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Spline Collocation for system of Fredholm and Volterra integro-differential equations

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Abstract. The spline collocation method is employed to solve a system of linear and nonlinear Fredholm and Volterra integro-differential equations. The solutions are collocated by cubic B-spline and the integrand is approximated by the Newton-Cotes formula. We obtain the unique solution for linear and nonlinear system $(nN + 3n) \times (nN + 3n)$ of integro-differential equations. This approximation reduces the system of integro-differential equations to an explicit system of algebraic equations. At the end, some examples are presented to illustrate the ability and simplicity of the method.

Keywords: System of Fredholm and Volterra integro-differential equations, Cubic B-spline, Newton-Cotes formula, Convergence analysis.
AMS Subject Classification: 41A15, 65R20.

1 Introduction

In this paper a spline collocation procedure is developed for the numerical solution of system of linear and nonlinear integro-differential equations of the Fredholm type

$$\sum_{r=0}^{m} Y^{(r)}(t)P_{jr}(t) + \int_a^b K_j(t,x,Y(x))dx + \psi_j(t,Y(t)) = g_j(t), \quad j = 1,\dots,n, \quad m \le 2, \quad t \in [a,b], \tag{1}$$

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and Volterra type

$$\sum_{r=0}^{m} Y^{(r)}(t)P_{jr}(t) + \int_a^t K_j(t,x,Y(x))dx + \psi_j(t,Y(t)) \; = \; g_j(t), \quad j = 1,\dots,n, \\ m \le 2, \quad t \in [a,b], \tag{2}$$

with boundary conditions

$$\sum_{r=0}^{m-1} [\alpha_{djr} y_j^{(r)}(a) + \beta_{djr} y_j^{(r)}(b)] = \gamma_{dj}, \quad j = 1,\dots,n, \; d = 0,\dots,m-1, \tag{3}$$

where $P_{jr}(t) = [p_{1jr}(t),\dots,p_{njr}(t)]^T$, and $\alpha_{djr}, \beta_{djr}$ and γ_{dj} are given real constants. The given K_j and ψ_j, are continuous and satisfy a uniform Lipschitz on $[a,b]$. $Y(t) = [y_1(t),\dots,y_n(t)]$ is unknown function and $g_j(t)$ and $P_{jr}(t)$ are the known functions. Boundary value problems of systems of nonlinear integro-differential equations have various practical applications in scientific fields such as population and polymer rheology [2, 10]. Several authors have proposed numerical methods to approximate the solutions of linear and nonlinear Fredholm and Volterra integro-differential equations, such as the sinc-collocation method [12, 15, 19], the variational iteration method [3, 18], the homotopy perturbation method [4], the formulation of the piecewise Tau method [1, 6, 8]. A simple operational approach, using the Adomian decomposition method, has been proposed for the numerical solution of systems of nonlinear Volterra integro-differential equations in [14]. This method leads to a system of linear algebraic equations. A global approximation to the solution of Fredholm and Volterra integral equation is constructed by means of the spline quadrature in [9, 11, 16, 17].

In this article, we consider the equations (1) and (2) with $n = 2$ and use the cubic B-spline collocation method to approximate the unknown function $Y(t) = [y_1(t), y_2(t)]$, and then apply the Newton-Cotes rule to approximate the obtained system of linear and nonlinear Fredholm and Volterra integro-differential equations of second kind. At the end, some examples are presented to illustrate the ability and simplicity of the method.

2 Cubic B-spline

We introduce the cubic B-spline space and basis functions to construct an interpolant $S_N = [s_1, s_2]$ to be used in the formulation of the cubic B-spline collocation method. Let $\pi = \{a = t_0 < t_1 < \cdots < t_N = b\}$ be a uniform partition of the interval $[a,b]$ with step size $h = \frac{b-a}{N}$. The cubic B-spline space is denoted by

$$S_3(\pi) = \left\{ s_i \in C^2[a,b] : s_i \mid_{[t_k,t_{k+1}]} \in P_3, \quad k = 0,1,\ldots,N, \quad i = 1,2 \right\},$$

where P_3 is the class of cubic polynomials. Following [13] we can define a cubic B-spline $s_i(t)$ of the form

