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Numerical solution of system of linear integral equations via improvement of block-pulse functions

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Abstract. In this article, a numerical method based on improvement of block-pulse functions (IBPFs) is discussed for solving the system of linear Volterra and Fredholm integral equations. By using IBPFs and their operational matrix of integration, such systems can be reduced to a linear system of algebraic equations. An efficient error estimation and associated theorems for the proposed method are also presented. Some examples are given to clarify the efficiency and accuracy of the method.

Keywords: system of linear integral equations, improvement of block-pulse functions, operational matrix, vector forms, error analysis.
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1 Introduction

Systems of linear integral equations and their solutions have great importance in science and engineering. Most physical problems, such as biological applications in population dynamics and genetics where impulses arise naturally or are caused by control, can be modeled by an integral equation or a system of these equations. The systems of integral equations are usually difficult to solve analytically, thus some numerical methods are applied to approximately solve them. Systems of linear Volterra and Fredholm inte-

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gral equations can be appeared in the following form, respectively

$$\mathbf{f}(x) = \mathbf{g}(x) + \int_0^x \mathbf{k}(x,y)\mathbf{f}(y)dy, \quad x \in D = [0,1), \tag{1}$$

and

$$\mathbf{f}(x) = \mathbf{g}(x) + \int_0^1 \mathbf{k}(x,y)\mathbf{f}(y)dy, \quad x \in D, \tag{2}$$

where $\mathbf{f}(x)$ is an unknown function and $\mathbf{g}(x)$ and $\mathbf{k}(x,y)$ are analytical function on D and $D \times D$, respectively, as follows

$$\mathbf{f}(x) = [f_1(x), f_2(x), \ldots, f_m(x)]^T \ , \tag{3}$$

$$\mathbf{g}(x) = [g_1(x), g_2(x), \ldots, g_m(x)]^T \ , \tag{4}$$

$$\mathbf{k}(x,y) = [k_{ij}(x,y)] \, , \quad i,j = 1,2,\ldots,m. \tag{5}$$

Recently, several numerical methods such as the homotopy perturbation method [24, 25], the Lagrange method [23], the modified homotopy perturbation method [8], the rationalized Haar functions method [10, 11], the differential transformation method [1], the Tau method [16], the variational iteration method [19], the Legendre matrix method [22], the Adomian method [5], the Galerkin method [12], the Bessel matrix method [21], and other methods [14–19] have been used for solving integral and integro-differential equations systems.

In this paper, IBPFs are introduced. Also, some theorems are proved for IBPFs method that show the results of this numerical expansions are more precise than the results of block pulse expansions. This functions are disjoint, orthogonal and complete. According to the disjointness of IBPFs, the joint terms will disappear in each subinterval when multiplication, division and some other operations are applied. Also, the orthogonality property of IBPFs will cause that the operational matrix to be a sparse matrix. The completeness of IBPFs guarantees that an arbitrary small mean square error can be obtained for a real bounded function, which has only a finite number of discontinuous point in the interval $x \in [0,1)$, by increasing the number of terms in the improved block pulse series.

The rest of paper is organized as follows: In Section 2, we describe IBPFs and their properties. In Section 3, we apply IBPFs for approximating the solution of system of linear Volterra and Fredholm integral equations. Convergence analysis is discussed in Section 4. Numerical results are given in Section 5 to illustrate the efficiency and accuracy of proposed method. Finally, Section 6 concludes the paper.

2 IBPFs and their properties

2.1 Definition of IBPFs

An $(n+1)$-set of IBPFs consists of $(n+1)$ functions which are defined over district D as follows

$$\phi_0(x) = \begin{cases} 1, & x \in [0, \frac{h}{2}), \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_i(x) = \begin{cases} 1, & x \in [(i-1)h + \frac{h}{2}, ih + \frac{h}{2}), \\ 0, & \text{otherwise}, \end{cases} \qquad i = 1, 2, \ldots, n-1,$$

$$\phi_n(x) = \begin{cases} 1, & x \in [1 - \frac{h}{2}, 1), \\ 0, & \text{otherwise}, \end{cases}$$

where n is an arbitrary positive integer and $h = \frac{1}{n}$.

The IBPFs are disjoint

$$\phi_i(x)\phi_j(x) = \begin{cases} \phi_i(x), & i = j, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

where $i, j = 0, 1, \ldots, n$ and are orthogonal to each other

$$\int_0^1 \phi_i(x)\phi_j(x)dx = \begin{cases} \frac{h}{2}, & i = j \in \{0, n\}, \\ h, & i = j \in \{1, 2, \ldots, n-1\}, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

where $x \in D$.

2.2 Vector forms

Consider the first $(n+1)$ terms of IBPFs and write them concisely as $(n+1)$-vector

$$\Phi_n(x) = [\phi_0(x), \phi_1(x), \ldots, \phi_n(x)]^T, \ x \in D. \tag{8}$$

Eq. (6) implies that

$$\Phi_n(x)\Phi_n^T(x) = \begin{bmatrix} \phi_0(x) & 0 & \ldots & 0 \\ 0 & \phi_1(x) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \phi_n(x) \end{bmatrix} = diag(\Phi_n(x)). \tag{9}$$

Now suppose that X be an $(n+1)$-vector. Hence by using Eq. (9) we obtain

$$\Phi_n(x)\Phi_n^T(x)X = \widetilde{X}\Phi_n(x),$$

where $\widetilde{X} = diag(X)$ is an $(n+1)\times(n+1)$ diagonal matrix.

2.3 Operational matrix

We have

$$\int_0^x \phi_0(y)dy = \begin{cases} x, & x \in [0, \frac{h}{2}), \\ \frac{h}{2}, & \text{otherwise.} \end{cases}$$

Note that $x = \frac{h}{4}$, at mid-point of $[0, \frac{h}{2})$. So we can approximate x, for $x \in [0, \frac{h}{2})$, by $\frac{h}{4}$.

$$\int_0^x \phi_i(x) = \begin{cases} 0, & x \in [0, (i-1)h + \frac{h}{2}), \\ x - (i-1)h - \frac{h}{2}, & x \in [(i-1)h + \frac{h}{2}, ih + \frac{h}{2}), \\ h, & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n-1$. Also, we can approximate $x - (i-1)h - \frac{h}{2}$, for $x \in [(i-1)h + \frac{h}{2}, ih + \frac{h}{2})$, by $\frac{h}{2}$.

$$\int_0^x \phi_n(x) = \begin{cases} x - 1 + \frac{h}{2}, & x \in [1 - \frac{h}{2}, 1), \\ 0 & \text{otherwise.} \end{cases}$$

So, we can approximate $x - 1 + \frac{h}{2}$, for $x \in [1 - \frac{h}{2}, 1)$, by $\frac{h}{4}$. Therefore, the integration of the vector $\Phi_n(x)$ defined in Eq. (8) can be approximated by

$$\int_0^x \Phi_n(y)dy \simeq P1\Phi_n(x), \tag{10}$$

where

$$P1 = \frac{h}{4}\begin{pmatrix} 1 & 2 & 2 & \ldots & 2 & 2 \\ 0 & 2 & 4 & \ldots & 4 & 4 \\ 0 & 0 & 2 & \ldots & 4 & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & 4 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Using Eq. (9), we get

$$\int_0^1 \Phi_n(y)\Phi_n^T(y)dy \simeq P2, \tag{11}$$

where

$$P2 = \frac{h}{2}\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

2.4 IBPFs expansions

A continues function $f(x) \in L^2(D)$ may be expanded by the IBPFs as

$$f(x) \simeq f_n(x) = \sum_{i=0}^{n} f_i\phi_i(x) = F_n^T\Phi_n(x) = \Phi_n^T(x)F_n \ , \tag{12}$$

where F_n is an $(n+1) \times 1$ vector given by

$$F_n = [f_0, f_1, \dots, f_n]^T,$$

and $\Phi_n(x)$ is defined in Eq. (8), and f_i is obtained as

$$f_i = \begin{cases} 2n\int_0^{\frac{h}{2}} f(x)dx, & i = 0, \\ n\int_{(i-1)h+\frac{h}{2}}^{ih+\frac{h}{2}} f(x)dx, & i = 1, 2, \dots, n-1, \\ 2n\int_{1-\frac{h}{2}}^{1} f(x)dx, & i = n. \end{cases} \tag{13}$$

