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The exponential functions of central-symmetric X-form matrices

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Abstract. It is well known that the matrix exponential function has practical applications in engineering and applied sciences. In this paper, we present some new explicit identities to the exponential functions of a special class of matrices that are known as central-symmetric X-form. For instance, $e^{\mathbf{A}t}$, $t^{\mathbf{A}}$ and $a^{\mathbf{A}t}$ will be evaluated by the new formulas in this particular structure. Moreover, upper bounds for the explicit relations will be given via subordinate matrix norms. Eventually, some numerical illustrations and applications are also adapted.

Keywords: central-symmetric matrix, matrix function, matrix exponential, Gamma and Beta matrix functions.
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1 Introduction

The initial value problem in the form

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t), \qquad \mathbf{y}(0) = \mathbf{c}, \tag{1}$$

where $\mathbf{A} \in \mathbb{C}^{n\times n}$, and $\mathbf{y}(t)$, $\mathbf{c} \in \mathbb{C}^n$ has the solution $\mathbf{y}(t) = e^{\mathbf{A}t}\mathbf{c}$. In the general case, with appropriate assumptions on the smoothness of $\mathbf{f}$, the

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solution of the inhomogeneous system

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{f}(t, y), \qquad \mathbf{y}(0) = \mathbf{c}, \tag{2}$$

satisfy

$$\mathbf{y}(t) = e^{\mathbf{A}t}\mathbf{c} + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{f}(s, y)ds,$$

which is an explicit formula for $\mathbf{y}$. These formulas do not essentially provide the best technique for the numerical solutions. Thus, the extensive literature on the numerical solutions of ordinary differential equations provide alternative techniques to approximate $e^{\mathbf{A}t}$ [2,7]. From the other point of view, it is known that the matrix exponential functions have significant roles in engineering and science such as nuclear magnetic resonance, Markov models, differential equations, exponential integrators, and Control theory [2, 6, 7]. Many numerical analysist investigated the computational methods for the matrix exponential functions, among them is the most important fundamental paper written by Moler and Van Loan [3]. They have proposed numerous approaches such as series method, Padé approximation method, scaling and squaring method, single ordinary differential equation (ODE) method, Cayley Hamilton method, Newton interpolation method, Vandermonde matrix method, inverse Laplace transforms method, Companion matrix method, matrix decomposition methods, and Krylov subspace methods. It is mentioned that the ODE's approach, scaling and squaring method, and Schur decomposition methods are more suitable methods. It must be emphasized that in MATLAB, the exponential function is approximated by scaling and squaring method.

The central-symmetric X-form matrices were first introduced by Nazari et. al in [4]. These matrices are in the following forms

$$\mathbf{A}_n = \begin{pmatrix} \alpha_n & & & & & & \beta_n \\ & \ddots & & & & \iddots & \\ & & \alpha_2 & & \beta_2 & & \\ & & & \alpha_1 & & & \\ & & \beta_2 & & \alpha_2 & & \\ & \iddots & & & & \ddots & \\ \beta_n & & & & & & \alpha_n \end{pmatrix} \in \mathbb{C}^{(2n-1)\times(2n-1)}, \tag{3}$$

and

$$\mathbf{B}_n = \begin{pmatrix} \alpha_n & & & & & & & \beta_n \\ & \ddots & & & & & \iddots & \\ & & \alpha_2 & & & \beta_2 & & \\ & & & \alpha_1 & \beta_1 & & & \\ & & & \beta_1 & \alpha_1 & & & \\ & & \beta_2 & & & \alpha_2 & & \\ & \iddots & & & & & \ddots & \\ \beta_n & & & & & & & \alpha_n \end{pmatrix} \in \mathbb{C}^{(2n)\times(2n)}. \qquad (4)$$

This paper is organized as follows. In Section 2, we develop some results about linear algebra operations for the central-symmetric X-form matrices. The main contributions of this paper are given in Section 3, by proposing several new formulas and by proving different theorems to the exponentials of the argued matrices. In Section 4, some examples are given for the capability of the matrix exponential functions such as $e^{\mathbf{A}t}$, $t^{\mathbf{A}}$, and $a^{\mathbf{A}t}$. Finally, concluding remarks are presented in Section 5.

2 Preliminary

This section is devoted to properties of the central-symmetric X-form matrices that are used in this paper. It is noted that some properties of this kind of matrices such as Dollitle factorization, determinant, inverse and singular values have been given in [4]. Firstly, we express the following theorem:

Theorem 1. *Let $\mathbf{A}$ and $\mathbf{B}$ be two central-symmetric X-form matrices. Then the following properties hold:*

1. *$\mathbf{A}^H$ or $\mathbf{A}^T$ are central-symmetric X-form matrices.*
2. *$\mathbf{AB}$ is a central-symmetric X-form matrix.*
3. *adj$(\mathbf{A})$ is a central-symmetric X-form matrix.*
4. *$\mathbf{A}^{-1}$ is a central-symmetric X-form matrix.*
5. *$\mathbf{A}^q$ and $\mathbf{A}^{1/q}$ for $(q > 0)$ are central-symmetric X-form matrices.*
6. *$e^{\mathbf{A}}$, $t^{\mathbf{A}}$ and $a^{\mathbf{A}}$ are central-symmetric X-form matrices.*
7. *The Gamma matrix function which is defined by $\Gamma(\mathbf{A}) = \int_0^1 e^{-t} t^{\mathbf{A}-\mathbf{I}} dt$ and the Beta matrix function which is defined by $\mathcal{B}(\mathbf{A}, \mathbf{B}) = \int_0^1 t^{\mathbf{A}-\mathbf{I}} (1-t)^{\mathbf{B}-\mathbf{I}} dt$, are both central-symmetric X-form matrices [5].*