$$s_i(t) = \sum_{k=-1}^{N+1} c_{k,i} \beta_k^3(t), \tag{4}$$

where

$$\beta_k^3(t) = \frac{1}{6h^3} \begin{cases} (t-t_{k-2})^3, & t \in [t_{k-2}, t_{k-1}], \\ h^3 + 3h^2(t-t_{k-1}) + 3h(t-t_{k-1})^2 - 3(t-t_{k-1})^3, & t \in [t_{k-1}, t_k], \\ h^3 + 3h^2(t_{k+1}-t) + 3h(t_{k+1}-t)^2 - 3(t_{k+1}-t)^3, & t \in [t_k, t_{k+1}], \\ (t_{k+2}-t)^3, & t \in [t_{k+1}, t_{k+2}], \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

satisfying the following interpolatory conditions

$$s_i(t_k) = y_i(t_k), \quad 0 \le k \le N, \quad i = 1,2,$$

and the boundary conditions

$$\mathcal{C}_1 : \quad s_i^{'}(t_0) = y_i^{'}(t_0), \quad s_i^{'}(t_N) = y_i^{'}(t_N), \quad i = 1,2,$$

or

$$\mathcal{C}_2 : \quad D^m s_i(t_0) = D^m s_i(t_N), \quad i = 1,2, \quad m = 1,2,$$

or

$$\mathcal{C}_3 : \quad s_i^{''}(t_0) = 0, \quad s_i^{''}(t_N) = 0, \quad i = 1,2. \tag{6}$$

3 The Collocation Method

3.1 Nonlinear Fredholm integro-differential equations system

In the given nonlinear Fredholm integro-differential Eq. (1) for $n = 2$, we can approximate the unknown function by cubic B-spline (4) as follows

$$\begin{cases} \displaystyle\sum_{r=0}^{m} \left(s_1^{(r)}(t) p_{11r}(t) + s_2^{(r)}(t) p_{21r}(t) \right) + \int_a^b K_1(t,x,s_1(x),s_2(x))dx \\ \qquad\qquad + \psi_1(t, s_1(t), s_2(t)) = g_1(t), \\ \displaystyle\sum_{r=0}^{m} \left(s_1^{(r)}(t) p_{12r}(t) + s_2^{(r)}(t) p_{22r}(t) \right) + \int_a^b K_2(t,x,s_1(x),s_2(x))dx \\ \qquad\qquad + \psi_2(t, s_1(t), s_2(t)) = g_2(t), \end{cases} \tag{7}$$

for $t \in [a,b]$ and $m \leq 2$. We now collocate Eq. (7) at collocation points $t_k = a + kh$, $h = (b-a)/N$, $k = 0, 1, \ldots, N$, and obtain

$$\begin{cases} \displaystyle\sum_{r=0}^{m} \Big(s_1^{(r)}(t_k) p_{11r}(t_k) + s_2^{(r)}(t_k) p_{21r}(t_k) \Big) + \int_a^b K_1(t_k, x, s_1(x), s_2(x)) dx \\ \qquad\qquad + \psi_1(t_k, s_1(t_k), s_2(t_k)) = g_1(t_k), \\ \displaystyle\sum_{r=0}^{m} \Big(s_1^{(r)}(t_k) p_{12r}(t_k) + s_2^{(r)}(t_k) p_{22r}(t_k) \Big) + \int_a^b K_2(t_k, x, s_1(x), s_2(x)) dx \\ \qquad\qquad + \psi_2(t_k, s_1(t_k), s_2(t_k)) = g_2(t_k), \end{cases} \tag{8}$$

for $k = 0, 1, \ldots, N$, and $m \leq 2$. To approximate the integro-differential Eq. (8), we use the Newton- Cotes formula (Simpson rule or Simpson's 3/8 rule) [5], then we get the following nonlinear system

$$\begin{cases} \displaystyle\sum_{r=0}^{m} \Big(s_1^{(r)}(t_k) p_{11r}(t_k) + s_2^{(r)}(t_k) p_{21r}(t_k) \Big) + h \sum_{i=0}^{N} w_{k,i} K_1(t_k, x_i, s_1(x_i), s_2(x_i)) \\ \qquad\qquad + \psi_1(t_k, s_1(t_k), s_2(t_k)) = g_1(t_k), \\ \displaystyle\sum_{r=0}^{m} \Big(s_1^{(r)}(t_k) p_{12r}(t_k) + s_2^{(r)}(t_k) p_{22r}(t_k) \Big) + h \sum_{i=0}^{N} w_{k,i} K_2(t_k, x_i, s_1(x_i), s_2(x_i)) \\ \qquad\qquad + \psi_2(t_k, s_1(t_k), s_2(t_k)) = g_2(t_k), \end{cases} \tag{9}$$