Similarly a function of two variables, $k(x, y) \in L^2(D \times D)$ can be approximated by IBPFs as follows

$$k(x, y) \simeq k_n(x, y) = \Phi_n^T(x)K_n\Phi_n(y), \tag{14}$$

where $\Phi_n(x)$ and $\Phi_n(y)$ are IBPFs vector of dimension $(n+1)$, and $K_n = [k_{ij}]$ is the $(n+1) \times (n+1)$ IBPFs coefficients matrix of $k(x, y)$.

3 Method of solution

3.1 System of linear Volterra integral equations

In this section, we solve system of linear Volterra integral equations of the form Eq. (1) by using IBPFs. We can rewrite Eq. (1) as follows

$$f_i(x) = g_i(x) + \sum_{j=1}^{m}\int_0^x k_{ij}(x, y)f_j(y)dy, \quad i = 1, 2, \dots, m. \tag{15}$$

We now approximate functions $f_i(x), g_i(x)$ and $k_{ij}(x,y),\ i,j=1,2,\ldots,m$, by IBPFs as follows

$$
\begin{aligned}
f_i(x) &\simeq \Phi_n^T(x)F_i,\\
g_i(x) &\simeq \Phi_n^T(x)G_i, \qquad (16)\\
k_{ij}(x,y) &\simeq \Phi_n^T(x)K_{ij}\Phi_n(y),
\end{aligned}
$$

where $i,j=1,2,\ldots,m$, $\Phi_n(x)$ is defined by (8), F_i, G_i and K_{ij} are IBPFs coefficients of $f_i(x), g_i(x)$ and $k_{ij}(x,y)$, respectively.

By substituting Eq. (16) in Eq. (15), we get

$$\Phi_n^T(x)F_i=\Phi_n^T(x)G_i+\sum_{j=1}^{m}\int_0^x \Phi_n^T(x)K_{ij}\Phi_n(y)\Phi_n^T(y)F_j dy, \quad i=1,2,\ldots,m,$$

therefore by Eq. (9), we have

$$\Phi_n^T(x)F_i=\Phi_n^T(x)G_i+\sum_{j=1}^{m}\Phi_n^T(x)K_{ij}\int_0^x diag(\Phi_n(y))dyF_j, \quad i=1,2,\ldots,m.$$

From Eq.(10), we have

$$\Phi_n^T(x)F_i=\Phi_n^T(x)G_i+\sum_{j=1}^{m}\Phi_n^T(x)K_{ij}AF_j, \quad i=1,2,\ldots,m, \qquad (17)$$

where

$$A=\begin{bmatrix} P1_0\Phi_n(x) & 0 & \ldots & 0\\ 0 & P1_1\Phi_n(x) & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & P1_n\Phi_n(x)\end{bmatrix},$$

and $P1_i$ is ith row of $P1$. On the other hand, using (6), we have

$$
\begin{aligned}
\Phi_n^T(x)K_{ij}A &= \Phi_n^T(x)\begin{bmatrix} K_{ij}^{00}P1_0\Phi_n(x) & K_{ij}^{01}P1_1\Phi_n(x) & \ldots & K_{ij}^{0n}P1_n\Phi_n(x)\\ K_{ij}^{10}P1_0\Phi_n(x) & K_{ij}^{11}P1_1\Phi_n(x) & \ldots & K_{ij}^{1n}P1_n\Phi_n(x)\\ \vdots & \vdots & \ddots & \vdots\\ K_{ij}^{n0}P1_0\Phi_n(x) & K_{ij}^{n1}P1_1\Phi_n(x) & \ldots & K_{ij}^{nn}P1_n\Phi_n(x)\end{bmatrix}\\
&= \begin{bmatrix} \sum_{l=0}^{n}K_{ij}^{l0}P1_{0l}\phi_l(x) & \sum_{l=0}^{n}K_{ij}^{l1}P1_{1l}\phi_l(x) & \ldots & \sum_{l=0}^{n}K_{ij}^{ln}P1_{nl}\phi_l(x)\end{bmatrix}\\
&= \Phi_n^T(x)B_{ij},
\end{aligned}
$$

where

$$B_{ij} = \begin{bmatrix} K_{ij}^{00} P1_{00} & K_{ij}^{01} P1_{10} & \dots & K_{ij}^{0n} P1_{n0} \\ K_{ij}^{10} P1_{01} & K_{ij}^{11} P1_{11} & \dots & K_{ij}^{1n} P1_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ K_{ij}^{n0} P1_{0n} & K_{ij}^{n1} P1_{1n} & \dots & K_{ij}^{nn} P1_{nn} \end{bmatrix}.$$

So, we can rewrite Eq. (17) as follows

$$\Phi_n^T(x) F_i = \Phi_n^T(x) G_i + \sum_{j=1}^{m} \Phi_n^T(x) B_{ij} F_j, \quad i = 1, 2, \dots, m,$$

or

$$F_i = G_i + \sum_{j=1}^{m} B_{ij} F_j, \quad i = 1, 2, \dots, m. \tag{18}$$

Let

$$\mathbf{F} = [F_1, F_2, \dots, F_m]^T,$$

$$\mathbf{G} = [G_1, G_2, \dots, G_m]^T, \tag{19}$$

$$\mathbf{B} = [B_{ij}], \quad i, j = 1, 2, \dots, m\ .$$

Therefore, Eq. (18) can be written as

$$F = G + BF.$$

After solving above linear system, we can find F and accordingly find F_i, $i = 1, 2, \dots, m$, so

$$f_i(x) \simeq \Phi_n^T(x) F_i.$$

3.2 System of linear Fredholm integral equations

In this section, we solve system of linear Fredholm integral equations of the form Eq. (2) by using IBPFs. We can rewrite Eq. (2) as follows

$$f_i(x) = g_i(x) + \sum_{j=1}^{m} \int_0^1 k_{ij}(x, y) f_j(y) dy, \quad i = 1, 2, \dots, m. \tag{20}$$

Substituting Eq. (16) in Eq. (20), we get

$$\Phi_n^T(x) F_i = \Phi_n^T(x) G_i + \sum_{j=1}^{m} \int_0^1 \Phi_n^T(x) K_{ij} \Phi_n(y) \Phi_n^T(y) F_j dy, \quad i = 1, 2, \dots, m.$$

Using Eq. (11), we get

$$\Phi_n^T(x)F_i = \Phi_n^T(x)G_i + \sum_{j=1}^{m} \Phi_n^T(x)K_{ij}P2F_j, \quad i = 1,2,\ldots,m.$$

