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On the numerical solution of integral equations of the fourth kind with higher index: differentiability and tractability index-3

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Abstract. In this paper, we consider a particular class of integral equations of the fourth kind and show that tractability and differentiability index of the given system are 3. Tractability and differentiability index are introduced based on the υ-smoothing property of a Volterra integral operator and index reduction procedure, respectively. Using the notion of index, we give sufficient conditions for the existence and uniqueness of the solutions for the index-3 system. Then, a numerical technique based on the Chebyshev polynomial collocation methods including the matrix-vector multiplication representation is proposed for the solution of these systems and the performance of the numerical scheme is illustrated by means of some test problems.

Keywords: Integral-algebraic equations, tractability index, differentiability index, orthogonal polynomial collocation methods, numerical treatment.
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1 Introduction

The general form of linear fourth kind integral equations or integral algebraic-equations (IAEs) is as:

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$$A(t)x(t) + (\nu x)(t) = f(t), \tag{1}$$

where the linear Volterra integral operators ν are given by:

$$(\nu\varphi)(t) = \int_0^t K(t,s)\varphi(s)ds, \qquad t \in I = [0,T], \tag{2}$$

the matrices $A, K \in \mathbb{R}^{d\times d}$ and $f \in \mathbb{R}^d (d \geq 2))$ are continuous. Moreover, we assume that

$$\det A(t) = 0, \quad \forall t \in I,$$

$\operatorname{rank}(A) \geq 1$ and $A(0)x(0) = f(0)$. Integral-algebraic equations of the form (1) arise in problems of identification of memory kernels in heat conduction and viscoelasticity (see [2, 5, 19] and a survey therein). The numerical solution of the integral-algebraic equations is discussed by several authors. Gear [6] introduced the theory of integral-algebraic equations and "index reduction procedure" for these systems. Brunner and Liang [14] analyzed collocation solutions for general systems of index-1 integral-algebraic equations which was based on the notions of the tractability index and the ν-smoothing property of a Volterra integral operator. Bulatov and Budnikova [3] constructed multistep methods to solve a certain class of linear IAEs based on the Adams quadratures rules and extrapolation formulas. Also, the numerical analysis of the two-dimensional IAEs has been investigated by Bulatov and Lima in [4]. Kauthen [12] applied the polynomial spline collocation method for a semi-explicit IAEs with index-1 and established global convergence as well as local superconvergence. Shiri et al.[18] studied the existence and uniqueness of the solution to IAEs using a new index definition and applied the well-known piecewise continuous collocation methods to solve this system numerically. Pishbin et al.[8, 10, 16, 17] proposed several efficient numerical algorithms to solve the index-1 and 2 IAEs and investigated convergence analysis of the numerical methods.

The present paper is devoted to the study of numerical solvability of the semi-explicit system of integral-algebraic equations with index-3. More precisely, we consider

$$\begin{cases} x(t) &= f(t) + (\nu_{11}x)(t) + (\nu_{12}y)(t) + (\nu_{13}z)(t), \\ y(t) &= g(t) + (\nu_{21}x)(t) + (\nu_{22}y)(t), \\ 0 &= h(t) + (\nu_{32}y)(t), \end{cases} \tag{3}$$

where the linear Volterra integral operators ν_{kl}, $(k, l = 1, 2, 3)$ are given by:

$$(\nu_{kl}\varphi)(t) = \int_0^t k_{kl}(t,s)\varphi(s)ds, \qquad t \in I = [0,T],$$

such that $x, f : I \to \mathbb{R}^{d_1}$, $\ y, g : I \to \mathbb{R}^{d_2}$, $\ z, h : I \to \mathbb{R}^{d_3}$. The matrix kernels $k_{ll}(.,.) \in L(\mathbb{R}^{d_l})$, $k_{12}(.,.) \in L(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$, $k_{13}(.,.) \in L(\mathbb{R}^{d_3}, \mathbb{R}^{d_1})$, $k_{21}(.,.) \in L(\mathbb{R}^{d_1}, \mathbb{R}^{d_2})$ and $k_{32}(.,.) \in L(\mathbb{R}^{d_2}, \mathbb{R}^{d_3})$ are assumed to be continuous.