Proof. The proof can be easily done by considering the elements of the matrices $\mathbf{A}$ and $\mathbf{B}$. □

The factors for $\mathbf{A}_n$ and $\mathbf{B}_n$ are as follows:

$$\mathbf{L}_{\mathbf{A}_n} = \begin{pmatrix} \sqrt{\alpha_n} & & & & & & \\ & \ddots & & & & & \\ & & \sqrt{\alpha_2} & & & & \\ & & & \sqrt{\alpha_1} & & & \\ & & \frac{\beta_2}{\sqrt{\alpha_2}} & & \sqrt{\frac{\alpha_2^2-\beta_2^2}{\alpha_2}} & & \\ & \iddots & & & & \ddots & \\ \frac{\beta_n}{\sqrt{\alpha_n}} & & & & & & \sqrt{\frac{\alpha_n^2-\beta_n^2}{\alpha_n}} \end{pmatrix},$$

and

$$\mathbf{L}_{\mathbf{B}_n} = \begin{pmatrix} \sqrt{\alpha_n} & & & & & \\ & \ddots & & & & \\ & & \sqrt{\alpha_1} & & & \\ & & \frac{\beta_1}{\sqrt{\alpha_1}} & \sqrt{\frac{\alpha_1^2-\beta_1^2}{\alpha_1}} & & \\ & \iddots & & & \ddots & \\ \frac{\beta_n}{\sqrt{\alpha_n}} & & & & & \sqrt{\frac{\alpha_n^2-\beta_n^2}{\alpha_n}} \end{pmatrix}.$$

The next two lemmas deal with the determinants and eigenvalues of $\mathbf{A}_n$ and $\mathbf{B}_n$ [4].

Lemma 1. *If $\mathbf{A}_n$ and $\mathbf{B}_n$ are odd and even dimensions central-symmetric X-form matrices respectively, then*

$$\det(\mathbf{A}_n) = \alpha_1 \prod_{i=2}^{n} (\alpha_i^2 - \beta_i^2), \tag{5}$$

$$\det(\mathbf{B}_n) = \prod_{i=1}^{n} (\alpha_i^2 - \beta_i^2).$$

Since $\mathbf{A}_n \in \mathbb{C}^{(2n-1)\times(2n-1)}$, it is straightforward to see that $p_{2n-1}(\lambda) = \det(\mathbf{A}_n - \lambda \mathbf{I}_{2n-1}) = 0$. Thus, according to Eq. (5), we have

$$\det(\mathbf{A}_n - \lambda \mathbf{I}_{2n-1}) = (\alpha_1 - \lambda_1) \prod_{i=2}^{n} \left[(\alpha_i - \lambda_i)^2 - \beta_i^2\right] = 0.$$

Consequently, the eigenvalues of $\mathbf{A}_n$ are given by $\lambda = \alpha_1, \lambda = \alpha_i \pm \beta_i, i = 2, \ldots, n$. The eigenvalues of $\mathbf{B}_n$ can be computed in a similar way.

Lemma 2. *The eigenvalues of $\mathbf{A}_n$ lie in the set $\{\alpha_1\} \cup \{\alpha_i \pm \beta_i, i \in \{2, \ldots, n\}\}$, and the eigenvalues of $\mathbf{B}_n$ lie in the set $\{\alpha_i \pm \beta_i, i \in \{1, \ldots, n\}\}$.*

Theorem 2. *The $1, 2, \infty$ and Frobenius norms of $\mathbf{A}_n$ and $\mathbf{B}_n$ are given as following:*

1. $\|\mathbf{A}_n\|_1 = \|\mathbf{A}_n\|_\infty = \max\limits_{2 \le i \le n} \{|\alpha_1|, |\alpha_i| + |\beta_i|\}$,
   $\|\mathbf{B}_n\|_1 = \|\mathbf{B}_n\|_\infty = \max\limits_{1 \le i \le n} \{|\alpha_i| + |\beta_i|\}$.

2. $\|\mathbf{A}_n\|_2 = \sqrt{\max\limits_{2 \le i \le n} \left\{(\alpha_i + \beta_i)^2, \alpha_1^2, (\alpha_i - \beta_i)^2)\right\}}$,
   $\|\mathbf{B}_n\|_2 = \sqrt{\max\limits_{1 \le i \le n} \left\{(\alpha_i + \beta_i)^2, (\alpha_i - \beta_i)^2)\right\}}$.

3. $\|\mathbf{A}_n\|_F = \sqrt{2 \sum\limits_{i=2}^{n} (\alpha_i^2 + \alpha_1^2 + \beta_i^2)}$,
   $\|\mathbf{B}_n\|_F = \sqrt{2 \sum\limits_{i=1}^{n} (\alpha_i^2 + \beta_i^2)}$.