Letting $\mathbf{C} = [K_{ij}P2]$, from Eq.(19) we have

$$F = G + CF.$$

After solving above linear system, we can find F and accordingly find $F_i;\ i = 1,2,\ldots,m$, so

$$f_i(x) \simeq \Phi_n^T(x)F_i.$$

4 Convergence analysis

In this section, we show that the method discussed in the previous section is convergent and its order of convergence is $O(\frac{1}{n})$. We define

$$\|f(x)\| = \left(\int_0^1 |f(x)|^2 dx\right)^{\frac{1}{2}},$$

and

$$\|\mathbf{f}(x)\| = \left(\sum_{i=1}^{m} \|f_i(x)\|^2\right)^{\frac{1}{2}}, \tag{21}$$

where $f(x) \in L^2(D)$ and $\mathbf{f}(x)$ is defined in Eq. (3) and

$$\|k(x,y)\| = \left(\int_0^1 \int_0^1 |k(x,y)|^2 dxdy\right)^{\frac{1}{2}},$$

and

$$\|\mathbf{k}(x,y)\| = \left(\sum_{i=1}^{m}\sum_{j=1}^{m} \|k_{ij}(x,y)\|^2\right)^{\frac{1}{2}}, \tag{22}$$

where $k(x,y) \in L^2(D \times D)$ and $\mathbf{k}(x,y)$ is defined in Eq. (5).

Also, for our purposes we will need the following theorems.

Theorem 1. *Let $f(x) \in L^2(D)$ and $f_n(x)$ be the IBPFs expansion of $f(x)$ that is defined as*

$$f_n(x) = \sum_{i=0}^{n} f_i\phi_i(x),$$

where f_i; $i = 0, 1, \ldots, n$, are defined in Eq. (13). Then the criterion of this approximation is that the mean square error between $f(x)$ and $f_n(x)$ in the interval $x \in D$

$$\int_0^1 (f(x) - f_n(x))^2 dx \ ,$$

achieves its minimum value and also we have

$$\int_0^1 f^2(x)dx = \sum_{i=0}^{\infty} f_i^2 ||\phi_i(x)||^2 \ .$$

Proof. It is an immediate consequence of theorem which is proved in [9]. □

Theorem 2. *Suppose $f(x)$ is continuous on D, differentiable on $(0,1)$, and there exists a positive scalar M such that $|f'(x)| \leqslant M$, for every $x \in D$. Then*

$$|f(b) - f(a)| \leqslant M|b-a|, \quad \forall a, b \in D.$$

Proof. See [18]. □

Theorem 3. *Suppose $f_n(x)$ is the IBPFs expansions of $f(x)$ that is defined as Eq. (12) and $f(x)$ is differentiable on D such that $|f'(x)| \leqslant M$. Also, assume that $e_n(x) = f(x) - f_n(x)$, then*

$$\|e_n(x)\| = O(h).$$

Proof. Suppose $x_0 = 0,\ x_i = ih - \frac{h}{2},\ i = 1, \ldots, n$ and $x_{n+1} = 1$. We define the error between $f(x)$ and its IBPFs expansion over every subinterval $I_i = [x_i, x_{i+1})$ as follows

$$e_{n,i}(x) = f(x) - f_i(x), \quad x \in I_i \ ,$$

where $i = 0, 1, \ldots, n$. By using mean value theorem for integral, we have

$$\|e_{n,0}(x)\|^2 = \int_0^{\frac{h}{2}} e_{n,0}^2(x)dx = \int_0^{\frac{h}{2}} (f(x) - f_0)^2 \, dx = \frac{h}{2} \left(f(\xi_0) - f_0\right)^2,$$

where $\xi_0 \in I_0$. Also, for $i = 1, 2, \ldots, n-1$, we have

$$\|e_{n,i}(x)\|^2 = \int_{ih-\frac{h}{2}}^{ih+\frac{h}{2}} e_{n,i}^2(x)dx = \int_{ih-\frac{h}{2}}^{ih+\frac{h}{2}} (f(x) - f_i)^2 \, dx = h \left(f(\xi_i) - f_i\right)^2,$$

where $\xi_i \in I_i$. Furthermore, we have

$$\|e_{n,n}(x)\|^2 = \int_{1-\frac{h}{2}}^{1} e_{n,n}^2(x)dx = \int_{1-\frac{h}{2}}^{1} (f(x) - f_n)^2 \, dx = \frac{h}{2} \left(f(\xi_n) - f_n\right)^2,$$

where $\xi_n \in I_n$. Using Eq. (13) and the mean value theorem, we have

$$f_i = \begin{cases} 2n\int_0^{\frac{h}{2}} f(x)dx = 2n\frac{h}{2}f(\eta_0) = f(\eta_0), & i = 0, \\ n\int_{(i-1)h+\frac{h}{2}}^{ih+\frac{h}{2}} f(x)dx = nhf(\eta_i) = f(\eta_i), & i = 1, 2, \ldots, n-1, \\ 2n\int_{1-\frac{h}{2}}^{1} f(x)dx = 2n\frac{h}{2}f(\eta_n) = f(\eta_n), & i = n, \end{cases}$$

where $\eta_i \in I_i, i = 0, 1, \ldots, n$. From the above equations and Theorem 2, we get

$$\|e_{n,i}(x)\|^2 =$$

$$\begin{cases} \frac{h}{2}\left(f(\xi_0) - f(\eta_0)\right)^2 \leqslant \frac{M^2h}{2}|\xi_0 - \eta_0|^2 \leqslant \frac{M^2h^3}{8}, & i = 0, \\ h\left(f(\xi_i) - f(\eta_i)\right)^2 \leqslant M^2h|\xi_i - \eta_i|^2 \leqslant M^2h^3, & i = 1, 2, \ldots, n-1, \\ \frac{h}{2}\left(f(\xi_n) - f(\eta_n)\right)^2 \leqslant \frac{M^2h}{2}|\xi_n - \eta_n|^2 \leqslant \frac{M^2h^3}{8}, & i = n. \end{cases} \quad (23)$$

We have

$$\begin{aligned} \|e_n(x)\|^2 &= \int_0^1 e_n^2(x)dx = \int_0^1 \left(\sum_{i=0}^n e_{n,i}(x)\right)^2 dx \\ &= \int_0^1 \left(\sum_{i=0}^n e_{n,i}^2(x)\right) dx + 2\sum_{i \leqslant j} \int_0^1 e_{n,i}(x)e_{n,j}(x)dx. \end{aligned}$$

Since for $i \neq j$, $I_i \cap I_j = \emptyset$, then

$$\|e_n(x)\|^2 = \int_0^1 \left(\sum_{i=0}^n e_{n,i}^2(x)\right) dx = \sum_{i=0}^n \|e_{n,i}(x)\|^2. \quad (24)$$