The outline of this paper is as follows. In Section 2, we firstly introduce tractability and differentiability index for system (3) and then obtain the sufficient conditions for the existence and uniqueness of the solutions of the IAEs (3). Scaled Chebyshev polynomial collocation method including the matrix-vector multiplication representation is applied to numerical solution of system (3) in Section 3. We conclude with the numerical illustrations in Section 4.

2 Index of IAEs

Similar to differential-algebraic equations (DAEs) (see [1, 7, 9, 11, 13, 15]), the concept of the index is the key to the theoretical and numerical analysis of the IAEs. In this section, we consider the definition of tractability and differentiability index to IAEs (3).

2.1 The tractability index of IAEs

Before introducing the definition of the tractability index of IAEs, we require the definition of υ-smoothing of the Volterra integral operator (2).

Definition 1. ([14]) The Volterra integral operator (2) corresponding to the kernel matrix $K(t,s) = \begin{pmatrix} k_{pq}(t,s) \\ p,q = 1,\cdots,d \end{pmatrix}$, with $d \geq 2$, is said to be υ-smoothing if there exist integers $\upsilon_{pq} \geq 1$ with $\upsilon = \max\limits_{1 \leq p,q \leq d} \{\upsilon_{pq}\}$ such that the following conditions hold:

1) $\frac{\partial^i k_{pq}(t,s)}{\partial t^i} \mid_{s=t} = 0, \quad t \in I, \qquad i = 0, \cdots, \upsilon_{pq} - 2,$

2) $\frac{\partial^{\upsilon_{pq}-1} k_{pq}(t,s)}{\partial t^{\upsilon_{pq}-1}} \mid_{s=t} \neq 0, \quad t \in I,$

3) $\frac{\partial^{\upsilon_{pq}} k_{pq}(t,s)}{\partial t^{\upsilon_{pq}}} \in C(D), \quad D = \{(t,s), 0 \leq s \leq t \leq T\}.$

We set $\upsilon_{pq} = 0$ when $k_{pq}(t,s) \equiv 0$.

A first-kind VIE $\nu u = f$ is called a υ-smoothing problem if ν is a υ-smoothing operator and $f \in C^{\upsilon}(I)$.

Now, we introduce the concept of index-μ tractability for a $(\upsilon + 1)$-smoothing problem of the form (1). Let

$$K^0 = K(t,s) \quad K_0 = K(t,t), \quad A_0 = A(t), \quad A_1 = A_0 + K_0 Q_0.$$

For $l \geq 0$, if $(K^l)_{pq} \mid_{s=t} \neq 0$, define $(K^{l+1})_{pq}(t,s) = 0$, otherwise

$$(K^{l+1})_{pq}(t,s) = \frac{\partial}{\partial t}(K^l)_{pq}(t,s).$$

We set $K_{l+1} = (K^{l+1})_{pq}(t,s) \mid_{s=t}$ $(p,q = 1,\cdots,d)$ and

$$\begin{aligned} A_{l+2} &= A_{l+1} + \sum_{i=0}^{l+1} K_i(\prod_{j=0}^{l-i} P_j)Q_{l-i+1}, \quad 0 \leq l \leq \upsilon - 1, \\ A_{l+2} &= A_{l+1} + \sum_{i=0}^{\upsilon} K_i(\prod_{j=0}^{l-i} P_j)Q_{l-i+1}, \quad l \geq \upsilon, \end{aligned}$$

where $Q_0 = Q_0(t)$ denotes a projector onto $\ker(A_0)$ and for $j \geq 1$, Q_j is a projector into $\ker A_j$ with $Q_jQ_k = 0$ $(k < j)$. Also, $P_j = I - Q_j$ with $\prod_{j=0}^{-1} P_j = 1$.

Definition 2. ([14]) Assume that the Volterra integral operator (2) is $(\upsilon+1)$-smoothing with $\upsilon \geq 0$. Then IAEs (1) is said to be index-μ tractable if all matrices $A_l = A_l(t),\ t \in I,\ l = 0,\cdots,\mu-1$, defined above, are singular with smooth null space and A_μ remains nonsingular at all points in I.

