Journal of Mathematical Modeling
Vol. 2, No. 1, 2014, pp. 90-106

JMM

Convergence of the multistage variational iteration method for solving a general system of ordinary differential equations

Mehdi Bastani[†] [*]

[†] *Young Researchers and Elite Club, Ardabil Branch, Islamic Azad University, Ardabil, Iran*
emails: bastani.mehdi@yahoo.com, bastani@azaruniv.edu

Abstract. In this paper, the multistage variational iteration method is implemented to solve a general form of the system of first-order differential equations. The convergence of the proposed method is given. To illustrate the proposed method, it is applied to a model for HIV infection of $CD4^+$ T cells and the numerical results are compared with those of a recently proposed method.

Keywords: Multistage variational iteration method, convergence, HIV infection of $CD4^+$ T cells, Adomian decomposition method.
AMS Subject Classification: 34A34, 65L05.

1 Introduction

In this paper, we consider the general system of ordinary differential equations of first order

$$\begin{cases} \dfrac{dx_i(t)}{dt} = \alpha_i + \displaystyle\sum_{j=1}^{m} \beta_{ij} x_j(t) + \sum_{j=1}^{m}\sum_{k=1}^{m} \gamma_{ijk} x_j(t) x_k(t), & i = 1, \ldots, m, \\ x_i(t_0) = c_i, & \end{cases} \tag{1}$$

[*]Corresponding author.
Received: 2 February 2014 / Revised: 23 March 2014 / Accepted: 27 March 2014.

where α_i's, β_{ij}'s, γ_{ijk}'s, c_i's $\in \mathbb{R}$. Many problems in engineering and science can be modelled by (1). For example, the mathematical model for HIV infection [5, 6, 18, 30, 38], the mathematical model for enzymatic reaction [34, 35], the Chen dynamical system [27], the Lotka-Volterra problem [3] and the epidemic model [2, 19].

The various kinds of equation (1) are approximately solved by numerical and analytical methods such as finite difference method [23], differential transform method [29], Adomian decomposition method [17], and homotopy-analysis method [24, 25]. As we know, the He's variational iteration method (VIM) [9, 10, 11, 12, 13, 14, 15] is a powerful device for solving various kind of problems. It has been successfully applied for solving various PDEs and ODEs [1, 4, 7, 26, 31]. Convergence of the VIM has been investigated under some conditions in [8, 28, 32, 33, 37].

In [8], Goh et al. have investigated the convergence of the VIM method for a system of ordinary differential equations which is an special case of (1). In this paper, we first investigate the convergence of the VIM to solve (1) and then apply the multistage version of the VIM (known as the MVIM) to a mathematical model for HIV infection. In fact we improve the application of the MVIM to model.

This paper is organized as follows. In section 2, we give a brief description of the VIM and the MVIM. Section 3, is devoted to application of the MVIM for solving system (1) and verifying its convergence. In Section 4, the application of the proposed method for solving a model for HIV infection of CD4$^+$ T cells is investigated. Some numerical results are given in Section 5. Concluding remarks are also given in section 5.

2 A brief description of the VIM and the multistage VIM

The VIM transforms the differential equation to a recurrence sequence of functions and the limit of the sequence, if exists, is considered as the solution of the differential equation. Consider the following differential equation

$$\mathcal{L}u(t) + \mathcal{N}u(t) = g(t),$$

where $\mathcal{L}$ is a linear operator, $\mathcal{N}$ is a nonlinear operator and $g(t)$ is an inhomogeneous term. Given an initial guess $u_0(t)$, a correctional functional as

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^{t} \lambda(\mathcal{L}u_n(s) + \mathcal{N}\widetilde{u}_n(s) - g(s))ds,$$

is made, where λ is a general Lagrangian multiplier [12] which can be identified optimally via the variational theory and the function $\widetilde{u}_n$ is a restricted variation which means $\delta\widetilde{u}_n = 0$. Obviously the successive approximations $u_j,\ j = 0, 1, \ldots,$ can be computed by determining λ. To find the optimal value of λ, we make correction functional stationary in the following form

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_{t_0}^{t} \lambda(\mathcal{L}u_n(s) + \mathcal{N}\widetilde{u}_n(s) - g(s))ds = 0,$$

which results the stationary conditions and consequently the optimal value of λ is obtained [37]. In fact the solution of the differential equation is considered as the fixed point of the following functional under the suitable choice of the initial term $u_0(t)$

$$u_{n+1}(t) = u_n(t) + \delta \int_{t_0}^{t} \lambda(\mathcal{L}u_n(s) + \mathcal{N}u_n(s) - g(s))ds, \quad n = 0, 1, 2, \ldots.$$