Substituting Eq. (23) into Eq. (24), we get

$$\|e_n(x)\|^2 \leqslant M^2h^2 - \frac{3M^2h^3}{4},$$

which completes the proof. □

Suppose that $e_n'(x)$ is the error between $f(x)$ and its BPFs expansion. From [9], it is clear that

$$\|e_n(x)\| \leqslant \|e_n'(x)\|.$$

Lemma 1. *Let* $\mathbf{g}(x)$ *be as defined in Eq. (4),* $\mathbf{g}_n(x)$ *be the IBPFs of* $\mathbf{g}(x)$ *and* $e_{\mathbf{g}}(x) = \mathbf{g}(x) - \mathbf{g}_n(x)$*. Then*

$$\|e_{\mathbf{g}}(x)\| = O(h).$$

Proof. From Eq. (21), we have

$$\|e_{\mathbf{g}}(x)\| = \left(\sum_{i=1}^{m} \|g_i(x) - g_{n,i}(x)\|^2\right)^{\frac{1}{2}},$$

and from Theorem 2, $\|g_i(x) - g_{n,i}(x)\| \leqslant C_i h$. Then

$$\|e_{\mathbf{g}}(x)\| \leqslant \left(\sum_{i=1}^{m} C_i^2 h^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{m} C_i^2\right)^{\frac{1}{2}} h = Ch, \tag{25}$$

which completes the proof. □

Theorem 4. *Let* $k_n(x,y)$ *be the IBPFs expansions of* $k(x,y)$ *defined as Eq. (14) and* $k(x,y)$ *be differentiable on* $D \times D$ *such that* $|k'(x,y)| \leqslant M$*. Also, assume that* $e_n(x,y) = k(x,y) - k_n(x,y)$*, then*

$$\|e_n(x,y)\| = O(h).$$

Proof. Suppose $x_0 = y_0 = 0$, $x_i = y_i = ih - \frac{h}{2}$, $i = 1, \dots, n$ and $x_{n+1} = y_{n+1} = 1$. We define the error between $k(x,y)$ and its IBPFs expansion over every subinterval $I_{i,j} = [x_i, x_{i+1}) \times [y_j, y_{j+1})$ as follows

$$e_{n,ij}(x,y) = k(x,y) - k_{ij}(x,y), \quad x \in I_{i,j}, \quad i,j = 0,1,\dots,n.$$

By using the mean value theorem for integral and similar to the proof of Theorem 3, we get

$$\|e_{n,ij}(x,y)\|^2 \leqslant \begin{cases} \frac{M^2h^4}{8}, & j = 0,n, \\ \frac{5M^2h^4}{8}, & j = 1,2,\dots,n-1, \end{cases} \tag{26}$$

for $i = 0, n$ and

$$\|e_{n,ij}(x,y)\|^2 \leqslant \begin{cases} \frac{5M^2h^4}{8}, & j = 0,n, \\ 2M^2h^4, & j = 1,2,\dots,n-1, \end{cases} \tag{27}$$

for $i = 1, 2, \ldots, n-1$. We have

$$\begin{aligned}
\|e_n(x,y)\|^2 &= \int_0^1 \int_0^1 e_n^2(x,y)dxdy = \int_0^1 \int_0^1 \left(\sum_{i=0}^{n} \sum_{j=0}^{n} e_{n,ij}(x,y) \right)^2 dxdy \\
&= \int_0^1 \int_0^1 \left(\sum_{i=0}^{n} \sum_{j=0}^{n} e_{n,ij}^2(x,y) \right) dxdy \\
&\quad + 2 \sum_{i\leqslant k} \sum_{j\leqslant l} \int_0^1 \int_0^1 e_{n,ij}(x,y) e_{n,kl}(x,y) dxdy.
\end{aligned}$$

Since for $i \neq k$ and $j \neq l$, we have $I_i \cap I_k = \emptyset$ and $I_j \cap I_l = \emptyset$, then

$$\begin{aligned}
\|e_n(x,y)\|^2 &= \int_0^1 \int_0^1 \left(\sum_{i=0}^{n} \sum_{j=0}^{n} e_{n,ij}^2(x,y) \right) dxdy \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n} \|e_{n,ij}(x,y)\|^2. \qquad (28)
\end{aligned}$$

Substituting Eqs. (26) and (27) into Eq. (28), we get

$$\|e_n(x,y)\|^2 \leqslant 2M^2h^2 - \frac{3M^2h^3}{2}.$$

□

Suppose $e'_n(x,y)$ be the error between $k(x,y)$ and its BPFs expansion. From [14], it is clear that

$$\|e_n(x,y)\| \leqslant \|e'_n(x,y)\|.$$

Lemma 2. *Let $\mathbf{k}(x,y)$ be as defined in Eq. (5), $\mathbf{k}_n(x,y)$ be the IBPFs of $\mathbf{k}(x,y)$ and $e_{\mathbf{k}}(x,y) = \mathbf{k}(x,y) - \mathbf{k}_n(x,y)$. Then*

$$\|e_{\mathbf{k}}(x,y)\| = O(h).$$

Proof. From Eq. (22), we have

$$\|e_{\mathbf{k}}(x,y)\| = \left(\sum_{i=1}^{m} \sum_{j=1}^{m} \|k_{ij}(x,y) - k_{n,ij}(x,y)\| \right)^{\frac{1}{2}}.$$

From Theorem 3, we conclude that $\|k_{ij}(x,y) - k_{n,ij}(x,y)\| \leqslant C_{ij}h$. Therefore

$$\|e_{\mathbf{k}}(x,y)\| \leqslant \left(\sum_{i=1}^{m}\sum_{j=1}^{m} C_{ij}^2 h^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{m}\sum_{j=1}^{m} C_{ij}^2\right)^{\frac{1}{2}} h = Ch. \tag{29}$$

□

Let the error of IBPFs be denoted by

$$E_n = \|\mathbf{f}(x) - \mathbf{f}_n(x)\|,\ x \in D,$$

where $\mathbf{f}(x)$ was defined in Eq. (3). Furthermore, assume the following hypotheses:

(M1) Let $\|\mathbf{f}(x)\| \leqslant N$ for $x \in D$;

(M2) Let $\|\mathbf{k}(x,y)\| \leqslant N'$ for $(x,y) \in D \times D$;

(M3) According to lemma 1 and 2, let

$$E_{\mathbf{g}} = \|e_{\mathbf{g}}(x)\| \leqslant Ch,$$

and

$$E_{\mathbf{k}} = \|e_{\mathbf{k}}(x,y)\| \leqslant C'h,$$

where C and C' are coefficients defined in Eqs. (25) and (29) and $\mathbf{g}(x)$ and $\mathbf{k}(x,y)$ were defined in Eqs. (4) and (5), respectively;

(M4) Let $N' + C'h < 1$.

Theorem 5. *Let $\mathbf{f}(x)$ and $\mathbf{f}_n(x)$ be the exact and approximate solutions of Eq. (1) or Eq. (2), respectively. Also assumptions (M1)-(M4) are satisfied. Then we have*

$$E_n \leqslant \frac{(C + C'N)h}{1 - N' - C'h}. \tag{30}$$

Proof. For the first case, from Eq. (1), we have

$$\mathbf{f}(x) - \mathbf{f}_n(x) = \mathbf{g}(x) - \mathbf{g}_n(x) + \int_0^x (\mathbf{k}(x,y)\mathbf{f}(y) - \mathbf{k}_n(x,y)\mathbf{f}_n(y))\, dy,$$

and therefore

$$E_n \leqslant E_{\mathbf{g}} + \|x\|\|\mathbf{k}(x,y)\mathbf{f}(y) - \mathbf{k}_n(x,y)\mathbf{f}_n(y)\|.$$

It is clear that $\|x\| \leqslant 1$, So

$$E_n \leqslant E_{\mathbf{g}} + \|\mathbf{k}(x,y)\mathbf{f}(y) - \mathbf{k}_n(x,y)\mathbf{f}_n(y)\|. \tag{31}$$

Also for the second case, from Eq. (2), we have

$$\mathbf{f}(x) - \mathbf{f}_n(x) = \mathbf{g}(x) - \mathbf{g}_n(x) + \int_0^1 \left(\mathbf{k}(x,y)\mathbf{f}(y) - \mathbf{k}_n(x,y)\mathbf{f}_n(y)\right) dy,$$

and therefore

$$E_n \leqslant E_{\mathbf{g}} + \|\mathbf{k}(x,y)\mathbf{f}(y) - \mathbf{k}_n(x,y)\mathbf{f}_n(y)\|.$$