for $k = 0, 1, \ldots, N$, with the boundary conditions,

$$\sum_{r=0}^{m-1} [\alpha_{djr} y_j^{(r)}(a) + \beta_{djr} y_j^{(r)}(b)] = \gamma_{dj}, \quad j = 1, 2, \quad d = 0, \ldots, m-1,$$

where $x_i = a + ih$, $i = 0, 1, \ldots, N$. We need more equations to obtain the unique solution for Eq. (9). Hence by associating Eq. (9) with (6), we get the following $(nN + 3n) \times (nN + 3n)$ nonlinear system (with $n = 2$)

$$\begin{cases} \displaystyle\sum_{r=0}^{m} (s_1^{(r)}(t_k) p_{11r}(t_k) + s_2^{(r)}(t_k) p_{21r}(t_k)) + h \sum_{i=0}^{N} w_{k,i} K_1(t_k, x_i, s_1(x_i), s_2(x_i)) \\ \qquad\qquad + \psi_1(t_k, s_1(t_k), s_2(t_k)) = g_1(t_k), \quad k = 0, 1, \ldots, N, \\ \displaystyle\sum_{r=0}^{m} (s_1^{(r)}(t_k) p_{12r}(t_k) + s_2^{(r)}(t_k) p_{22r}(t_k)) + h \sum_{i=0}^{N} w_{k,i} K_2(t_k, x_i, s_1(x_i), s_2(x_i)) \\ \qquad\qquad + \psi_2(t_k, s_1(t_k), s_2(t_k)) = g_2(t_k), \quad k = 0, 1, \ldots, N, \\ \displaystyle\sum_{r=0}^{m-1} [\alpha_{djr} y_j^{(r)}(a) + \beta_{djr} y_j^{(r)}(b)] = \gamma_{dj}, \quad j = 1, 2, \quad d = 0, \ldots, m-1, \\ D^m s_j(t_0) = D^m s_j(t_N), \qquad m = 1, 2, \quad j = 1, 2, \end{cases} \tag{10}$$

where $w_{k,i}$'s represent the weights for a quadrature rule of Newton-Cotes type. By solving the above nonlinear system, we can determine the coefficients in Eq. (4) and by setting coefficients in (4), we obtain the approximate solutions for Eq. (1).

3.2 Nonlinear Volterra integro-differential equations system

Now we consider the system of nonlinear Volterra integro-differential equations

$$\begin{cases} \sum_{r=0}^{m} \left(y_1^{(r)}(t)p_{11r}(t) + y_2^{(r)}(t)p_{21r}(t) \right) + \int_a^t K_1(t,x,y_1(x),y_2(x))dx \\ \qquad\qquad +\psi_1(t,y_1(t),y_2(t)) = g_1(t), \\ \sum_{r=0}^{m} \left(y_1^{(r)}(t)p_{12r}(t) + y_2^{(r)}(t)p_{22r}(t) \right) + \int_a^t K_2(t,x,y_1(x),y_2(x))dx \\ \qquad\qquad +\psi_2(t,y_1(t),y_2(t)) = g_2(t), \end{cases} \tag{11}$$

where $x,t \in [a,b]$ and $m \leq 2$, with the boundary conditions

$$\sum_{r=0}^{m-1} [\alpha_{djr} y_j^{(r)}(a) + \beta_{djr} y_j^{(r)}(b)] = \gamma_{dj}, \quad j = 1,2, \quad d = 0,\dots,m-1.$$