Now, considering IAEs (3) with $d_1 = d_2 = d_3 = 1$, we have

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad K^0(t,s) = \begin{pmatrix} k_{11}(t,s) & k_{12}(t,s) & k_{13}(t,s) \\ k_{21}(t,s) & k_{22}(t,s) & 0 \\ 0 & k_{32}(t,s) & 0 \end{pmatrix}.$$

Let Volterra integral operator of system (3) be 1-smoothing, then from Definition 1 $k_{ij}(t,t) \neq 0$ $(i,j = 1,2,3)$ and $K_0 = K^0(t,t) \neq 0$. We can take

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The corresponding matrix

$$\begin{aligned} A_1 = A_0 + K_0Q_0 = &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ + &\begin{pmatrix} k_{11}(t,t) & k_{12}(t,t) & k_{13}(t,t) \\ k_{21}(t,t) & k_{22}(t,t) & 0 \\ 0 & k_{32}(t,t) & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & k_{13}(t,t) \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

is singular. Since $K^0 \mid_{s=t} \neq 0$, define $K^1(t,s) = 0$, and $K_1 = 0$. Also, we have

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ k_{13}^{-1} & 0 & 0 \end{pmatrix}, \quad P_0 = I - Q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$A_2 = A_1 + K_0 P_0 Q_1 = \begin{pmatrix} 1 + k_{11}(t,t) & 0 & k_{13}(t,t) \\ k_{21}(t,t) & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We observe that here again A_2 is singular. In the sequel, taking

$$Q_2 = \begin{pmatrix} 0 & -k_{21}^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{(1+k_{11})}{k_{21}k_{13}} & 0 \end{pmatrix}, \quad P_1 = I - Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -k_{13}^{-1} & 0 & 1 \end{pmatrix},$$

it follows that

$$A_3 = A_2 + K_0(P_0 P_1) Q_2 = \begin{pmatrix} 1 + k_{11}(t,t) & k_{12}(t,t) & k_{13}(t,t) \\ k_{21}(t,t) & 1 + k_{22}(t,t) & 0 \\ 0 & k_{32}(t,t) & 0 \end{pmatrix}.$$

We then find that

$$\det(A_3) = k_{32}(t,t) k_{21}(t,t) k_{13}(t,t) \neq 0,$$

and from Definition 2, the tractability index is 3.

Example 1. Consider IAEs (3) with $d_1 = d_2 = d_3 = 1$ and let $k_{ij}(t,t) \neq 0$ $(i,j = 1,2)$, $k_{32}(t,s) = (t - s)$. From Definition 1, we have

$$k_{32}(t,s) \mid_{s=t} = 0, \quad \frac{\partial k_{32}(t,s)}{\partial t} \mid_{s=t} = 1,$$

then $\nu_{32} = 2$ and the Volterra integral operator of system (3) is 2-smoothing. Also, we have

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad K_0 = K^0(t,t) = \begin{pmatrix} k_{11}(t,t) & k_{12}(t,t) & k_{13}(t,t) \\ k_{21}(t,t) & k_{22}(t,t) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus we can take $Q_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The corresponding matrix

$$A_1 = A_0 + K_0 Q_0 = \begin{pmatrix} 1 & 0 & k_{13}(t,t) \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

is singular. Since for $(p,q=1,2)$, $(K^0)_{pq}\mid_{s=t}\neq 0$, define $(K^1)_{pq}(t,s)=0$. On the other hand, $(K^0)_{32}\mid_{s=t}=0$, then $(K^1)_{32}(t,s)=\frac{\partial}{\partial t}(K^0)_{32}(t,s)=1$. Thus, K_1 and Q_1 can be defined as:

$$K_1=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix},\quad Q_1=\begin{pmatrix}1&0&0\\0&0&0\\k_{13}^{-1}&0&0\end{pmatrix}.$$