Proof. Parts (1) and (3) are obvious. In order to prove part (2), it is sufficient to consider the maximum eigenvalue of the matrices $\mathbf{A}_n^T \mathbf{A}_n$ and $\mathbf{B}_n^T \mathbf{B}_n$ that are both central-symmetric X-form matrices. □

3 Main results

Suppose that $f(z)$ is an analytic function over a closed contour Γ which encircles $\sigma(\mathbf{A})$, denotes the set of eigenvalues of matrix $\mathbf{A}$. A function of matrix is defined by using Cauchy integral definition [1]:

$$f(\mathbf{A}) = \frac{1}{2\pi i} \oint_\Gamma f(\xi)(\xi \mathbf{I} - \mathbf{A})^{-1} d\xi.$$

The entries of $(\xi \mathbf{I} - \mathbf{A})^{-1}$ are analytic on Γ, thus $f(\mathbf{A})$ is analytic in a neighborhood of $\sigma(\mathbf{A})$. The exponential of a matrix is one of the most applicable functions that is deduced in the following definition:

Definition 1. [1, 7] The matrix exponential for $\mathbf{A} \in \mathbb{C}^{n \times n}$ is defined by $e^{\mathbf{A}t} = \sum\limits_{k=0}^{\infty} \dfrac{(\mathbf{A}t)^k}{k!}$.

In the sequel, we provide new formulations for the exponential functions of central symmetric X-form matrices. In addition, error analysis of exponential functions are performed with details.

3.1 Explicit formula for $e^{\mathbf{A}t}$

In this subsection, we are going to derive an explicit formula for the exponential functions of central-symmetric X-form matrices. For this purpose, we first have the following theorem:

Theorem 3. *Assume that $\mathbf{A}_n \in \mathbb{C}^{(2n-1)\times(2n-1)}$ and $\mathbf{B}_n \in \mathbb{C}^{(2n)\times(2n)}$ are central-symmetric X-form matrices. Then, the exponential functions of $\mathbf{A}_n t$ and $\mathbf{B}_n t$ are given by*

$$e^{\mathbf{A}_n t} = \begin{pmatrix} \varphi_n & & & & & & \psi_n \\ & \ddots & & & & ⋰ & \\ & & \varphi_2 & & \psi_2 & & \\ & & & \zeta_1 & & & \\ & & \psi_2 & & \varphi_2 & & \\ & ⋰ & & & & \ddots & \\ \psi_n & & & & & & \varphi_n \end{pmatrix},$$

$$e^{\mathbf{B}_n t} = \begin{pmatrix} \varphi_n & & & & & & & \psi_n \\ & \ddots & & & & & ⋰ & \\ & & \varphi_2 & & & \psi_2 & & \\ & & & \varphi_1 & \psi_1 & & & \\ & & & \psi_1 & \varphi_1 & & & \\ & & \psi_2 & & & \varphi_2 & & \\ & ⋰ & & & & & \ddots & \\ \psi_n & & & & & & & \varphi_n \end{pmatrix},$$

where, $\zeta_1 = e^{\alpha_1 t}$, and for $i = 1, \ldots, n$:

$$\varphi_i = e^{\alpha_i t}\cosh(\beta_i t),$$

$$\psi_i = e^{\alpha_i t}\sinh(\beta_i t).$$

Proof. To prove the assertion, we split the matrix $\mathbf{A}_n$ into two matrices $\mathbf{D}_n$ and $\mathbf{D}'_n$ such that $\mathbf{A}_n = \mathbf{D}_n + \mathbf{D}'_n$, whereas

$$\mathbf{D}_n = \operatorname{diag}_{(2n-1)}(\alpha_n, \ldots, \alpha_2, \alpha_1, \alpha_2, \ldots, \alpha_n),$$

and

$$\mathbf{D}'_n = \operatorname{antidiag}_{(2n-1)}(\beta_n, \ldots, \beta_2, 0, \beta_2, \ldots, \beta_n).$$

Furthermore, the powers of $\mathbf{D}'_n$ for $q \in \{2, 4, 6, \ldots\}$ are

$$(\mathbf{D}'_n)^q = \operatorname{diag}_{(2n-1)}(\beta_n^q, \ldots, \beta_2^q, 0, \beta_2^q, \ldots, \beta_n^q),$$

and for $q \in \{1,3,5,\dots\}$ are

$$(\mathbf{D}'_n)^q = \mathrm{antidiag}_{(2n-1)}(\beta_n^q,\dots,\beta_2^q,0,\beta_2^q,\dots,\beta_n^q).$$

Accordingly, from series definition for the matrix exponential, we have

$$e^{\mathbf{D}'_n t} = \sum_{k=0}^{\infty}\frac{(\mathbf{D}'_n t)^k}{k!} = \sum_{k\in\{2n\}}\frac{(\mathbf{D}'_n t)^k}{k!} + \sum_{k\in\{2n+1\}}\frac{(\mathbf{D}'_n t)^k}{k!}$$

$$= \begin{pmatrix} \xi_n & & & & & & \eta_n \\ & \ddots & & & & ⋰ & \\ & & \xi_2 & & \eta_2 & & \\ & & & 1 & & & \\ & & \eta_2 & & \xi_2 & & \\ & ⋰ & & & & \ddots & \\ \eta_n & & & & & & \xi_n \end{pmatrix},$$

wherein,

$$\xi_i = \sum_{k=0}^{\infty}\frac{(\beta_i t)^{2k}}{(2k)!} = \cosh(\beta_i t), \quad \eta_i = \sum_{k=0}^{\infty}\frac{(\beta_i t)^{2k+1}}{(2k+1)!} = \sinh(\beta_i t),$$

for $i = 2,3,\dots n$. In addition, as we know $e^{\mathbf{\Lambda}} = \mathrm{diag}(e^{\lambda_1},\dots,e^{\lambda_n})$, then

$$e^{\mathbf{D}_n t} = \mathrm{diag}_{(2n-1)}(e^{\alpha_n t},\dots,e^{\alpha_2 t},e^{\alpha_1 t},e^{\alpha_2 t},\dots,e^{\alpha_n t}).$$