In the MVIM, to solve the differential equation in the interval $[t_0, T]$, the interval is first partitioned into m subintervals $[t_0, t_1], [t_1, t_2], \ldots, [t_{m-1}, t_m]$ where $t_m = T$. In the ith stage, by using the computed solution in the previous stage, the approximate solution at t_{i-1} is computed. Then the approximate solution in the interval $[t_{i-1}, t_i]$ is computed via the iteration formula

$$u_{n+1}(t) = u_n(t) + \int_{t_{i-1}}^{t} \lambda(\mathcal{L}u_n(s) + \mathcal{N}u_n(s) - g(s))ds, \quad n = 0, 1, 2, \ldots,$$

where $t \in [t_{i-1}, t_i]$.

Numerical results presented for several problems show that the MVIM is usually more reliable than the VIM, especially for ordinary differential equations (for example see [8]). However, this method is affected by some problems. Since we are concerned with symbolic computations, if λ has a complicated form, then after a few iterations the computation of $u_{i,n+1}$ (the value of u_{n+1} in the interval $[t_{i-1}, t_i]$, $i = 1, \ldots, n$) would be very difficult or even impossible. Therefore, to overcome on this problem, we should perform only a very few iterations (e.g., one or two) of the method in each subinterval and use a large number of subintervals. Another way to improve the MVIM is to choose the linear and nonlinear terms carefully. More precisely, to obtain a tractable Lagrangian multiplier we may consider a linear term as a nonlinear one (as we will do in this paper).

3 Application of the VIM to system (1)

For the sake of the simplicity, let

$$F_i(x_1(t),\ldots,x_m(t)) = \alpha_i + \sum_{j=1}^{m} \beta_{ij} x_j(t) + \sum_{j=1}^{m}\sum_{k=1}^{m} \gamma_{ijk} x_j(t) x_k(t).$$

Then, the ODEs in (1) take the following form

$$\frac{dx_i(t)}{dt} = F_i(x_1(t),\ldots,x_m(t)), \qquad i = 1,\ldots,m.$$

According to the VIM we write down the corresponding correction functional as

$$\begin{aligned} x_i^{(n+1)}(t) &= x_i^{(n)}(t) \\ &\quad + \int_{t_0}^{t} \lambda_i(s) \left[\frac{dx_i^{(n)}(s)}{ds} - F_i(\tilde{x}_1^{(n)}(s),\ldots,\tilde{x}_m^{(n)}(s)) \right] ds, \quad i = 1,\ldots,m, \end{aligned}$$

where $x_i^{(0)}(t)$, $i = 1,\ldots,m$ are the initial approximations. Taking variation with respect to the $x_i^{(n)}(t)$ and noticing that $\delta x_i^{(n)}(t_0) = 0$, for $i = 1,\ldots,m$, we have

$$\begin{aligned} \delta x_i^{(n+1)}(t) &= \delta x_i^{(n)}(t) + \delta \int_{t_0}^{t} \lambda_i(s) \left[\frac{dx_i^{(n)}(s)}{ds} - F_i(\tilde{x}_1^{(n)}(s),\ldots,\tilde{x}_m^{(n)}(s)) \right] ds \\ &= \delta x_i^{(n)}(t) + \lambda_i(s) \delta x_i^{(n)}(s)|_{s=t} + \int_{t_0}^{t} \lambda_i'(s) \delta x_i^{(n)}(s) ds. \end{aligned}$$

Therefore, the following stationary conditions

$$\begin{cases} \lambda_i'(s)|_{s=t} & = 0, \\ 1 + \lambda_i(s)|_{s=t} & = 0, \end{cases} \qquad i = 1,\ldots,m.$$