We replace the solutions of Eq. (11) by the cubic B-spline and by collocating Eq. (11) at collocation points $t_k = a + kh$, $h = (t-a)/N$, $k = 0,1,\dots,N$, we get

$$\begin{cases} \sum_{r=0}^{m} \left(s_1^{(r)}(t_k)p_{11r}(t_k) + s_2^{(r)}(t_k)p_{21r}(t_k) \right) + \int_a^{t_k} K_1(t_k,x,s_1(x),s_2(x))dx \\ \qquad\qquad +\psi_1(t_k,s_1(t_k),s_2(t_k)) = g_1(t_k), \\ \sum_{r=0}^{m} \left(s_1^{(r)}(t_k)p_{12r}(t_k) + s_2^{(r)}(t_k)p_{22r}(t_k) \right) + \int_a^{t_k} K_2(t_k,x,s_1(x),s_2(x))dx \\ \qquad\qquad +\psi_2(t_k,s_1(t_k),s_2(t_k)) = g_2(t_k), \end{cases} \tag{12}$$

for $k = 1,\dots,N$, and $m \leq 2$. To approximate the integral Eq.(12), we use the Newton- Cotes formula (Simpson rule or Simpson's 3/8 rule), then get the following nonlinear system

$$\begin{cases} \sum_{r=0}^{m} \left(s_1^{(r)}(t_k)p_{11r}(t_k) + s_2^{(r)}(t_k)p_{21r}(t_k) \right) + h\sum_{i=0}^{k} w_{k,i} K_1(t_k,x_i,s_1(x_i),s_2(x_i)) \\ \qquad\qquad +\psi_1(t_k,s_1(t_k),s_2(t_k)) = g_1(t_k), \\ \sum_{r=0}^{m} \left(s_1^{(r)}(t_k)p_{12r}(t_k) + s_2^{(r)}(t_k)p_{22r}(t_k) \right) + h\sum_{i=0}^{k} w_{k,i} K_2(t_k,x_i,s_1(x_i),s_2(x_i)) \\ \qquad\qquad +\psi_2(t_k,s_1(t_k),s_2(t_k)) = g_2(t_k), \end{cases} \tag{13}$$

where $k = 1,\dots,N$, with the boundary conditions,

$$\sum_{r=0}^{m-1} [\alpha_{djr} y_j^{(r)}(a) + \beta_{djr} y_j^{(r)}(b)] = \gamma_{dj}, \quad j = 1,2, \quad d = 0,\dots,m-1.$$

We need more equations to obtain the unique solution for Equation (13). Hence, by associating Equation (13) with (6) we obtain the following $(2N+$

6) $\times$ (2N + 6) nonlinear system

$$\begin{cases} \sum_{r=0}^{m}(s_1^{(r)}(t_k)p_{11r}(t_k)+s_2^{(r)}(t_k)p_{21r}(t_k))+h\sum_{i=0}^{k}w_{k,i}K_1(t_k,x_i,s_1(x_i),s_2(x_i)) \\ \qquad +\psi_1(t_k,s_1(t_k),s_2(t_k))=g_1(t_k), \quad k=1,\ldots,N, \\ \sum_{r=0}^{m}(s_1^{(r)}(t_k)p_{12r}(t_k)+s_2^{(r)}(t_k)p_{22r}(t_k))+h\sum_{i=0}^{k}w_{k,i}K_2(t_k,x_i,s_1(x_i),s_2(x_i)) \\ \qquad +\psi_2(t_k,s_1(t_k),s_2(t_k))=g_2(t_k), \quad k=1,\ldots,N, \\ \sum_{r=0}^{m-1}[\alpha_{djr}y_j^{(r)}(a)+\beta_{djr}y_j^{(r)}(b)]=\gamma_{dj}, \quad j=1,2, \quad d=0,\ldots,m-1, \\ D^m s_j(t_0)=D^m s_j(t_N), \quad m=1,2, \quad j=1,2. \end{cases} \tag{14}$$

By solving the above nonlinear system, we can determine the coefficients in Eq. (4) and by setting coefficients in (4), we obtain the approximate solutions for Eq. (11).

4 Error analysis

In this section, we consider the error analysis of the system of nonlinear Volterra integro-differential equations of the second kind. Let $S_N = [s_1,\ldots,s_n]$ be the approximation of $Y=[y_1,\ldots,y_n]$. We firs recall the following definition from [13].

Definition 1. Let $s(t)$ be the cubic B-spline interpolates $y\in C^4[a,b]$, then for all admissible h, there exists a constant $M_j<\infty$, independent of h, such that

$$\|D^j(y-s)\|_2 \le M_j\|y^{(4)}\|_2 h^{4-j-1/2}, \quad j=0,\ldots,3 \ ,$$

where $M_j = 2/j!$, $j=0,\ldots,3$, and D^j is the j-th derivative. If $p = 4-j-1/2$ is the largest number for which such an inequality holds, then p is called the order of convergence of the method.