Moreover, according to the facts that $e^{\mathbf{D}_n+\mathbf{D}'_n} = e^{\mathbf{D}_n}e^{\mathbf{D}'_n}$ and $\mathbf{D}_n\mathbf{D}'_n = \mathbf{D}'_n\mathbf{D}_n$, we have

$$e^{\mathbf{A}_n t} = e^{\mathbf{D}_n t+\mathbf{D}'_n t} = e^{\mathbf{D}_n t}e^{\mathbf{D}'_n t}.$$

This further gives

$$e^{\mathbf{A}_n t} = \begin{pmatrix} e^{\alpha_n t}\cosh(\beta_n t) & & & & & & e^{\alpha_n t}\sinh(\beta_n t) \\ & \ddots & & & & ⋰ & \\ & & e^{\alpha_2 t}\cosh(\beta_2 t) & & e^{\alpha_2 t}\sinh(\beta_2 t) & & \\ & & & e^{\alpha_1 t} & & & \\ & & e^{\alpha_2 t}\sinh(\beta_2 t) & & e^{\alpha_2 t}\cosh(\beta_2 t) & & \\ & ⋰ & & & & \ddots & \\ e^{\alpha_n t}\sinh(\beta_n t) & & & & & & e^{\alpha_n t}\cosh(\beta_n t) \end{pmatrix}.$$

By a similar technique, we have

$$e^{\mathbf{B}_n t} = \begin{pmatrix} e^{\alpha_n t}\cosh(\beta_n t) & & & & & e^{\alpha_n t}\sinh(\beta_n t) \\ & \ddots & & & ⋰ & \\ & & e^{\alpha_1 t}\cosh(\beta_1 t) & e^{\alpha_1 t}\sinh(\beta_1 t) & & \\ & & e^{\alpha_1 t}\sinh(\beta_1 t) & e^{\alpha_1 t}\cosh(\beta_1 t) & & \\ & ⋰ & & & \ddots & \\ e^{\alpha_n t}\sinh(\beta_n t) & & & & & e^{\alpha_n t}\cosh(\beta_n t) \end{pmatrix},$$

which completes the proof. □

Example 1. Assume that $\mathbf{A} = t\left(\begin{smallmatrix} \mathbf{0} & \mathbf{J}_n \\ \mathbf{J}_n & \mathbf{0} \end{smallmatrix}\right)$ a matrix $2n \times 2n$, where $\mathbf{J}_n$ is exchange matrix, that is defined by

$$\mathbf{J}_n = (\mathbf{j}_{ij})_{n\times n} = \begin{cases} 1, & j = n-i+1, \\ 0, & j \neq n-i+1. \end{cases}$$

Hence, we obtain the following block form of the matrix exponential, similar as

$$e^{\mathbf{A}t} = \begin{pmatrix} (\cosh t)\mathbf{I}_n & (\sinh t)\mathbf{J}_n \\ (\sinh t)\mathbf{J}_n & (\cosh t)\mathbf{I}_n \end{pmatrix}_{(2n)\times(2n)}.$$

In this part, we are interested to give an upper bound for the exponential functions of central-symmetric X-form matrices. According to references [1, 7], consider the following sets:

$$\alpha(\mathbf{A}) = \min\{\mathrm{Re}(z) : z \in \sigma(\mathbf{A})\}, \quad \beta(\mathbf{A}) = \max\{\mathrm{Re}(z) : z \in \sigma(\mathbf{A})\}.$$

Now, by utilizing the Schur decomposition of $\mathbf{A}$, we have

$$\|e^{\mathbf{A}t}\| \leq e^{\alpha(\mathbf{A})t} \sum_{k=0}^{n-1} \frac{(\|\mathbf{A}\|\sqrt{n}t)^k}{k!}.$$

If we consider $\Phi(\mathbf{A}_n) = \max_{2\leq i\leq n}\{\alpha_i + \beta_i, \alpha_1\}$, then after simplifying, we get

$$\|e^{\mathbf{A}_n t}\|_\infty \leq e^{\alpha(\mathbf{A})t} \sum_{k=0}^{n-1} \frac{(\|\mathbf{A}\|_\infty\sqrt{n}t)^k}{k!} \leq e^{\alpha(\mathbf{A})t} \sum_{k=0}^{n-1} \frac{(\Phi(\mathbf{A}_n)\sqrt{n}t)^k}{k!}.$$

Furthermore, if we consider $\Psi(\mathbf{B}_n) = \max_{1\leq i\leq n}\{\alpha_i + \beta_i\}$, then we deduce that

$$\|e^{\mathbf{B}_n t}\|_\infty \leq e^{\alpha(\mathbf{A})t} \sum_{k=0}^{n-1} \frac{(\|\mathbf{A}\|_\infty\sqrt{n}t)^k}{k!} \leq e^{\alpha(\mathbf{A})t} \sum_{k=0}^{n-1} \frac{(\Psi(\mathbf{B}_n)\sqrt{n}t)^k}{k!}.$$

Since, $\|\mathbf{A}_n\|_\infty = \|\mathbf{A}_n\|_1$ and $\|\mathbf{B}_n\|_\infty = \|\mathbf{B}_n\|_1$, the results can be expressed for the 1-norm, too.