So Eq. (31) is true in the both cases. Now, according to assumptions (M1)-(M3), we have

$$\begin{aligned} \|\mathbf{k}(x,y)\mathbf{f}(y) - \mathbf{k}_n(x,y)\mathbf{f}_n(y)\| &\leqslant \|\mathbf{k}(x,y)\|\; E_n + E_{\mathbf{k}}\left(E_n + \|\mathbf{f}(x)\|\right) \\ &\leqslant N' E_n + C' h\left(E_n + N\right). \end{aligned} \tag{32}$$

Also from assumptions (M3), Eqs. (31) and (32), we have

$$E_n \leqslant (C + C'N)h + (N' + C'h)E_n.$$

Therefore according to (M4), Eq. (30) is satisfied and this completes the proof. Also we have $E_n = O(h)$. □

Lemma 3. *Suppose $\mathbf{f}(x)$ and $\mathbf{f}_n(x)$ are the exact and approximate solution of Eq. (1) or Eq. (2), respectively, where $\mathbf{f}(x)$ was defined in Eq. (3) and*

$$\mathbf{f}_n(x) = \left[f_{1,n}(x), f_{2,n}(x), \ldots, f_{m,n}(x)\right]^T .$$

Then

$$e_{i,n} = \|f_i(x) - f_{i,n}(x)\| = O(h). \tag{33}$$

Proof. From Theorem 4, we have $E_n \leqslant Ch$ and according to Eq. (21) we have

$$e_{i,n} \leqslant E_n \leqslant Ch.$$

□

Table 1: Numerical results for Example 1 ($n = 8,\ 16$).

Nodes x	Exact solution	Present method	
		$n=8$	$n=16$
0.0	(1.0,2.0,0.00)	(1.00068,2.09408,0.03384)	(1.00017,2.04696,0.01627)
0.1	(1.0,2.3,0.12)	(1.00330,2.37638,0.15868)	(1.00080,2.37535,0.15686)
0.2	(1.0,2.6,0.28)	(1.00391,2.75169,0.37768)	(1.00087,2.56287,0.25843)
0.3	(1.0,2.9,0.48)	(1.00391,2.75169,0.37768)	(1.00105,2.93799,0.50857)
0.4	(1.0,3.2,0.72)	(1.00477,3.12733,0.65979)	(1.00117,3.12558,0.65713)
0.5	(1.0,3.5,1.00)	(1.00612,3.50347,1.00526)	(1.00150,3.50086,1.00131)
0.6	(1.0,3.8,1.32)	(1.00831,3.88035,1.41441)	(1.00205,3.87632,1.40828)
0.7	(1.0,4.1,1.68)	(1.01195,4.25841,1.88776)	(1.00244,4.06416,1.63535)
0.8	(1.0,4.4,2.08)	(1.01195,4.25841,1.88776)	(1.00359,4.44011,2.13678)
0.9	(1.0,4.7,2.52)	(1.01799,4.63840,2.42623)	(1.00443,4.62830,2.41120)

5 Numerical experiments

In this section, several examples are presented to demonstrate the effectiveness of our approach. In this regard, we have reported the values of the exact solution $f(x)$, the approximate solution $f_n(x); n = 8;\ 16;\ 32;\ 64$ and the L^2-norms of errors that have been calculated by using

$$E_{2,n} = \left(\int_0^1 |f(x) - f_n(x)|^2 dx \right)^{\frac{1}{2}}, \quad x \in D.$$

at the selected points of the given interval in tables and figures. All computations is preformed in Matlab. In order to show the convergence rate of this numerical method, we introduce the following notation

$$E^{n,2n} = \log_2 \frac{\|f(x) - f_n(x)\|}{\|f(x) - f_{2n}(x)\|}.$$

The following problems have been tested.

Example 1. Consider the following system of linear Volterra integral equations [20]

$$\begin{cases} f_1(x) = 1 - x^2 - x^3 - \frac{1}{5}x^5 - \frac{1}{3}x^6 + \displaystyle\int_0^x y f_2(y)dy + \int_0^x y^3 f_3(y)dy, \\ f_2(x) = 2 + 3x - \frac{1}{2}x^2 - \frac{1}{4}x^4 - \frac{2}{5}x^5 + \displaystyle\int_0^x y f_1(y)dy + \int_0^x y^2 f_3(y)dy, \\ f_3(x) = x + 2x^2 - x^3 - \frac{1}{2}x^4 - \frac{3}{5}x^5 + \displaystyle\int_0^x x^2 f_1(y)dy + \int_0^x y^3 f_2(y)dy, \end{cases}$$

where $x \in [0, 1)$. The exact solution is $f_1(x) = 1$, $f_2(x) = 2 + 3x$ and $f_3(x) = x+2x^2$. Table 1 and 2 illustrates the comparison between the exact solution and numerical solution given by the proposed method (IBPFs) for different values of n. Also, we compare the infinity-norm of absolute error computed by the present method, block pulse functions method (BPFs) [13] and rationalized Haar functions method (RHFs) [15] in Table 3. Furthermore, for this example, numerical convergence rate for proposed method is tabulated in Table 4. Graph of the absolute errors for $f_1(x)$, $f_2(x)$ and $f_2(x)$ are shown in Figures 1, 2 and 3, respectively.

Table 2: Numerical results for Example 1 ($n = 32,\ 64$).

Nodes x	Present method	
	$n = 32$	$n = 64$
0.0	(1.00004,2.02346,0.00797)	(1.00001,2.01172,0.00395)
0.1	(1.00019,2.28133,0.11148)	(1.00005,2.28127,0.11137)
0.2	(1.00022,2.56259,0.25797)	(1.00005,2.60940,0.28569)
0.3	(1.00026,2.93762,0.50800)	(1.00006,2.89065,0.47319)
0.4	(1.00031,3.21891,0.73657)	(1.00008,3.21879,0.73639)
0.5	(1.00037,3.50021,1.00033)	(1.00009,3.50005,1.00008)
0.6	(1.00047,3.78155,1.29928)	(1.00012,3.78132,1.29894)
0.7	(1.00061,4.06291,1.63345)	(1.00016,4.10948,1.69206)
0.8	(1.00089,4.43815,2.13380)	(1.00021,4.39078,2.06713)
0.9	(1.00123,4.71967,2.55021)	(1.00031,4.71898,2.54917)

Table 3: Approximate L^2-norm of absolute error for Example 1.