Theorem 1. *The approximate method*

$$\sum_{r=0}^{m} S_N^{(r)}(t_k)P_{jr}(t_k)+h\sum_{i=0}^{k}w_{k,i}K_j(t_k,x_i,S_N(x_i))+\psi_j(t_k,S_N(t_k)) \;=\; g_j(t_k), \\ j=1,\ldots,n, \quad k=1,\ldots,N, \quad m\le 2, \tag{15}$$

for the solution of the system of nonlinear Volterra integro-differential Eq. (2) is converges and the error bounded is

$$\|E_{N_k}^{(m)}\| \le \frac{1}{|P_{jm}(t_k)|}\sum_{r=0}^{m-1}|E_{N_k}^{(r)}||P_{jr}(t_k)| + \frac{hWL_j}{|P_{jm}(t_k)|}\sum_{i=0}^{k}|E_{N_i}| + \frac{L_j^*|E_{N_k}|}{|P_{jm}(t_k)|} \\ + \frac{|E_j(h,t_k)|}{|P_{jm}(t_k)|}.$$

Proof. We know that at $t_k = a + kh$, $h = (t-a)/N$, $k = 1, \ldots, N$, the corresponding approximate solution for the system of nonlinear Volterra integro-differential equation (2) is

$$\sum_{r=0}^{m} S_N^{(r)}(t_k)P_{jr}(t_k) + h\sum_{i=0}^{k} w_{k,i}K_j(t_k, x_i, S_N(x_i)) + \psi_j(t_k, S_N(t_k)) = g_j(t_k),$$
$$j = 1, \ldots, n, \quad k = 1, \ldots, N, \quad m \leq 2. \tag{16}$$

By discretizing (2) and approximating the integrand by the Newton-Cotes formula, we obtain

$$\sum_{r=0}^{m} Y^{(r)}(t_k)P_{jr}(t_k) + h\sum_{i=0}^{k} w_{k,i}K_j(t_k, x_i, Y(x_i)) + \psi_j(t_k, Y(t_k)) = g_j(t_k)$$
$$+E_j(h, t_k), \quad j = 1, \ldots, n, \quad k = 1, \ldots, N, \tag{17}$$

where

$$E_j(h, t_k) = \int_a^{t_k} K_j(t_k, x, Y(x))dx - h\sum_{i=0}^{k} w_{k,i}K_j(t_k, x_i, Y(x_i)),$$

$j = 1, \ldots, n$. By subtracting (17) from (16) and using interpolatory conditions of cubic B-spline, we get

$$\sum_{r=0}^{m}[S_N^{(r)}(t_k) - Y^{(r)}(t_k)]P_{jr}(t_k) + h\sum_{i=0}^{k} w_{k,i}[K_j(t_k, x_i, S_N(x_i) - K_j(t_k, x_i, Y(x_i))]$$
$$+[\psi_j(t_k, S_N(t_k)) - \psi_j(t_k, Y(t_k))] = -E_j(h, t_k),$$
$$k = 1, \ldots, N, \quad j = 1, \ldots, n, \quad m \leq 2.$$

Let $W = \max_{i,k}|w_{k,i}|$ and $S_N^{(r)}(t_k) = S_{N_k}^{(r)}$, $Y^{(r)}(t_k) = Y_k^{(r)}$, $k = 1, \ldots, N$, and suppose that K_j, ψ_j, $j = 1, \ldots, n$, satisfy a Lipschitz condition in its third argument of the form

$$|K_j(t, x, S_N) - K_j(t, x, Y)| \leq L_j|S_N - Y|, \qquad |\psi_j(t, S_N) - \psi_j(t, Y)| \leq L_j^*|S_N - Y|,$$

where L_j and L_j^* are independent of t, x, S_N and Y. Then, we get

$$|S_{N_k}^{(m)} - Y_k^{(m)}||P_{jm}(t_k)| \leq \sum_{r=0}^{m-1}|S_{N_k}^{(r)} - Y_k^{(r)}||P_{jr}(t_k)| + hWL_j\sum_{i=0}^{k}|S_N(x_i) - Y(x_i)|$$
$$+L_j^*|S_{N_k} - Y_k| + |E_j(h, t_k)|.$$