3.2 Explicit formula for $t^{\mathbf{A}}$

In primitive Calculus, it is well known that $t^a = e^{a\ln t}$. Then for matrix $\mathbf{A}$ we deduced that

$$t^{\mathbf{A}} = e^{(\ln t)\mathbf{A}}, \quad t > 0.$$

In the following theorem, we identify some important properties of $t^{\mathbf{A}}$. This theorem can be easily proved by some properties of the exponential functions. Thus, we omited the proof to save space.

Theorem 4. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two square matrices. Then the following properties hold:*

1. $t^{\mathbf{0}_{n\times n}} = \mathbf{I}_n$ *and* $t^{\mathbf{I}_n} = t\mathbf{I}_n$.
2. $t^{\mathbf{A}+\mathbf{B}} = t^{\mathbf{A}}t^{\mathbf{B}}$, *provided* $\mathbf{AB} = \mathbf{BA}$.
3. *If* $\mathbf{P}$ *is nonsingular matrix, then* $t^{\mathbf{PAP}^{-1}} = \mathbf{P}t^{\mathbf{A}}\mathbf{P}^{-1}$.
4. $t^{-\mathbf{A}}t^{\mathbf{A}} = t^{\mathbf{A}}t^{-\mathbf{A}} = \mathbf{I}_n$.
5. $(t^{\mathbf{A}})^H = t^{\mathbf{A}^H}$, *it follows that if* $\mathbf{A}$ *is Hermitian, then* $t^{\mathbf{A}}$ *is also Hermitian, and if* $\mathbf{A}$ *is skew-Hermitian, then* $t^{\mathbf{A}}$ *is unitary.*
6. $(t^{\mathbf{A}})^T = t^{\mathbf{A}^T}$, *it follows that if* $\mathbf{A}$ *is symmetric, then* $t^{\mathbf{A}}$ *is also symmetric, and if* $\mathbf{A}$ *is skew-symmetric, then* $t^{\mathbf{A}}$ *is orthogonal.*
7. $\det(t^{\mathbf{A}}) = \exp\left((\ln t)\, Tr(\mathbf{A})\right)$.
8. $\frac{d}{dt}(t^{\mathbf{A}}) = \mathbf{A}t^{-1}t^{\mathbf{A}}$.
9. $t^{\mathbf{A}\otimes\mathbf{I}_n} = t^{\mathbf{A}} \otimes t\mathbf{I}_n, \quad t^{\mathbf{I}_n\otimes\mathbf{B}} = t\mathbf{I}_n \otimes t^{\mathbf{B}}$.
10. $t^{\mathbf{A}\oplus\mathbf{B}} = t(t^{\mathbf{A}} \otimes t^{\mathbf{B}})$.

It should be noted that Kronecker product of matrices $\mathbf{A}$ and $\mathbf{B}$ is defined by $\mathbf{A}\otimes\mathbf{B} = a_{ij}\mathbf{B}$, and alternatively Kronecker sum of matrices $\mathbf{A}$ and $\mathbf{B}$ is defined by $\mathbf{A}\oplus\mathbf{B} = (\mathbf{I}\otimes\mathbf{A}) + (\mathbf{B}\otimes\mathbf{I})$. Now, explicit formulas for computing $t^{\mathbf{A}_n}$ and $t^{\mathbf{B}_n}$ are given in the following theorem.

Theorem 5. *Let* $\mathbf{A}_n \in \mathbb{C}^{(2n-1)\times(2n-1)}$ *and* $\mathbf{B}_n \in \mathbb{C}^{(2n)\times(2n)}$ *be central-symmetric* X*-form matrices. Then, the functions* $t^{\mathbf{A}_n}$ *and* $t^{\mathbf{B}_n}$ *are computed by*

$$t^{\mathbf{A}_n} = \begin{pmatrix} \varphi_n & & & & & & \psi_n \\ & \ddots & & & & \iddots & \\ & & \varphi_2 & & \psi_2 & & \\ & & & \zeta_1 & & & \\ & & \psi_2 & & \varphi_2 & & \\ & \iddots & & & & \ddots & \\ \psi_n & & & & & & \varphi_n \end{pmatrix},$$

$$t^{\mathbf{B}_n} = \begin{pmatrix} \varphi_n & & & & & & & \psi_n \\ & \ddots & & & & & \iddots & \\ & & \varphi_2 & & & \psi_2 & & \\ & & & \varphi_1 & \psi_1 & & & \\ & & & \psi_1 & \varphi_1 & & & \\ & & \psi_2 & & & \varphi_2 & & \\ & \iddots & & & & & \ddots & \\ \psi_n & & & & & & & \varphi_n \end{pmatrix},$$

where, $\zeta_1 = e^{\alpha_1(\ln t)}$*, and for* $i = 1, \ldots, n$*, we have*

$$\varphi_i = e^{\alpha_i(\ln t)} \cosh(\beta_i(\ln t)),$$

$$\psi_i = e^{\alpha_i(\ln t)} \sinh(\beta_i(\ln t)).$$

Proof. The proof is similar to that of Theorem 3 and omitted here. □

Example 2. Let $\mathbf{A}$ be the 4×4 matrix [8]:

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{\pi}{2} & 0 & \pi \\ -\frac{\pi}{2} & 1 & -\pi & 0 \\ 0 & 0 & 1 & \frac{5\pi}{2} \\ 0 & 0 & -\frac{5\pi}{2} & 1 \end{pmatrix}.$$