Now, from Definition 2, we have

$$\begin{aligned}A_2 &= \begin{pmatrix}1&0&k_{13}(t,t)\\0&1&0\\0&0&0\end{pmatrix}\\
&+\begin{pmatrix}k_{11}(t,t)&k_{12}(t,t)&k_{13}(t,t)\\k_{21}(t,t)&k_{22}(t,t)&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\k_{13}^{-1}&0&0\end{pmatrix}\\
&+\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\\
&=\begin{pmatrix}1+k_{11}(t,t)&0&k_{13}(t,t)\\k_{21}(t,t)&1&0\\0&0&0\end{pmatrix},\end{aligned}$$

which is singular. We can take Q_2 as a projector into $ker A_2$ with $Q_2Q_k=0$ $(k<2)$ in the following form

$$Q_2=\begin{pmatrix}0&-k_{21}^{-1}&0\\0&1&0\\0&\frac{(1+k_{11})}{k_{21}k_{13}}&0\end{pmatrix},\quad P_1=I-Q_1=\begin{pmatrix}0&0&0\\0&1&0\\-k_{13}^{-1}&0&1\end{pmatrix}.$$

The corresponding matrix

$$\begin{aligned}A_3 &= A_2+K_0(P_0P_1)Q_2+K_1P_0Q_1\\
&=\begin{pmatrix}1+k_{11}(t,t)&k_{12}(t,t)&k_{13}(t,t)\\k_{21}(t,t)&1+k_{22}(t,t)&0\\0&0&0\end{pmatrix},\end{aligned}$$

is singular. Taking $Q_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$, it follows that

$$\begin{aligned}A_4 &= A_3+K_0(P_0P_1P_2)Q_3+K_1(P_0P_1)Q_2\\
&=\begin{pmatrix}1+k_{11}(t,t)&k_{12}(t,t)&k_{13}(t,t)\\k_{21}(t,t)&1+k_{22}(t,t)&0\\0&1&0\end{pmatrix}.\end{aligned}$$

Since A_4 is nonsingular, the tractability index is 4.

2.2 The differentiability index of IAEs

The differentiability index is based on the “index reduction procedure” i.e. a differentiation process of the algebraic constraints which yields a system of regular VIEs. Differentiating the third equation of (3) with respect to t and substituting y from its second equation gives

$$0 = \bar{h}_1(t) + \int_0^t \bar{K}_{31}(t,s)x(s)ds + \int_0^t \bar{K}_{32}(t,s)y(s)ds, \tag{4}$$

where $\bar{h}_1(t) = h'(t) + k_{32}(t,t)g(t)$, $\bar{K}_{31}(t,s) = k_{32}(t,t)k_{21}(t,s)$, $\bar{K}_{32}(t,s) = k_{32}(t,t)k_{22}(t,s) + \frac{\partial k_{32}(t,s)}{\partial t}$. We differentiate Eq. (4) with respect to t and substitute x, y from the first and second equations of (3), so we obtain

$$0 = \bar{h}_2(t) + \int_0^t H_{31}(t,s)x(s)ds + \int_0^t H_{32}(t,s)y(s)ds + \int_0^t H_{33}(t,s)z(s)ds, \tag{5}$$

where

$$\begin{aligned}
\bar{h}_2(t) &= \bar{h}_1' + \bar{K}_{31}f + \bar{K}_{32}g, \\
H_{31}(t,s) &= \bar{K}_{31}k_{11} + \bar{K}_{32}k_{21} + \frac{\partial \bar{K}_{31}}{\partial t}, \\
H_{32}(t,s) &= \bar{K}_{31}k_{12} + \bar{K}_{32}k_{22} + \frac{\partial \bar{K}_{32}}{\partial t}, \\
H_{33}(t,s) &= \bar{K}_{31}k_{13}.
\end{aligned}$$

Assume that $H_{33}(t,t) \neq 0$. Differentiating (5) and inserting x, y from the first and second equations of (3), lead to the following second-kind integral equation

$$z(t) = \bar{h}_3(t) + \int_0^t \hat{H}_{31}(t,s)x(s)ds + \int_0^t \hat{H}_{32}(t,s)y(s)ds + \int_0^t \hat{H}_{33}(t,s)z(s)ds, \tag{6}$$

where $\bar{h}_3(t), \hat{H}_{31}(t,s), \hat{H}_{32}(t,s)$ and $\hat{H}_{33}(t,s)$ can be easily computed. Now, (6) together with the first and second equations of (3) are as a regular system of Volterra equations. We observe that the number of analytical differentiations of (3) until it can be formulated as a regular system of Volterra integral equations is three. Thus the differentiability index is 3.

