{n+1}) - V(S_j,t_n)}{\Delta t} + rS_j\frac{V(S_{j+1},t_n) - V(S_j,t_{n+1})}{\Delta S}$$
$$+\frac{1}{2}\sigma^2 S_j^2\frac{V(S_{j-1},t_n) - 2V(S_j,t_{n+1}) + V(S_{j+1},t_n)}{\Delta S^2} - rV(S_j,t_{n+1}). \tag{9}$$

By Taylor's expansion, we have

$$V(S_j,t_{n+1}) = V(S_j,t_n) + \Delta t\left(\frac{\partial V}{\partial t}\right)_j^n + \frac{\Delta t^2}{2}\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n + \frac{\Delta t^3}{6}\left(\frac{\partial^3 V}{\partial t^3}\right)_j^n + \cdots,$$

$$V(S_{j+1},t_n) = V(S_j,t_n) + \Delta S\left(\frac{\partial V}{\partial S}\right)_j^n + \frac{\Delta S^2}{2}\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n + \frac{\Delta S^3}{6}\left(\frac{\partial^3 V}{\partial S^3}\right)_j^n + \cdots,$$

$$V(S_{j-1},t_n) = V(S_j,t_n) - \Delta S\left(\frac{\partial V}{\partial S}\right)_j^n + \frac{\Delta S^2}{2}\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n - \frac{\Delta S^3}{6}\left(\frac{\partial^3 V}{\partial S^3}\right)_j^n + \cdots,$$

substituting these equations into the expression for T_j^n then gives

$$\begin{aligned} T_j^n =& \left(-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV\right)_j^n \\ & - \left(rj\Delta t + \sigma^2 j^2\Delta t + r\Delta t\right)\left(\frac{\partial V}{\partial t}\right)_j^n + \frac{1}{2}rj\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n \\ & - \left(\frac{1}{2}\Delta t + \frac{1}{2}rj^2\Delta t^2 + \frac{1}{2}\sigma^2 j^2\Delta t^2 + \frac{1}{2}r\Delta t^2\right)\left(\frac{\partial^2 V}{\partial t^2}\right)_j^n + \cdots. \end{aligned} \tag{10}$$

But V is the solution of the Black-Scholes equation so

$$\left(-\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}-rV\right)_j^n=0.$$

Therefore the principle part of the local truncation error is

$$-(rj\Delta t+\sigma^2j^2\Delta t+r\Delta t)\left(\frac{\partial V}{\partial t}\right)_j^n+\frac{1}{2}rj\Delta S^2\left(\frac{\partial^2 V}{\partial S^2}\right)_j^n.$$

Hence

$$T_j^n=\mathcal{O}(\Delta t+\Delta S^2).$$

□

3 Numerical Results

To illustrate the advantage of the designed new positive explicit scheme, we again consider the Black-Scholes equation (1) with $\sigma^2 \ll r$. Proposed nonstandard scheme provides an accurate spurious oscillation free solution and is positivity preserving, see Figure 3.

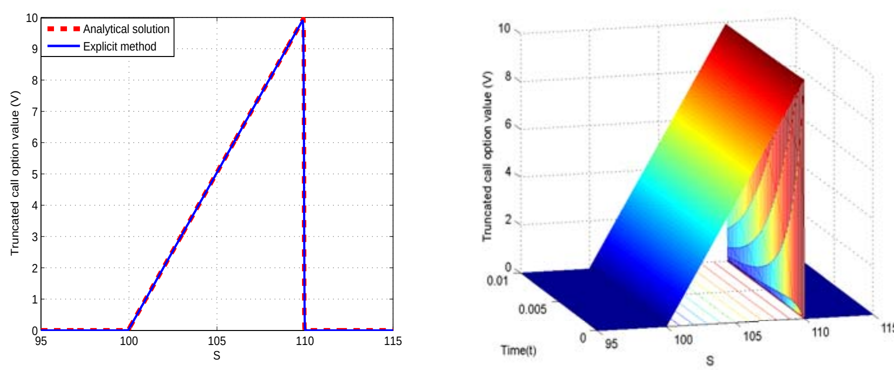

Figure 3: Truncated call option value for nonstandard explicit method with $\Delta s = 0.01$, $\Delta t = 10^{-6}$. parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 0.01$, $S_{max} = 120$.

As we can see in Figure 4, similar behavior is observed when (5) is used with larger time steps.

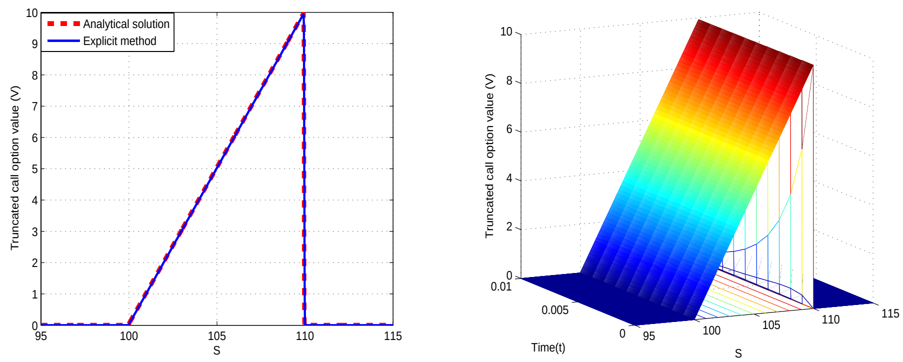

Figure 4: Truncated call option value for nonstandard explicit method with $\Delta s = 0.01$, $\Delta t = 10^{-3}$. parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 0.01$, $S_{max} = 120$.

4 Conclusions and discussion

Within the strategy of the nonstandard discretization of spatial derivatives, we have presented an alternative scheme to the classical explicit ones that prevents the occurrence of spurious oscillations and negative values where even very small negative values are unacceptable. The main advantages of the new scheme is that in addition to the unconditionally positivity property, it performs well for larger time steps. Future works will include extending the new scheme to the nonlinear Black-Scholes equation.

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