Hence, the Lagrange multiplier can be readily identified by $\lambda_i = -1$, for $i = 1,\ldots,m$. This gives the following iteration formula

$$\begin{aligned} x_i^{(n+1)}(t) &= x_i^{(n)}(t) \\ &\quad - \int_{t_0}^{t} \left[\frac{dx_i^{(n)}(s)}{ds} - F_i(x_1^{(n)}(s),\ldots,x_m^{(n)}(s)) \right] ds, \quad i = 1,\ldots,m, \end{aligned}$$

which is equivalent to

$$x_i^{(n+1)}(t) = x_i^{(n)}(t_0) + \int_{t_0}^{t} F_i(\tilde{x}_1^{(n)}(s),\ldots,\tilde{x}_m^{(n)}(s)) ds, \qquad i = 1,\ldots,m,$$

or

$$x_i^{(n+1)}(t) = x_i^{(n)}(t_0) + \int_{t_0}^{t} \left(\alpha_i + \sum_{j=1}^{m} \beta_{ij} x_j^{(n)}(s) + \sum_{j=1}^{m} \sum_{k=1}^{m} \gamma_{ijk} x_j^{(n)}(s) x_k^{(n)}(s) \right) ds, \quad (2)$$

for $i = 1, \ldots, m$.

We now state and prove the next theorem concerning the convergence of the iteration formula (2).

Theorem 1. *Let $\Omega = [t_0, t_1]$ and $x_i \in C(\Omega)$, for $i = 1, \ldots, m$. Assume that the initial guess is chosen to be the initial condition. If $x_i^{(n)} \in C(\Omega)$ for each i and $n \geq 0$, and $\{x_i^{(n)}\}_{n=0}^{\infty}$, is uniformly bounded sequence on Ω for each i, then $x_i^{(n)}$ defined by (2) converges to x_i, for $i = 1, \ldots, m$.*

Proof. Obviously from equation (2), we have $x_i^{(n+1)}(t_0) = x_i^{(n)}(t_0)$, $i = 1 \ldots, m$. Therefore, $x_i^{(n)}(t_0) = x_i(t_0)$ for $n \geq 1$ and $i = 1, \ldots, m$. On the other hand, Eq. (1) may be written as

$$x_i(s) = x_i(t_0) + \int_{t_0}^{t} \left(\alpha_i + \sum_{j=1}^{m} \beta_{ij} x_j(s) + \sum_{j=1}^{m} \sum_{k=1}^{m} \gamma_{ijk} x_j(s) x_k(s) \right) ds,$$

for $i = 1, \ldots, m$. Now, from Eqs. (2) and (3), we get

$$\begin{aligned} E_i^{(n+1)}(t) &= \int_{t_0}^{t} \Bigg(\sum_{j=1}^{m} \beta_{ij} E_j^{(n)}(s) \\ &\qquad + \sum_{j=1}^{m} \sum_{k=1}^{m} \gamma_{ijk} \left(x_j^{(n)}(s) x_k^{(n)}(s) - x_j(s) x_k(s) \right) \Bigg) ds \\ &= \int_{t_0}^{t} \Bigg(\sum_{j=1}^{m} \beta_{ij} E_j^{(n)}(s) \\ &\qquad + \sum_{j=1}^{m} \sum_{k=1}^{m} \gamma_{ijk} \left(x_j^{(n)}(s) E_k^{(n)}(s) + E_j^{(n)}(s) x_k^{(n)}(s) \right) \Bigg) ds, \end{aligned}$$

where $E_i^{(n+1)} = E_i^{(n)}(t) = x_i^{(n)}(t) - x_i(t)$, for $n = 0, 1, \ldots$, and $i = 1, \ldots, m$. Now, from the uniform boundedness of the sequences $\{x_i^{(n)}\}_{n=0}^{\infty}$,

$i = 1, \ldots, m$, there exist constants M_i such that $|x_i^{(n)}(t)| \leq M_i$ on Ω, for each i. On the other hand, since x_i, $i = 1 \ldots, m$ are continuous on Ω, there exist constants N_i such that $|x_i(t)| \leq N_i$, $i = 1, \ldots, m$ on Ω. Therefore, we have

$$\left|E_i^{(n+1)}(t)\right| \leq \int_{t_0}^{t} \left(\sum_{j=1}^{m} |\beta_{ij}| \, |E_j^{(n)}(s)| + \sum_{k=1}^{m} \left(\left|E_k^{(n)}(s)\right| \sum_{j=1}^{m} |\gamma_{ijk}| \left|x_j^{(n)}(s)\right| \right) + \sum_{j=1}^{m} \left(\left|E_j^{(n)}(s)\right| \sum_{k=1}^{m} |\gamma_{ijk}| \left|x_k^{(n)}(s)\right| \right) \right) ds.$$

On the other hand, we have

$$\sum_{j=1}^{m} |\gamma_{ijk}| \left|x_j^{(n)}(t)\right| \leq \sum_{j=1}^{m} |\gamma_{ijk}| \, M_j =: c_{ik},$$
$$\sum_{k=1}^{m} |\gamma_{ijk}| \left|x_k^{(n)}(t)\right| \leq \sum_{k=1}^{m} |\gamma_{ijk}| \, M_k =: c'_{ij}.$$