It can be observed that

$$e^{\mathbf{A}+\mathbf{A}^T} = \begin{pmatrix} e^2\cosh(\pi) & 0 & 0 & e^2\sinh(\pi) \\ 0 & e^2\cosh(\pi) & -e^2\sinh(\pi) & 0 \\ 0 & -e^2\sinh(\pi) & e^2\cosh(\pi) & 0 \\ e^2\sinh(\pi) & 0 & 0 & e^2\cosh(\pi) \end{pmatrix}$$

$$= \begin{pmatrix} 85.6536 & 0 & 0 & 85.3343 \\ 0 & 85.6536 & -85.3343 & 0 \\ 0 & -85.3343 & 85.6536 & 0 \\ 85.3343 & 0 & 0 & 85.6536 \end{pmatrix}.$$

Notice that $e^{\mathbf{A}+\mathbf{A}^T}$ is a central symmetric X-form matrix. Furthermore, by employing Theorem 5, we obtain

$$t^{\mathbf{A}+\mathbf{A}^T} = \begin{pmatrix} \frac{t^{2+\pi}+t^{2-\pi}}{2} & 0 & 0 & \frac{t^{2+\pi}-t^{2-\pi}}{2} \\ 0 & \frac{t^{2+\pi}+t^{2-\pi}}{2} & -\frac{t^{2+\pi}-t^{2-\pi}}{2} & 0 \\ 0 & -\frac{t^{2+\pi}-t^{2-\pi}}{2} & \frac{t^{2+\pi}+t^{2-\pi}}{2} & 0 \\ \frac{t^{2+\pi}-t^{2-\pi}}{2} & 0 & 0 & \frac{t^{2+\pi}+t^{2-\pi}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} t^2\cosh(\pi\ln t) & 0 & 0 & t^2\sinh(\pi\ln t) \\ 0 & t^2\cosh(\pi\ln t) & -t^2\sinh(\pi\ln t) & 0 \\ 0 & -t^2\sinh(\pi\ln t) & t^2\cosh(\pi\ln t) & 0 \\ t^2\sinh(\pi\ln t) & 0 & 0 & t^2\cosh(\pi\ln t) \end{pmatrix}.$$

In what follows, we give upper bound for the matrices $t^{\mathbf{A}_n}$ and $t^{\mathbf{B}_n}$. Since $t^a = e^{a\ln t}$, let $\Phi(\mathbf{A}_n) = \max_{2\le i\le n}\{\alpha_i + \beta_i, \alpha_1\}$, then

$$\|t^{\mathbf{A}_n}\|_\infty = \|e^{\mathbf{A}_n \ln t}\| \le t^{\alpha(\mathbf{A})} \sum_{k=0}^{n-1} \frac{(\Phi(\mathbf{A}_n)\sqrt{n}\ln t)^k}{k!}, \quad t \ge 1,$$

$$\|t^{\mathbf{A}_n}\|_\infty = \|e^{-\mathbf{A}_n(-\ln t)}\| \le t^{\beta(\mathbf{A})} \sum_{k=0}^{n-1} \frac{(\Phi(\mathbf{A}_n)\sqrt{n}\ln t)^k}{k!}, \quad 0 < t \le 1.$$

Moreover, let $\Psi(\mathbf{B}_n) = \max_{1\le i\le n}\{\alpha_i + \beta_i\}$, we then have

$$\|t^{\mathbf{B}_n}\|_\infty \le t^{\alpha(\mathbf{A})} \sum_{k=0}^{n-1} \frac{(\Psi(\mathbf{B}_n)\sqrt{n}\ln t)^k}{k!}, \quad t \ge 1,$$

$$\|t^{\mathbf{B}_n}\|_\infty \le t^{\beta(\mathbf{A})} \sum_{k=0}^{n-1} \frac{(\Psi(\mathbf{B}_n)\sqrt{n}\ln t)^k}{k!}, \quad 0 < t \le 1.$$

The results is also valid for 1-norm.

3.3 Explicit formula for $a^{\mathbf{A}t}$

In calculus, it is known that $a^t = e^{t\ln a}$. Then for the matrix $\mathbf{A}$, we can write

$$a^{\mathbf{A}t} = e^{(\ln a)\mathbf{A}t}, \quad a > 0.$$

In the next theorem, some valuable properties of $a^{\mathbf{A}t}$ will be characterized. They can be easily proved by some properties of the exponential functions. Thus, we omited the proof.

Theorem 6. *Let $\mathbf{A}$ and $\mathbf{B}$ be two square matrices and $a > 0$. Then the following properties hold:*