Since $|P_{jm}(t_k)| \neq 0$, then we have

$$\|E_{N_k}^{(m)}\| \leq \frac{1}{|P_{jm}(t_k)|}\sum_{r=0}^{m-1}|E_{N_k}^{(r)}||P_{jr}(t_k)| + \frac{hWL_j}{|P_{jm}(t_k)|}\sum_{i=0}^{k}|E_{N_i}| + \frac{L_j^*|E_{N_k}|}{|P_{jm}(t_k)|}$$
$$+\frac{|E_j(h, t_k)|}{|P_{jm}(t_k)|},$$

where $E_{N_k}^{(r)} = S_{N_k}^{(r)} - Y_k^{(r)}, \ \ k = 1, \ldots, N$. Now, since by assumption both the quadrature error and the function approximate error are zero in the limit , then $\lim \max |E_j(h, t_k)| = 0$, when $h \to 0$. Therefore, the second term in the previous equation is zero and the first and third terms tend to zero due to interpolating $Y(t)$ by cubic B-spline. Thus for a fixed k, we get

$$|E_{N_k}^{(m)}| \to 0 \ \text{ as } \ h \to 0, m \leq 2,$$

which completes the proof. □

5 Numerical examples

In order to test the applicability of the presented method, we consider four examples of the system of linear and nonlinear Volterra and Fredholm integro-differential equations with the boundary conditions. We solve them for several values of N and absolute errors are reported in Tables. The RMS error in the solutions

$$E_{RMS} = \sqrt{\frac{1}{N} \sum_{i=0}^{N} [s(x_i) - y(x_i)]^2},$$

is computed by our purposed method where $y(t)$ is the exact solution and $s(t)$ is the approximated solution of integral equation. All computations are performed using Mathematica version 8.

Example 1. *([14])* Consider the following linear Fredholm integro-differential equation with exact solution $y_1(t) = 3t^2 + 1$, $y_2(t) = t^3 + 2t - 1$,

$$\begin{aligned} y_1''(t) &= \frac{3t}{10} + 6 - \int_0^1 2xt(y_1(x) - 3y_2(x))dx, \\ y_2''(t) &= 15t + \frac{4}{5} - \int_0^1 3(2t + x^2)(y_1(x) - 2y_2(x))dx, \end{aligned}$$

with the boundary conditions $y_1(0) = 1$, $y_2(0) = -1$, $y_1(1) = 4$, and $y_2(1) = 2$. This system has been solved by our method with $N = 10, 30, 60$, the absolute errors at the particular grid points and the RMS errors are given in Table 1, which shows that the error in the solutions for our method decreases by reducing the values of h.

Table 1: The error $\|E\|$ for the solution of Example 1 at particular points.

t	$N = 10$	$N = 30$	$N = 60$
		Results for $y_1(x)$	
0	0	2.22E − 16	0
0.1	3.23E − 07	4.0E − 09	2.50E − 10
0.2	2.58E − 06	3.20E − 08	2.0E − 09
0.3	8.73E − 06	1.08E − 07	6.75E − 09
0.4	2.06E − 05	2.56E − 07	1.60E − 08
0.5	4.04E − 05	5.0E − 07	3.13E − 08
0.6	6.98E − 05	8.64E − 07	5.40E − 08
0.7	1.11E − 04	1.37E − 06	8.58E − 08
0.8	1.65E − 04	2.05E − 06	1.28E − 07
0.9	2.35E − 04	1.51E − 06	1.82E − 07
1	3.23E − 04	4.0E − 06	2.50E − 07
RMS error	1.22E − 04	2.92E − 06	9.73E − 08
		Results for $y_2(x)$	
0	0	2.22E − 16	0
0.1	8.40E − 06	1.04E − 07	6.50E − 09
0.2	3.70E − 05	4.58E − 07	2.86E − 08
0.3	9.08E − 05	1.12E − 06	7.03E − 08
0.4	1.75E − 04	2.17E − 06	1.35E − 07
0.5	2.95E − 04	3.65E − 06	2.28E − 07
0.6	4.55E − 04	5.63E − 06	3.52E − 07
0.7	6.61E − 04	8.18E − 06	5.11E − 07
0.8	9.17E − 04	1.14E − 05	7.09E − 07
0.9	1.23E − 03	8.11E − 06	9.51E − 07
1	1.60E − 03	1.98E − 05	1.24E − 06
RMS error	6.55E − 04	1.52E − 05	5.19E − 07