Remark 1. *It is obvious that the tractability index of the 1-smoothing system (3) is 3 if and only if $H_{33}(t,t)$ is nonsingular. This implies that if $\upsilon = 1$ and $\mu = 3$, the differentiability index equals the tractability index.*

Applying the conditions of existence and uniqueness of solutions related to the IAEs of index-2 (Theorem 1 [10]), differentiability and tractability indices, the following theorem gives the relevant conditions for the investigation of the unique solution of IAEs (3):

Theorem 1. *Let $m \geq 0$ and assume that*
1. $k_{1l} \in C^m(D)$ for $l = 1, 2, 3$,
2. $k_{2l} \in C^{m+1}(D)$ for $l = 1, 2$,
3. $k_{32} \in C^{m+2}(D)$ and $|\det(k_{32}(t,t)k_{21}(t,t)k_{13}(t,t))| \geq k_0 > 0$,
4. $f \in C^m(D)$, $g \in C^{m+1}(D)$, $h \in C^{m+2}(D)$ and $h(0) = 0$.
Then the integral-algebraic equations (3) possesses a unique solution $x, y, z \in C^m(I)$.

Proof. Under appropriate regularity assumptions, we observe that the equation (4) together with the first and second equations of (3) are as the integral-algebraic equations of index-2 which has been defined in [10]. Then proof is completed by appealing [10, Theorem 1]. $\square$

2.3 The numerical treatment

We consider the scaled Chebyshev polynomials as

$$\hat{T}_i^N = \delta_i T_i(x), \qquad i = 0, 1, \ldots, N-1,$$

where T_i is the Chebyshev polynomial of degree i and

$$\delta_i = \begin{cases} \frac{1}{\sqrt{N}}, & i = 0, \\ \sqrt{2}\delta_0, & i = 1, \ldots, N-1. \end{cases}$$

For any $u(x) \in C[-1, 1]$, we can define the projection I_N as the interpolating polynomial associated with the scaled Chebyshev polynomials

$$(I_N u)(x) = \sum_{k=1}^{N} b_k \hat{T}_{k-1}^N(x), \tag{7}$$

where the coefficients b_k are determined by the interpolating conditions:

$$u(\tau_i^N) = \sum_{k=1}^{N} b_k \hat{T}_{k-1}^N(\tau_i^N), \tag{8}$$

and $\left\{\tau_i^N = \cos\left(\frac{(2i-1)\pi}{2N}\right)\right\}_{i=1}^N$ are the Chebyshev-Gauss quadrature points. Assume that $B = [b_1, \ldots, b_N]^T$, $U = [u(\tau_1), \ldots, u(\tau_N)]^T$ and

$C = [\hat{T}_{i-1}^N(\tau_j^N)]_{i,j=1}^N$, then the relation (8) can be written in compact form as $U = C^T B$. Note that C is the cosine transform matrix, which is orthogonal [20], we have $B = CU$ and the approximation of $u(x)$ can be written as

$$u(x) \approx (I_N u)(x) = \sum_{k=1}^{N} [CU]_k \hat{T}_{k-1}^N(x) = B^T \hat{\mathbf{D}}(x) = U^T C^T \hat{\mathbf{D}}(x), \tag{9}$$

where $\hat{\mathbf{D}}(.) = (\hat{T}_0^N(.), \cdots, \hat{T}_{N-1}^N(.))^T$.