Therefore,

$$\begin{aligned}
\left|E_i^{(n+1)}(t)\right| &\leq \int_{t_0}^{t} \left(\sum_{j=1}^{m} |\beta_{ij}| \, |E_j^{(n)}(s)| + \sum_{k=1}^{m} (c_{ik} + c'_{ik}) \left|E_k^{(n)}(s)\right| \right) ds \\
&\leq \int_{t_0}^{t} \left(\sum_{j=1}^{m} \left(|\beta_{ij}| + c_{ij} + c'_{ij}\right) |E_j^{(n)}(s)| \right) ds \\
&\leq \int_{t_0}^{t} \left(\sum_{j=1}^{m} \bar{c}_{ij} |E_j^{(n)}(s)| \right) ds
\end{aligned}$$

where $\bar{c}_{ij} = |\beta_{ij}| + c_{ij} + c'_{ij}$. Now, by using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
\left|E_i^{(n+1)}(t)\right| &\leq \int_{t_0}^{t} \left(\sum_{j=1}^{m} \bar{c}_{ij} |E_j^{(n)}(s)| \right) ds \\
&\leq \int_{t_0}^{t} \left(\sum_{j=1}^{m} \bar{c}_{ij}^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{m} |E_j^{(n)}(s)|^2 \right)^{\frac{1}{2}} ds.
\end{aligned}$$

Let

$$S_n(t) = \left(\sum_{j=1}^{m} |E_j^{(n)}(t)|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \tilde{c}_i = \left(\sum_{j=1}^{m} \bar{c}_{ij}^2 \right)^{\frac{1}{2}}.$$

Therefore, for $i = 1, \ldots, m$, we have

$$\left| E_i^{(n+1)}(t) \right| \leq \tilde{c}_i \int_{t_0}^{t} S_n(s) ds.$$

It is straightforward to verify that

$$S_{n+1}(t) \leq c \int_{t_0}^{t} S_n(s) ds,$$

where

$$c = \left(\sum_{j=1}^{m} \tilde{c}_{ij}^2 \right)^{\frac{1}{2}}.$$

Now, letting $M = \max_{t \in \Omega} S_0(t)$, we proceed as follows

$$\begin{aligned}
S_1(t) &\leq c \int_{t_0}^{t} S_0(s) ds \leq M \int_{t_0}^{t} ds = cM(t - t_0), \\
S_2(t) &\leq c \int_{t_0}^{t} S_1(s) ds \leq c \int_{t_0}^{t} cM(s - t_0) ds = c^2 M \frac{(t - t_0)^2}{2}, \\
S_3(t) &\leq c \int_{t_0}^{t} S_2(s) ds \leq c \int_{t_0}^{t} c^2 M \frac{(s - t_0)^2}{2} ds = c^3 M \frac{(t - t_0)^3}{3!}, \\
&\vdots \\
S_n(t) &\leq c \int_{t_0}^{t} S_{n-1}(s) ds \leq c \int_{t_0}^{t} c^{n-1} M \frac{(s - t_0)^{n-1}}{(n-1)!} ds = c^n M \frac{(t - t_0)^n}{n!}.
\end{aligned}$$

Therefore

$$S_n(t) \leq M \frac{(c(t - t_0))^n}{n!} \leq M \frac{(ct_1)^n}{n!} \to 0,$$

as n tends to infinity and this completes the proof. □

4 Application of the method to a model for HIV infection of CD4$^+$ T cells

The Human Immunodeficiency Virus (HIV) is a retrovirus that targets the CD4$^+$ T lymphocytes, which are the most abundant white blood cells of the immune system. Although HIV infects other cells also, it wreaks the most havoc on the CD4$^+$ T cells by causing their decline and destruction, thus decreasing the resistance of the immune system [36].