1. $a^{\mathbf{0}_{n\times n}} = \mathbf{I}_n$ *and* $a^{\mathbf{I}_n t} = a^t\mathbf{I}_n$.
2. *If $\mathbf{P}$ is nonsingular matrix, then* $a^{\mathbf{PAP}^{-1}} = \mathbf{P}a^{\mathbf{A}}\mathbf{P}^{-1}$.
3. $a^{(\mathbf{A}+\mathbf{B})t} = a^{\mathbf{A}t}a^{\mathbf{B}t}$, *provided* $\mathbf{AB} = \mathbf{BA}$.
4. $a^{\mathbf{A}(t+s)} = a^{\mathbf{A}t}a^{\mathbf{A}s}$.
5. $a^{-\mathbf{A}t}a^{\mathbf{A}t} = a^{\mathbf{A}t}a^{-\mathbf{A}t} = \mathbf{I}_n$.
6. $(a^{\mathbf{A}})^H = a^{\mathbf{A}^H}$, *it follows that if $\mathbf{A}$ is Hermitian, then $a^{\mathbf{A}}$ is Hermitian too, and if $\mathbf{A}$ is skew-Hermitian, then $a^{\mathbf{A}}$ is unitary.*
7. $(a^{\mathbf{A}})^T = a^{\mathbf{A}^T}$, *it follows that if $\mathbf{A}$ is symmetric, then $a^{\mathbf{A}}$ is symmetric too, and if $\mathbf{A}$ is skew-symmetric, then $a^{\mathbf{A}}$ is orthogonal.*
8. $\det(a^{\mathbf{A}}) = exp((\ln a)\,Tr(\mathbf{A}))$.
9. $\frac{d}{dt}(a^{\mathbf{A}t}) = \ln(a)\mathbf{A}a^{\mathbf{A}t}$.
10. $a^{\mathbf{A}\otimes\mathbf{I}_n} = a^{\mathbf{A}} \otimes a\mathbf{I}_n$, *and* $a^{\mathbf{I}_n\otimes\mathbf{B}} = a\mathbf{I}_n \otimes a^{\mathbf{B}}$.
11. $a^{\mathbf{A}\oplus\mathbf{B}} = a(a^{\mathbf{A}} \otimes a^{\mathbf{B}})$.

Consequently, in the following theorem, we compute $a^{\mathbf{A}_n t}$ and $a^{\mathbf{B}_n t}$:

Theorem 7. *Let* $\mathbf{A}_n \in \mathbb{C}^{(2n-1)\times(2n-1)}$ *and* $\mathbf{B}_n \in \mathbb{C}^{(2n)\times(2n)}$ *be central-symmetric* X*-form matrices. Then, the functions* $a^{\mathbf{A}_n t}$ *and* $a^{\mathbf{B}_n t}$ *are computed by*

$$a^{\mathbf{A}_n t} = \begin{pmatrix} \varphi_n & & & & & & \psi_n \\ & \ddots & & & & ⋰ & \\ & & \varphi_2 & & \psi_2 & & \\ & & & \zeta_1 & & & \\ & & \psi_2 & & \varphi_2 & & \\ & ⋰ & & & & \ddots & \\ \psi_n & & & & & & \varphi_n \end{pmatrix},$$

$$a^{\mathbf{B}_n t} = \begin{pmatrix} \varphi_n & & & & & & & \psi_n \\ & \ddots & & & & & ⋰ & \\ & & \varphi_2 & & & \psi_2 & & \\ & & & \varphi_1 & \psi_1 & & & \\ & & & \psi_1 & \varphi_1 & & & \\ & & \psi_2 & & & \varphi_2 & & \\ & ⋰ & & & & & \ddots & \\ \psi_n & & & & & & & \varphi_n \end{pmatrix},$$

wherein, $\zeta_1 = e^{\alpha_1 t(\ln a)}$, *and for* $i = 1, \ldots, n$*:*

$$\varphi_i = e^{\alpha_i t(\ln a)} \cosh(\beta_i t(\ln a)),$$

$$\psi_i = e^{\alpha_i t(\ln a)} \sinh(\beta_i t(\ln a)).$$

Proof. It can be proved similar to Theorem 3 and omitted here. □

Example 3. Considering matrix $\mathbf{A}$ in Example 2, after application and simplification of relations, we obtain

$$a^{\mathbf{A}+\mathbf{A}^T} = \begin{pmatrix} \frac{a^{2t+\pi t}+a^{2t-\pi t}}{2} & 0 & 0 & \frac{a^{2t+\pi t}-a^{2t-\pi t}}{2} \\ 0 & \frac{a^{2t+\pi t}+a^{2t-\pi t}}{2} & -\frac{a^{2t+\pi t}-a^{2t-\pi t}}{2} & 0 \\ 0 & -\frac{a^{2t+\pi t}-a^{2t-\pi t}}{2} & \frac{a^{2t+\pi t}+a^{2t-\pi t}}{2} & 0 \\ \frac{a^{2t+\pi t}-a^{2t-\pi t}}{2} & 0 & 0 & \frac{a^{2t+\pi t}+a^{2t-\pi t}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} a^{2t}\cosh(\pi t \ln a) & 0 & 0 & a^{2t}\sinh(\pi t \ln a) \\ 0 & a^{2t}\cosh(\pi t \ln a) & -a^{2t}\sinh(\pi t \ln a) & 0 \\ 0 & -a^{2t}\sinh(\pi t \ln a) & a^{2t}\cosh(\pi t \ln a) & 0 \\ a^{2t}\sinh(\pi t \ln a) & 0 & 0 & a^{2t}\cosh(\pi t \ln a) \end{pmatrix},$$

which is central symmetric X-form matrix.

4 Applications

This section is devoted to applications of the exponential function of a square matrix. In order to achieve this purpose, we consider the central-symmetric X form matrices.