For the sake of applying the theory of orthogonal polynomials, we use the change of variables

$$s = \frac{T}{2}(\eta + 1), \qquad t = \frac{T}{2}(\tau + 1), \qquad -1 \leq \eta \leq \tau \leq 1,$$

to rewrite the system (3) as follows:

$$A\widehat{X}(\tau) = \widehat{G}(\tau) + \int_{-1}^{\tau} \widehat{K}(\tau, \eta)\widehat{X}(\eta)d\eta, \qquad \tau \in [-1, 1] \tag{10}$$

where $\widehat{X}(\tau) = (\widehat{x}, \widehat{y}, \widehat{z})^T, \widehat{G}(\tau) = (\widehat{f}, \widehat{g}, \widehat{h})^T$, $A = \mathrm{diag}\Big(I_{d_1}, I_{d_2}, O_{d_3}\Big) \in L(\mathbb{R}^d)$ is a singular block matrix and $\widehat{K}(\tau, \eta) = \{\widehat{k}_{ij}(\tau, \eta)\}_{i,j=1}^d$. From (9)

$$(I_N \widehat{K})(\tau_m, \eta) = \Big\{\widehat{H}_{ij}^T\Big\}_{i,j=1}^d \bigotimes C^T \hat{\mathbf{D}}(\eta) = \Big\{\widehat{H}_{ij}^T\Big\}_{i,j=1}^d \bigotimes (C^T V W), \tag{11}$$

where $\widehat{H}_{ij} = [\widehat{k}_{ij}(\tau_m, \tau_1), \cdots, \widehat{k}_{ij}(\tau_m, \tau_N)]^T$, V is the coefficient matrix of the scaled Chebyshev polynomials $\{\hat{T}_k^N\}_{k=0}^{N-1}$ and $W = (1, \eta, \cdots, \eta^{N-1})^T$.

Using the discrete expansion of $\widehat{X}(\eta)$, we can write

$$\widehat{X}_N(\eta) = \left\{\sum_{k=1}^{N} (\widehat{x}_i)_k \hat{T}_{k-1}^N(\eta)\right\}_{i=1}^d = \{\widehat{x}_i\}_{i=1}^d \bigotimes (VW), \tag{12}$$

where $\widehat{x}_i = [(\widehat{x}_i)_1, \cdots, (\widehat{x}_i)_N]$ and $\bigotimes$ represents Kronecker product of matrices. Inserting collocation Gauss points $\{\tau_m\}_{m=1}^N$ and the above approximations into (10), we obtain:

$$\begin{aligned} A\widehat{X}_N(\tau_m) &= \widehat{G}(\tau_m) + \int_{-1}^{\tau_m} (I_N \widehat{K}(\tau_m, \eta))\widehat{X}_N(\eta)d\eta = \widehat{G}(\tau_m) \\ &+ \int_{-1}^{\tau_m} \Big(\Big\{\widehat{H}_{ij}^T\Big\}_{i,j=1}^d \bigotimes (C^T V W)\Big)\Big(\{\widehat{x}_i\}_{i=1}^d \bigotimes (VW)\Big) d\eta, \end{aligned} \tag{13}$$

Let M be a block sparse matrix of the form $M = \left(M_{N-1}^{(k)}\right)_{k=0}^{N-1}$ with

$$M_j^{(k)} = \begin{pmatrix} \overbrace{0 & \dots & 0}^{k} & 1 & 0 & \dots & 0 & \overbrace{0 & \dots & 0}^{j-k} \\ 0 & \dots & 0 & 0 & 1 & \dots & \vdots & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & & \ddots & 0 & \vdots & \dots & \vdots \\ 0 & \dots & 0 & \underbrace{0 & \dots & 0 & 1}_{N} & 0 & \dots & 0 \end{pmatrix},$$

such that $M_j^{(k)}$ is a $(N)\times(N+j)$ sparse matrix. Applying the Lemma 1 from [10] for the matrix vector multiplication representation of $(I_N\widehat{K})(\tau_m,\eta)\widehat{X}_N(\eta)$, the relation (13) can be written as:

$$\widehat{A}\widehat{X}_N(\tau_m) = \widehat{G}(\tau_m) + \left(\sum_{j=1}^{d}(\widehat{H}_{ij}^T V\Big((\widehat{x}_j V)\bigotimes M\Big)Q_m\right)_{i=1}^{d}, \qquad (m = 1,\dots,N) \tag{14}$$

where $Q_m = \displaystyle\int_{-1}^{\tau_m} W' d\eta$ and $W' = (1,\eta,\cdots,\eta^{2N-2})^T$. Finally, by substituting (12) into (14), we end up with a linear system of algebraic equations for the unknown coefficients $\{\widehat{x}_i\}_{i=1}^d$.