Methods	$E_{2,n}$
Method of [15]	
$n = 8$	(1.25e-2,1.27e-1,1.31e-1)
$n = 16$	(3.02e-3,5.68e-2,6.11e-2)
$n = 32$	(6.93e-4,3.01e-2,3.16e-2)
$n = 64$	(1.87e-4,2.10e-2,1.96e-2)
Method of [13]	
$n = 8$	(1.06e-2,1.09e-1,1.16e-1)
$n = 16$	(2.67e-3,5.42e-2,5.80e-2)
$n = 32$	(6.67e-4,2.71e-2,2.90e-2)
$n = 64$	(1.67e-4,1.40e-2,1.48e-2)
Present method	
$n = 8$	(8.89e-3,1.03e-1,1.09e-1)
$n = 16$	(2.44e-3,5.29e-2,5.63e-2)
$n = 32$	(6.37e-4,2.67e-2,2.86e-2)
$n = 64$	(1.63e-4,1.35e-2,1.44e-2)

Table 4: Numerical convergence rate for Example 1.

m	$E^{n,2n}$		
	for $f_1(x)$	for $f_2(x)$	for $f_3(x)$
4	1.7360	0.9267	0.9116
8	1.8685	0.9656	0.9578
16	1.9342	0.9831	0.9790
32	1.9670	0.9916	0.9894

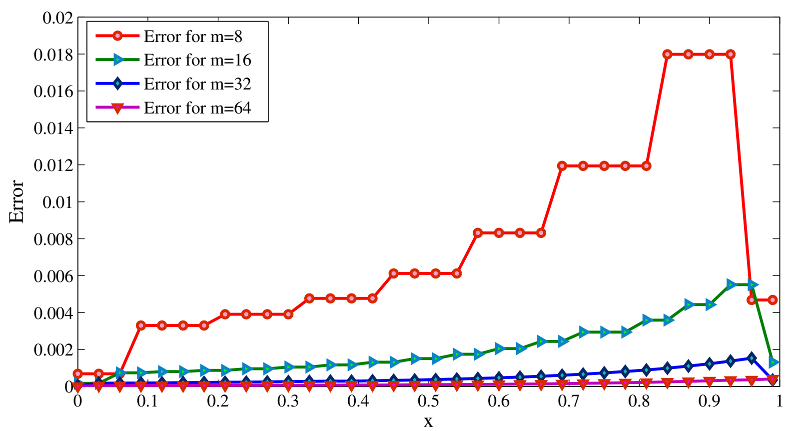

Figure 1: Absolute value of error, Example 1 for $f_1(x)$.

Example 2. Consider the following system of linear Volterra integral equations [4]:

$$\begin{cases} f_1(x) = 1 - \frac{x^2}{2} + \displaystyle\int_0^x \left(f_1(y) + ye^y f_2(y)\right) dy \ , \\ f_2(x) = 1 + \frac{x^2}{2} + \displaystyle\int_0^x \left(-ye^{-y} f_1(y) - f_2(y)\right) dy \ , \end{cases}$$

where $x \in [0, 1)$. The exact solution is $f_1(x) = e^x$ and $f_2(x) = e^{-x}$. Table 5 illustrates the comparison between the exact solution and numerical solution given by the proposed method (IBPFs) for different values of n. Also, we compare the infinity-norm of absolute error computed by the present method, BPFs [4] and RHFs [15] method in Table 6. Furthermore, for this example, numerical convergence rate for proposed method is tabulated in Table 7. Graph of the absolute errors for $f_1(x)$ and $f_2(x)$ are shown in Figures 4 and 5, respectively.

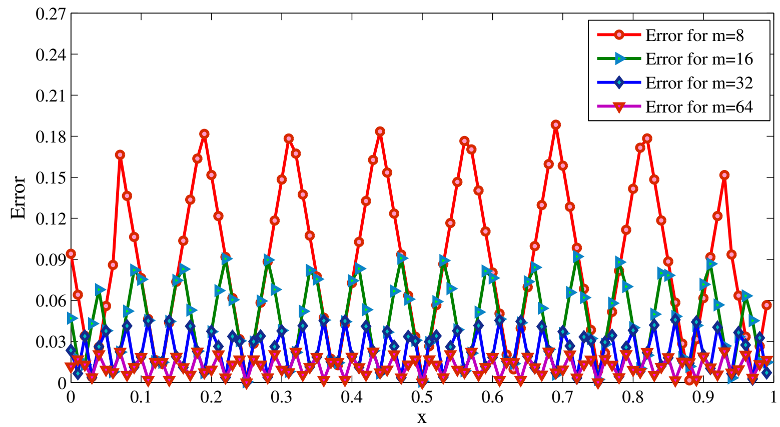

Figure 2: Absolute value of error, Example 1 for $f_2(x)$.

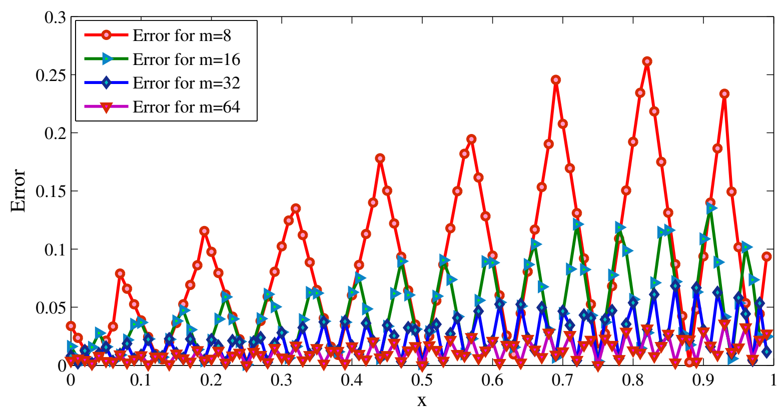

Figure 3: Absolute value of error, Example 1 for $f_3(x)$.

Table 5: Numerical results for Example 2.

Nodes x, Exact	Present method			
	$n=8$	$n=16$	$n=32$	$n=64$
0.0, (1.0,1.0)	(1.03260,0.96939)	(1.01596,0.98454)	(1.00789,0.99223)	(1.00393,0.99610)
0.1, (1.10517,0.90484)	(1.14747,0.87722)	(1.14120,0.87780)	(1.10128,0.90839)	(1.09984,0.90931)
0.2, (1.22140,0.81873)	(1.32221,0.76300)	(1.21968,0.82151)	(1.21308,0.82471)	(1.22902,0.81371)
0.3, (1.34986,0.74082)	(1.32221,0.76300)	(1.39260,0.71933)	(1.37969,0.72491)	(1.35167,0.73977)
0.4, (1.49182,0.67032)	(1.52126,0.66293)	(1.48767,0.67297)	(1.51917,0.65795)	(1.51014,0.66190)
0.5, (1.64872,0.60653)	(1.74720,0.57499)	(1.69669,0.58868)	(1.67240,0.59705)	(1.66049,0.60165)
0.6, (1.82212,0.54881)	(2.00282,0.49749)	(1.93339,0.51440)	(1.84065,0.54164)	(1.82560,0.54681)
0.7, (2.01375,0.49659)	(2.29115,0.42904)	(2.06312,0.48060)	(2.02531,0.49121)	(2.03877,0.48901)
0.8, (2.22554,0.44933)	(2.29115,0.42904)	(2.34752,0.41901)	(2.29969,0.43091)	(2.24086,0.44426)
0.9, (2.45960,0.40657)	(2.61538,0.36847)	(2.50313,0.39097)	(2.52876,0.39037)	(2.50165,0.39709)

Table 6: Approximate L^2-norm of absolute error for Example 2.