Example 2. *([7])* Consider the following linear Volterra integro-differential equation with exact solution $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$,

$$y_1''(t) = -1 - y_1(t) + \cos(t) + \int_0^t y_2(x)dx, \tag{18}$$

$$y_2''(t) = -y_2(t) + \sin(t) - \int_0^t y_1(x)dx \tag{19}$$

with the boundary conditions $y_1(0) = 1$, $y_2(0) = 0$, $y_1(1) = \cos(1)$ and $y_2(1) = \sin(1)$. The approximate solutions are calculated for different values of $N = 10, 30, 60$, the absolute errors at the particular grid points and the RMS errors are given in Table 2.

Table 2: The error $\|E\|$ in solution of Example 2 at particular points.

t	$N = 10$	$N = 30$	$N = 60$
	Results for $y_1(x)$		
0	2.22E − 16	5.55E − 15	2.26E − 14
0.1	4.15E − 06	4.62E − 07	1.15E − 07
0.2	1.65E − 05	1.84E − 06	4.59E − 07
0.3	3.68E − 05	4.10E − 06	1.03E − 06
0.4	6.48E − 05	7.21E − 06	1.80E − 06
0.5	9.97E − 05	1.11E − 05	2.77E − 06
0.6	1.41E − 04	1.57E − 05	3.92E − 06
0.7	1.87E − 04	2.01E − 05	6.65E − 06
0.8	2.39E − 04	2.66E − 05	7.28E − 06
0.9	2.93E − 04	2.91E − 05	8.17E − 06
1	3.51E − 04	3.25E − 05	9.77E − 06
RMS error	1.83E − 04	1.91E − 05	4.68E − 06
	Results for $y_2(x)$		
0	4.35E − 16	2.42E − 15	1.02E − 14
0.1	1.94E − 07	1.61E − 08	3.89E − 09
0.2	1.21E − 06	1.24E − 07	3.08E − 08
0.3	3.85E − 06	4.13E − 07	1.03E − 07
0.4	8.87E − 06	6.17E − 07	2.41E − 07
0.5	1.69E − 05	9.66E − 07	4.67E − 07
0.6	2.86E − 05	1.86E − 06	7.89E − 07
0.7	4.44E − 05	3.16E − 06	9.11E − 07
0.8	6.47E − 05	4.92E − 06	1.79E − 06
0.9	8.96E − 05	7.17E − 06	2.48E − 06
1	1.19E − 04	9.94E − 06	3.31E − 06
RMS error	5.44E − 05	5.46E − 06	1.33E − 06

Example 3. *([20])* Consider the following nonlinear Fredholm integro-differential equation

$$-y_1''(t)t + y_1'(t)\frac{t}{2} - y_1(t)e^{y_2(t)} + \int_0^1 (x+t)(y_1^2(x) + y_2^2(x))dx = g_1(t)$$
$$y_1'(t)\frac{-t}{3} - y_2''(t)t + y_2'(t) + \sin(y_1(t)) + \int_0^1 xt(y_1^2(x) - y_2^2(x))dx = g_2(t),$$

with the the boundary conditions $y_1(0) = 0$, $y_2(0) = 0$, $y_1(1) = 0$ and $y_2(1) = 0$, where $g_1(t)$ and $g_2(t)$ are chosen such that the exact solution is $y_1(t) = \sin(\pi t)$ and $y_2(t) = t^2 - t$. The approximate solutions are calculated for different values of $N = 10, 30, 60$, and the absolute errors at the particular grid points and the RMS errors are given in Table 3 .