Example 4. Consider the initial value problem

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t),$$

whenever $\mathbf{y}(0) = (1, \ldots, 1)^t \in \mathbb{R}^{202}$ and $\mathbf{A}$ is 202×202 central-symmetric X-form matrix as follows:

$$\mathbf{A} = \begin{pmatrix} 201 & & & & & & & & 200 \\ & \ddots & & & & & & ⋰ & \\ & & 5 & & & & 4 & & \\ & & & 3 & & 2 & & & \\ & & & & 1 \quad 1 & & & & \\ & & & & 1 \quad 1 & & & & \\ & & & 2 & & 3 & & & \\ & & 4 & & & & 5 & & \\ & ⋰ & & & & & & \ddots & \\ 200 & & & & & & & & 201 \end{pmatrix},$$

Thus, according to Theorem 3, we obtain the following solution:

$$\mathbf{y}(t) = \begin{pmatrix} G_{201}^{200} & & & & & & H_{201}^{200} \\ & \ddots & & & & ⋰ & \\ & & G_3^2 & & H_3^2 & & \\ & & & G_1^1 \quad H_1^1 & & & \\ & & & H_1^1 \quad G_1^1 & & & \\ & & H_3^2 & & G_3^2 & & \\ & ⋰ & & & & \ddots & \\ H_{201}^{200} & & & & & & G_{201}^{200} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

where

$$G_\alpha^\beta = e^{\alpha t}\cosh(\beta t), \quad H_\alpha^\beta = e^{\alpha t}\sinh(\beta t).$$

After multiplication and simplification of the last expression, we have:

$$\mathbf{y}(t) = \left(e^{401t}, \ldots, e^{5t}, e^{2t}, e^{2t}, e^{5t}, \ldots, e^{401t}\right)^t.$$

Example 5. In this example we illustrate the computation of the Gamma and Beta matrix functions. In order to reach this aim, consider 3×3 central-symmetric X-form matrices

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Hence, for $0 < t < 1$, it is concluded that

$$t^{\mathbf{A}-\mathbf{I}} = t^{\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}} = \exp\begin{pmatrix} 2\ln t & 0 & \ln t \\ 0 & \ln t & 0 \\ \ln t & 0 & 2\ln t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{t(t^2+1)}{2} & 0 & \frac{t(t^2-1)}{2} \\ 0 & t & 0 \\ \frac{t(t^2-1)}{2} & 0 & \frac{t(t^2+1)}{2} \end{pmatrix},$$

$$(1-t)^{\mathbf{B}-\mathbf{I}} = (1-t)^{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}}$$

$$= \exp\begin{pmatrix} \ln(1-t) & 0 & \ln(1-t) \\ 0 & 4\ln(1-t) & 0 \\ \ln(1-t) & 0 & \ln(1-t) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{(t-1)^2+1}{2} & 0 & \frac{(t-1)^2+1}{2} \\ 0 & (t-1)^4 & 0 \\ \frac{(t-1)^2+1}{2} & 0 & \frac{(t-1)^2+1}{2} \end{pmatrix}.$$

Consequently, we obtain

$$\Gamma(\mathbf{A}) = \int_0^1 e^{-t} \begin{pmatrix} \frac{t(t^2+1)}{2} & 0 & \frac{t(t^2-1)}{2} \\ 0 & t & 0 \\ \frac{t(t^2-1)}{2} & 0 & \frac{t(t^2+1)}{2} \end{pmatrix} dt$$

$$= \begin{pmatrix} \frac{7}{2} - 9e^{-1} & 0 & \frac{5}{2} - 7e^{-1} \\ 0 & 1 - 2e^{-1} & 0 \\ \frac{5}{2} - 7e^{-1} & 0 & \frac{7}{2} - 9e^{-1} \end{pmatrix},$$

and

$$\mathcal{B}(\mathbf{A},\mathbf{B}) = \int_0^1 \begin{pmatrix} \frac{t(t^2+1)}{2} & 0 & \frac{t(t^2-1)}{2} \\ 0 & t & 0 \\ \frac{t(t^2-1)}{2} & 0 & \frac{t(t^2+1)}{2} \end{pmatrix} \begin{pmatrix} \frac{(t-1)^2+1}{2} & 0 & \frac{(t-1)^2+1}{2} \\ 0 & (t-1)^4 & 0 \\ \frac{(t-1)^2+1}{2} & 0 & \frac{(t-1)^2+1}{2} \end{pmatrix} dt$$

$$= \int_0^1 \begin{pmatrix} \frac{t(t^3-2t^3+t^2-1)}{2} & 0 & \frac{t(t^3-2t^3+t^2+1)}{2} \\ 0 & t(t-1)^4 & 0 \\ \frac{t(t^3-2t^3+t^2+1)}{2} & 0 & \frac{t(t^3-2t^3+t^2-1)}{2} \end{pmatrix} dt$$

$$= \begin{pmatrix} \frac{31}{120} & 0 & -\frac{29}{120} \\ 0 & \frac{1}{30} & 0 \\ -\frac{29}{120} & 0 & \frac{31}{120} \end{pmatrix}.$$

It should be mentioned that $\Gamma(\mathbf{A})$ and $\mathcal{B}(\mathbf{A},\mathbf{B})$ are also central symmetric X-form matrices.

5 Conclusions

In this paper, we explored various properties of the especial forms of matrices that are called central symmetric X-form matrices. It could be observed that the proposed class of matrices are not particular case of centrosymmetric matrices. In spite of the existence of many procedures that can compute the exponential of a matrix, according to our knowledge there is not any relation which can evaluate the matrix exponential of central symmetric X-form matrices explicitly. The most important merit of the proposed formulas are avoiding the computation of Jordan form or Schur decomposition of matrices. Moreover, upper bounds of the exponential approximations are given with details by using $\|e^{\mathbf{A}t}\|$. At the end, an initial value problem and also Gamma and Beta matrix functions by central symmetric X-form matrices have shown in examples.

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