Methods	$E_{2,n}$
Method of [15]	
$n = 8$	(6.47e-2,7.89e-2)
$n = 16$	(8.13e-2,4.72e-2)
$n = 32$	(4.12e-2,2.07e-2)
$n = 64$	(2.34e-2,6.28e-2)
Method of [4]	
$n = 8$	(5.61e-2,7.17e-2)
$n = 16$	(7.94e-2,2.89e-2)
$n = 32$	(3.71e-2,1.35e-2)
$n = 64$	(2.03e-2,5.97e-3)
Present method	
$n = 8$	(1.44e-1,4.02e-2)
$n = 16$	(7.13e-2,2.17e-2)
$n = 32$	(3.54e-2,1.13e-2)
$n = 64$	(1.77e-2,5.74e-3)

Table 7: Numerical convergence rate for Example 2.

m	$E^{n,2n}$	
	for $f_1(x)$	for $f_2(x)$
4	1.0162	0.7896
8	1.0159	0.8916
16	1.0096	0.9449
32	1.0052	0.9722

Example 3. Consider the following system of linear Fredholm integral equations [13]:

$$\begin{cases} f_1(x) = \frac{11}{6}x + \frac{11}{15} - \int_0^1 (x+y) f_1(y) dy - \int_0^1 (x - 2y^2) f_2(y) dy \ , \\ f_2(x) = \frac{5}{4}x^2 + \frac{1}{4}x - \int_0^1 xy^2 f_1(y) dy - \int_0^1 x^2 y f_2(y) dy \ , \end{cases}$$

where $x \in [0, 1)$. The exact solution is $f_1(x) = x$ and $f_2(x) = x^2$. Table 8 illustrates the comparison between the exact solution and numerical solution given by the proposed method (IBPFs) for different values of n. Also,

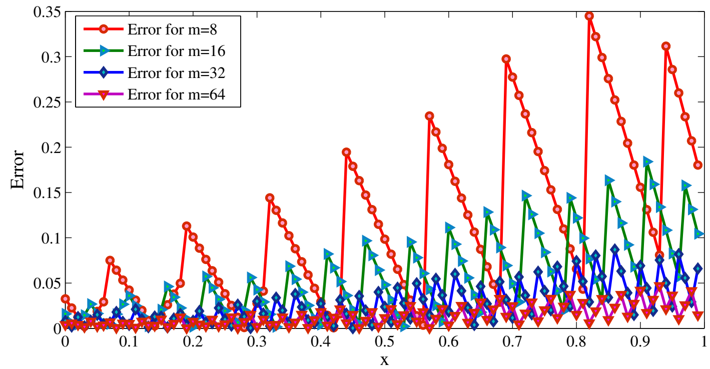

Figure 4: Absolute value of error, Example 2 for $f_1(x)$.

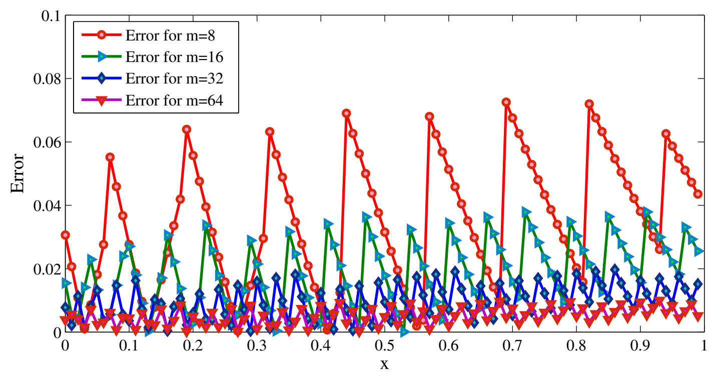

Figure 5: Absolute value of error, Example 2 for $f_2(x)$.

we compare the infinity-norm of absolute error computed by the present method, BPFs [13] and RHFs [15] method in Table 9. Furthermore, for this example, numerical convergence rate for proposed method is tabulated in Table 10. Graph of the absolute errors for $f_1(x)$ and $f_2(x)$ are shown in Figures 6 and 7 respectively.

Example 4. Consider the following system of linear Fredholm integral equations [13]:

$$\begin{cases} f_1(x) = 2e^x + \frac{e^{x+1}-1}{x+1} - \int_0^1 e^{x-y} f_1(y)dy - \int_0^1 e^{(x+2)y} f_2(y)dy \ , \\ \\ f_2(x) = e^x + e^{-x} + \frac{e^{x+1}-1}{x+1} - \int_0^1 e^{xy} f_1(y)dy - \int_0^1 e^{x+y} f_2(y)dy \ , \end{cases}$$

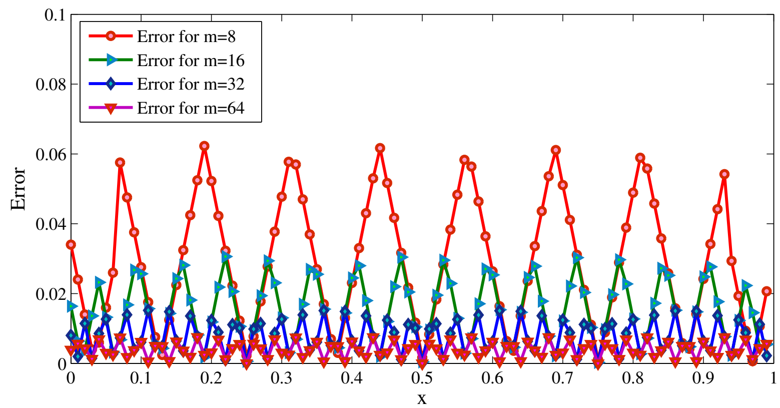

Figure 6: Absolute value of error, Example 3 for $f_1(x)$.

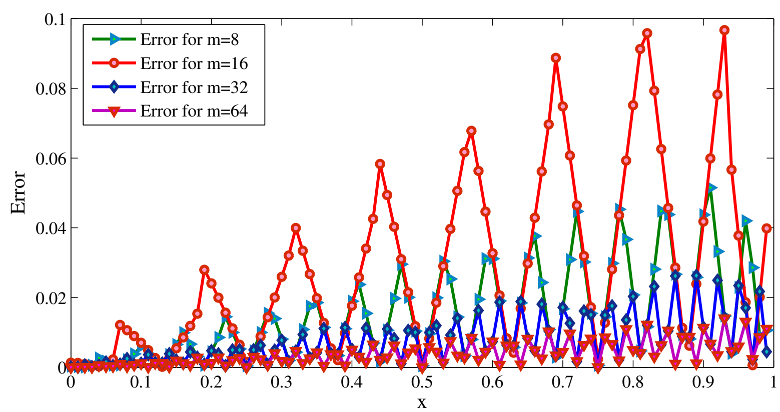

Figure 7: Absolute value of error, Example 3 for $f_2(x)$.

Table 8: Numerical results for Example 3.

Nodes x, Exact	Present method			
	$n = 8$	$n = 16$	$n = 32$	$n = 64$
0.0, (0.0,0.00)	(0.03402,0.00133)	(0.01638,0.00033)	(0.00801,0.00008)	(0.00396,0.00002)
0.1, (0.1,0.01)	(0.12755,0.01704)	(0.12568,0.01598)	(0.09393,0.00888)	(0.09380,0.00881)
0.2, (0.2,0.04)	(0.25226,0.06405)	(0.18815,0.03553)	(0.18767,0.03525)	(0.20317,0.04128)
0.3, (0.3,0.09)	(0.25226,0.06405)	(0.31307,0.09807)	(0.31265,0.09776)	(0.29691,0.08816)
0.4, (0.4,0.16)	(0.37697,0.14233)	(0.37553,0.14106)	(0.40638,0.16515)	(0.40628,0.16507)
0.5, (0.5,0.25)	(0.50168,0.25189)	(0.50045,0.25048)	(0.50012,0.25012)	(0.50003,0.25003)
0.6, (0.6,0.36)	(0.62639,0.39272)	(0.62537,0.39116)	(0.59385,0.35267)	(0.59378,0.35257)
0.7, (0.7,0.49)	(0.75110,0.56482)	(0.68784,0.47322)	(0.68759,0.47280)	(0.70315,0.49442)
0.8, (0.8,0.64)	(0.75110,0.56482)	(0.81276,0.66078)	(0.81257,0.66031)	(0.79689,0.63505)
0.9, (0.9,0.81)	(0.87581,0.76820)	(0.87522,0.76628)	(0.90630,0.82146)	(0.90626,0.82133)

Table 9: Approximate L^2-norm of absolute error for Example 3.