Table 3: The error $\|E\|$ in solution of Example 3 at particular points

t	$N=10$	$N=30$	$N=60$
	Results for $y_1(x)$		
0.05	0	0	0
0.15	1.12E − 02	1.26E − 03	3.16E − 04
0.25	8.50E − 03	9.57E − 04	2.39E − 04
0.35	5.39E − 03	6.09E − 04	1.52E − 04
0.45	2.42E − 03	2.74E − 04	6.88E − 05
0.55	1.62E − 05	7.54E − 07	2.49E − 10
0.65	1.64E − 03	1.82E − 04	4.56E − 05
0.75	2.33E − 03	2.62E − 04	6.54E − 05
0.85	2.15E − 03	2.41E − 04	6.02E − 05
0.95	1.27E − 03	1.42E − 04	3.55E − 05
RMS error	5.22E − 03	6.28E − 04	4.68E − 05
	Results for $y_2(x)$		
0.05	3.47E − 18	0	4.33E − 19
0.15	1.36E − 07	1.82E − 09	1.15E − 10
0.25	4.03E − 07	5.14E − 09	3.23E − 10
0.35	6.88E − 07	8.67E − 09	5.43E − 10
0.45	9.37E − 07	1.17E − 08	7.35E − 10
0.55	1.11E − 06	1.39E − 08	8.69E − 10
0.65	1.18E − 06	1.47E − 08	9.23E − 10
0.75	1.13E − 06	1.40E − 08	8.77E − 10
0.85	9.21E − 07	1.15E − 08	7.17E − 10
0.95	5.51E − 07	6.85E − 09	4.28E − 10
RMS error	8.12E − 07	1.01E − 08	6.35E − 10

Example 4. *([1])* Consider the following nonlinear Volterra integro-differential equation with exact solution $y_1(t) = t + e^t$ and $y_2(t) = t - e^t$,

$$\begin{aligned} y_1''(t) + \frac{1}{2}y_2'^2(t) - \frac{1}{2}\int_0^t (y_1^2(x) + y_2^2(x))dx &= 1 - \frac{1}{3}t^3, \\ y_1(t)t + y_2''(t) - \frac{1}{4}\int_0^t (y_1^2(x) - y_2^2(x))dx &= -1 + t^2, \end{aligned}$$

with the boundary conditions $y_1(0) = 1$, $y_2(0) = -1$, $y_1'(0) = 2$ and $y_2'(0) = 0$. This system has been solved by our method with $N = 10, 30, 60$. The absolute errors at the particular grid points and the RMS errors are given in Table 4.

Table 4: The error $\|E\|$ in solution of Example 4 at particular points.

t	$N = 10$	$N = 30$	$N = 60$
		Results for $y_1(x)$	
0	0	2.22E − 16	0
0.1	4.35E − 06	4.78E − 07	1.11E − 07
0.2	1.78E − 05	1.96E − 06	4.92E − 07
0.3	4.10E − 05	4.55E − 06	1.14E − 06
0.4	7.45E − 05	8.27E − 06	2.07E − 06
0.5	7.18E − 04	1.31E − 05	3.29E − 06
0.6	1.73E − 04	1.93E − 05	4.82E − 06
0.7	2.38E − 04	2.65E − 05	6.63E − 06
0.8	3.14E − 04	3.48E − 05	8.72E − 06
0.9	3.97E − 04	4.42E − 05	1.10E − 05
1	4.87E − 04	5.42E − 05	1.35E − 05
RMS error	2.18E − 04	2.43E − 05	7.33E − 06
		Results for $y_2(x)$	
0	2.22E − 16	0	0
0.1	4.37E − 06	4.79E − 07	1.19E − 07
0.2	1.79E − 05	1.98E − 06	4.96E − 07
0.3	4.17E − 05	4.62E − 06	1.15E − 06
0.4	7.75E − 05	8.53E − 06	2.13E − 06
0.5	1.24E − 04	1.38E − 05	3.46E − 06
0.6	1.87E − 04	2.07E − 05	5.18E − 06
0.7	2.65E − 04	2.94E − 05	7.36E − 06
0.8	3.62E − 04	4.02E − 05	1.0E − 05
0.9	4.79E − 04	5.31E − 05	1.32E − 05
1	6.20E − 04	1.35E − 05	1.72E − 05
RMS error	2.58E − 04	2.86E − 05	6.19E − 06

6 Conclusion

The spline collocation method was used to solve the system of linear and nonlinear integro-differential equations with boundary conditions of the Fredholm and Volterra types. Some examples have been given to show the effectiveness of the proposed method. The absolute errors in the solutions of these examples show that this method is efficient.

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