3 Numerical examples

In this section, we consider two numerical examples in order to illustrate the validity of the proposed technique. All the computations were performed using Mathematica® software. For analyzing the behavior of the error representations, we define the weighted L_w^2-norm by

$$\|e\|_{L_w^2(-1,1)} = \Big(\int_{-1}^{1}|e|^2 w(x)dx\Big)^{\frac{1}{2}},$$

where $w(x) = \frac{1}{\sqrt{1-x^2}}$.

Example 2. Consider the integral-algebraic equations of index-3 in the form of (3) with $d_1 = d_2 = d_3 = 1$:

$$AX(t) = F(t) + \int_0^t K(t,s)X(s)ds, \quad t\in[0,1], \tag{15}$$

Table 1: $L^2_w(-1,1)$ errors for Example 2.

N	$\\|\hat{x}-\hat{x}_N\\|_{L^2_w}$	$\\|\hat{y}-\hat{y}_N\\|_{L^2_w}$	$\\|\hat{z}-\hat{z}_N\\|_{L^2_w}$
3	8.12×10^{-3}	5.47×10^{-3}	1.98×10^{-1}
4	1.66×10^{-3}	3.59×10^{-4}	3.12×10^{-2}
5	8.68×10^{-4}	2.84×10^{-5}	1.88×10^{-3}
6	2.01×10^{-4}	3.64×10^{-6}	1.03×10^{-3}
7	1.68×10^{-6}	2.88×10^{-8}	8.89×10^{-6}
8	1.00×10^{-7}	9.54×10^{-10}	4.26×10^{-7}
9	3.45×10^{-9}	4.03×10^{-11}	2.93×10^{-8}

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$K(t,s) = \begin{pmatrix} t+2s+1 & (s+2)^2 & s^2+t^2+2 \\ \sin(t+s+2) & s^2+1 & 0 \\ 0 & \cos(s+2) & 0 \end{pmatrix},$$

$$X(t) = \begin{pmatrix} x(t), y(t), z(t) \end{pmatrix}^T, \quad F(t) = \begin{pmatrix} f(t), g(t), h(t) \end{pmatrix}^T,$$

and $f(t), g(t), h(t)$ such that the exact solution is $X(t) = \begin{pmatrix} e^t, \sinh t, \cosh t \end{pmatrix}^T$.

Let $\widehat{X}_N = (\hat{x}_N, \hat{y}_N, \hat{z}_N)$ denote the approximation of the exact solution $\widehat{X} = (\hat{x}, \hat{y}, \hat{z})$ which is given by (12). We apply the proposed collocation scheme for the integral-algebraic equations (15) and report the weighted L^2_w-norm of errors for several values of N in Table 1.

Example 3. Consider the IAEs of index-3 in the form of (3) with $d_1 = d_2 = 2$ and $d_3 = 1$:

$$AX(t) = F(t) + \int_0^t K(t,s)X(s)ds, \quad t \in [0,1],$$

where

$$A = \begin{pmatrix} I_{4\times 4} & \mathbf{0}_{4\times 1} \\ \mathbf{0}_{1\times 4} & 0 \end{pmatrix}, \qquad K(t,s) = \begin{pmatrix} k_{11}(t,s) & k_{12}(t,s) & k_{13}(t,s) \\ k_{21}(t,s) & k_{22}(t,s) & \mathbf{0}_{2\times 1} \\ \mathbf{0}_{1\times 2} & k_{32}(t,s) & 0 \end{pmatrix},$$

Table 2: $L^2_w(-1,1)$ errors for Example 3.