Methods	$E_{2,n}$
Method of [15]	
$n = 8$	(3.98e-2,4.44e-2)
$n = 16$	(2.21e-2,2.58e-2)
$n = 32$	(1.04e-2,1.53e-2)
$n = 64$	(1.77e-2,5.74e-3)
Method of [13]	
$n = 8$	(3.61e-2,4.16e-2)
$n = 16$	(1.80e-2,2.08e-2)
$n = 32$	(9.02e-3,1.04e-2)
$n = 64$	(4.51e-3,5.21e-3)
Present method	
$n = 8$	(3.44e-2,3.88e-2)
$n = 16$	(1.76e-2,2.01e-2)
$n = 32$	(8.92e-3,1.02e-2)
$n = 64$	(4.48e-3,5.16e-3)

Table 10: Numerical convergence rate for Example 3.

m	$E^{n,2n}$	
	for $f_1(x)$	for $f_2(x)$
4	0.9258	0.8896
8	0.9651	0.9471
16	0.9829	0.9741
32	0.9915	0.9872

where $x \in [0, 1)$. The exact solution is $f_1(x) = e^x$ and $f_2(x) = e^{-x}$. Table 11 illustrates the comparison between the exact solution and numerical solution given by the proposed method (IBPFs) for different values of n. Also, we compare the infinity-norm of absolute error computed by the present method, BPFs [13] and RHFs [15] method in Table 12. Furthermore, for this example, numerical convergence rate for proposed method is tabulated in Table 13. Graph of the absolute errors for $f_1(x)$ and $f_2(x)$ are shown in Figures 8 and 9, respectively.

Table 11: Numerical results for Example 4.

Nodes x, Exact	Present method			
	$n=8$	$n=16$	$n=32$	$n=64$
0.0, (1.0,1.0)	(1.01970,0.97502)	(1.01257,0.98603)	(1.00703,0.99261)	(1.00371,0.99620)
0.1, (1.10517,0.90484)	(1.12106,0.88873)	(1.12992,0.88415)	(1.09747,0.91093)	(1.09808,0.91062)
0.2, (1.22140,0.81873)	(1.27114,0.78496)	(1.20290,0.83067)	(1.20537,0.82945)	(1.22501,0.81628)
0.3, (1.34986,0.74082)	(1.27114,0.78496)	(1.36330,0.73323)	(1.36592,0.73203)	(1.34542,0.74324)
0.4, (1.49182,0.67032)	(1.44131,0.69334)	(1.45134,0.68889)	(1.50022,0.66655)	(1.50093,0.66625)
0.5, (1.64872,0.60653)	(1.63418,0.61243)	(1.64484,0.60809)	(1.64772,0.60693)	(1.64847,0.60663)
0.6, (1.82212,0.54881)	(1.85279,0.54095)	(1.86412,0.53677)	(1.80972,0.55264)	(1.81050,0.55235)
0.7, (2.01375,0.49659)	(2.10060,0.47776)	(1.98448,0.50431)	(1.98764,0.50321)	(2.01977,0.49513)
0.8, (2.22554,0.44933)	(2.10060,0.47776)	(2.24901,0.44514)	(2.25237,0.44411)	(2.21830,0.45083)
0.9, (2.45960,0.40657)	(2.38148,0.42189)	(2.39422,0.41820)	(2.47380,0.40437)	(2.47471,0.40412)

Table 12: Approximate L^2-norm of absolute error for Example 4.

Methods	$E_{2,n}$
Method of [15]	
$n=8$	(6.94e-2,2.85e-2)
$n=16$	(3.46e-2,1.33e-2)
$n=32$	(1.88e-2,6.17e-3)
$n=64$	(8.36e-3,3.21e-3)
Method of [13]	
$n=8$	(6.69e-2,2.45e-2)
$n=16$	(3.26e-2,1.20e-2)
$n=32$	(1.62e-2,5.94e-3)
$n=64$	(8.07e-3,2.98e-3)
Present method	
$n=8$	(6.26e-2,2.29e-2)
$n=16$	(3.15e-2,1.16e-2)
$n=32$	(1.59e-2,5.85e-3)
$n=64$	(8.00e-3,2.94e-3)

Table 13: Numerical convergence rate for Example 4.

m	$E^{n,2n}$	
	for $f_1(x)$	for $f_2(x)$
4	1.0126	0.9954
8	0.9890	0.9831
16	0.9871	0.9855
32	0.9913	0.9909

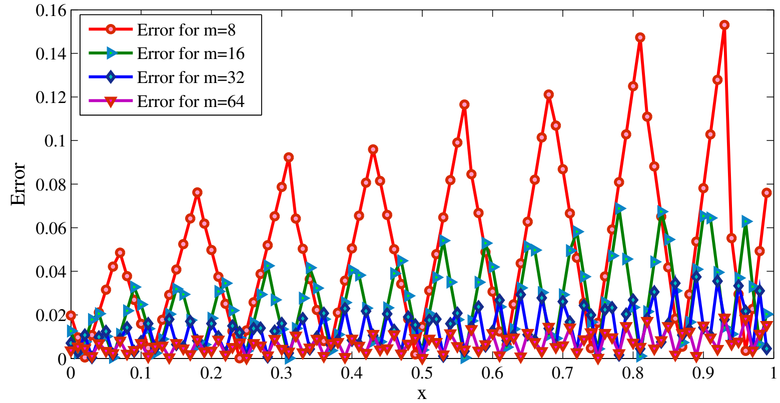

Figure 8: Absolute value of error, Example 4 for $f_1(x)$.

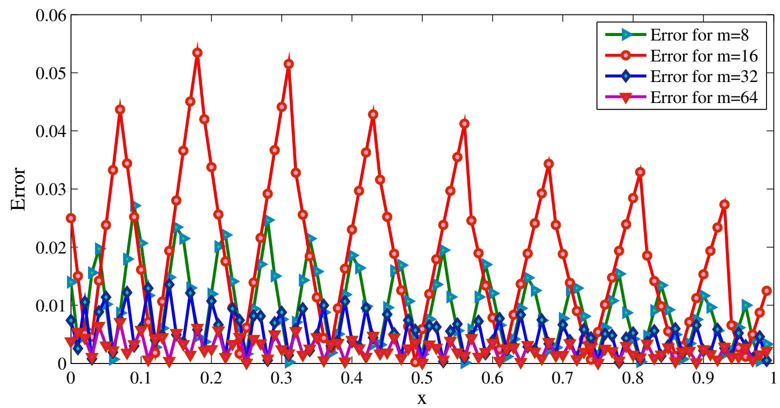

Figure 9: Absolute value of error, Example 4 for $f_2(x)$.

6 Conclusion

The IBPFs, together with operational matrices of integration $P1$ and $P2$, are used to obtain the solution of system of linear Volterra and Fredholm integral equations. The present method reduces a system of linear Volterra and Fredholm integral equations into a system of algebraic equations. The matrices $P1$ and $P2$ have many zeros, hence the method is computationally very attractive. Also, we have shown that our approach is convergence and its order of convergent is $O(\frac{1}{n})$. For the approximate solution of the given examples, we plot absolute value of error to discover applicability and accuracy of the proposed method. Comparisons of the results of applying IBPFs, BPFs and RHFs methods reveals that the new technique is effective and convenient. This method can be easily extended and applied for solving systems of nonlinear Volterra and Fredholm integral equations.

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