N	$\Vert\hat{x}_1-\hat{x}_{1N}\Vert_{L^2_w}$	$\Vert\hat{x}_2-\hat{x}_{2N}\Vert_{L^2_w}$	$\Vert\hat{y}_1-\hat{y}_{1N}\Vert_{L^2_w}$	$\Vert\hat{y}_2-\hat{y}_{2N}\Vert_{L^2_w}$	$\Vert\hat{z}-\hat{z}_N\Vert_{L^2_w}$
3	2.98×10^{-2}	3.01×10^{-2}	4.85×10^{-3}	2.82×10^{-3}	6.86×10^{-2}
4	3.74×10^{-3}	5.08×10^{-3}	2.85×10^{-3}	1.48×10^{-3}	1.54×10^{-2}
5	2.44×10^{-4}	3.12×10^{-4}	5.73×10^{-4}	3.22×10^{-4}	1.10×10^{-3}
6	3.05×10^{-5}	3.13×10^{-5}	4.06×10^{-5}	2.09×10^{-5}	2.35×10^{-4}
7	2.59×10^{-5}	2.95×10^{-5}	3.14×10^{-5}	1.85×10^{-5}	1.56×10^{-4}
8	3.62×10^{-6}	3.79×10^{-6}	4.50×10^{-6}	2.71×10^{-6}	2.01×10^{-5}
9	2.61×10^{-7}	1.79×10^{-7}	4.32×10^{-7}	1.31×10^{-7}	5.11×10^{-6}

such that

$$k_{11}(t,s)=\begin{pmatrix} e^{2t-s} & t^2+s \\ e^{t+s} & s^2+3t \end{pmatrix},\qquad k_{12}(t,s)=\begin{pmatrix} t^2+4 & \sin(t+s) \\ \cos(t+1) & s^2+t^2+2 \end{pmatrix},$$

$$k_{13}(t,s)=\begin{pmatrix} t+s^2+4 \\ t^2+4 \end{pmatrix},\qquad k_{21}(t,s)=\begin{pmatrix} e^{t^2+1} & t^4+s^2+1 \\ (t+4)^2+1 & (s+t+1)^2 \end{pmatrix},$$

$$k_{22}(t,s)=\begin{pmatrix} t+1 & s+t^4+2 \\ e^t+4 & s^2+t+2 \end{pmatrix},\quad k_{32}(t,s)=\begin{pmatrix} (t^2+1)^2\,, & s^2+t^4+4 \end{pmatrix},$$

$$X(t)=\begin{pmatrix} x_1(t),x_2(t),y_1(t),y_2(t),z(t) \end{pmatrix}^T,$$
$$F(t)=\begin{pmatrix} f_1(t),f_2(t),g_1(t),g_2(t),h(t) \end{pmatrix}^T,$$

and F such that the exact solution is $x_1(t)=te^t,\ x_2(t)=t^2+1,\ y_1(t)=\frac{t}{t^2+1},\ y_2(t)=2t+1,\ z(t)=\cos(2t)$. Let $\widehat{X}_N=(\hat{x}_{1N},\hat{x}_{2N},\hat{y}_{1N},\hat{y}_{2N},\hat{z}_N)$ be the collocation approximations of the solutions $\widehat{X}=(\hat{x}_1,\hat{x}_2,\hat{y}_1,\hat{y}_2,\hat{z})$ that is given by (12). The L^2_w errors for different values of N are reported in Table 2.

The proposed collocation scheme based on the Chebyshev polynomials is a spectral method. However, we note that, spectral methods are global methods such that the computation at any given point depends not only on information at neighboring points, but also on information from the entire domain. Due to the smoothness of the exact solutions for the two previous examples, the spectral accuracy presented in Tables 1 and 2 have been obtained.

4 Conclusion

This work has been concerned with the scaled Chebyshev collocation method for the numerical solution of the special integral-algebraic equations of the semi-explicit form. We showed that tractability and differentiability index of given IAEs system are identical. The existence and uniqueness theorem related to the IAEs of index-3 was introduced. The extension of our analysis for two-dimensional IAE systems is left as a future work.

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