Mathematical models have become important tools in analyzing the dynamics of HIV infection [16, 20, 21, 22, 36, 39]. The model which we investigate in this paper is given by the following system of ordinary differential equations [5, 6, 18, 38, 30]:

$$\begin{cases} \dfrac{dT}{dt} = p - \alpha T + rT(1 - \dfrac{T+I}{T_{\max}}) - kVT, \\ \dfrac{dI}{dt} = kVT - \beta I, \\ \dfrac{dV}{dt} = N\beta I - \gamma V, \end{cases} \tag{4}$$

with the initial conditions:

$$T(0) = r_{1,0}, \quad I(0) = r_{2,0}, \quad V(0) = r_{3,0},$$

where $T(t)$ and $I(t)$ represent, respectively, the concentration of healthy CD4^+ T cells and infected CD4^+ T cells at time t and $V(t)$ represents the concentration of free HIV at time t. As we see system (4) is a special of Eq. (1).

The parameters p, α, r, k, β, N, γ and $T_{\max}$ are assuming only positive values. These parameters are defined as follows: p is the source of CD4^+ T cells precursors, α is the natural turn-over rate of CD4^+ T cells, r is their growth rate, and $T_{\max}$ is their carrying capacity, k is the infection rate, β is a blanket death term for infected cells, γ is the lytic death rate for infected cells and N viral particles are released by each lysing cell.

Recently, Ongun in [30] proposed the Laplace Adomian decomposition method (LADM) and the LADM-Padé for solving system (4). To investigate this model, we apply the MVIM for computing an approximate solution to (4).

According the results presented in Section 3, the VIM for solving (4) can be written as

$$\begin{cases} T_{n+1}(t) = T_n(t_0) - \displaystyle\int_{t_0}^{t} \Big[-p + \alpha T_n(s) \\ \qquad\qquad -rT_n(s)\left(1 - \dfrac{T_n(s)+I_n(s)}{T_{\max}}\right) + kV_n(s)T_n(s)\Big] ds, \\ I_{n+1}(t) = I_n(t_0) - \displaystyle\int_{t_0}^{t} \left[-kV_n(s)T_n(s) + \beta I_n(s)\right] ds, \\ V_{n+1}(t) = V_n(t_0) - \displaystyle\int_{t_0}^{t} \left[-\beta N I_n(s) + \gamma V_n(s)\right] ds. \end{cases} \tag{5}$$

Remark 1. *Although, the terms $-\alpha T$, $-\beta I$ and $-\gamma V$ in (4) are linear, we identify the multipliers approximately considering them as restricted variations. If we consider them as linear terms, then the Lagrange multipliers would be $\lambda_1(s) = -e^{\alpha(s-t)}$, $\lambda_2(s) = -e^{\beta(s-t)}$ and $\lambda_3(s) = -e^{\gamma(s-t)}$. In this case, the expressions for T_n, I_n and V_n for $n \geq 2$ become very complicated and computing a highly accurate solution would be difficult.*

5 Numerical examples

All the computations presented in this section have been obtained by the Maple software. We consider the following two sets of data

$$
\begin{array}{lllll}
(i) & p = 0.1, & \alpha = 0.02, & \beta = 0.3, & r = 3, \\
 & \gamma = 2.4, & k = 0.0027, & T_{\max} = 1500, & N = 10,
\end{array}
$$

$$
\begin{array}{lllll}
(ii) & p = 2.2, & \alpha = 0.2, & \beta = 0.5, & r = 0.02, \\
 & \gamma = 2.8, & k = 0.027, & T_{\max} = 1300, & N = 20,
\end{array}
$$

for Eq. (4). For both set of data, let $\Omega = [0, 1]$ and the initial conditions be

$$T(0) = 0.1, \quad I(0) = 0, \quad V(0) = 0.1.$$

The first set of data has been investigated by Ongun in [30]. We use the MVIM to solve Eq. (4). To do this, we partition the interval $[0, 1]$ into ten subintervals and apply six iterations of the VIM in each subinterval. Numerical results of the first set of data are given in Tables 1-3 and for the second set of data in Tables 4-6. To show the efficiency of the proposed method we also give the approximate solution computed by the fourth order Runge-Kutta (RK4) method with stepsize 0.01 and the Laplace Adomian decomposition method (LADM). As we observe, the numerical results presented in these tables show the proposed method provides a highly accurate solution to Eq. (4) and are better than those of the LADM method.

For more investigation, since the exact solution of the model is not in hand, we assume that the solution computed by the RK4 method is enough accurate and consider it as the exact solution. Then the plot of $\log_{10}$ of the errors for the computed solutions by MVIM and LADM are displayed in Figure 1 and Figure 2. These figures indicate that the new method is better than the LADM.

Table 1: Numerical comparison of $T(t)$ for the first set of data.

t	LADM	RK4	MVIM
0.0	0.1	0.1	0.1
0.2	0.2088073298	0.2088080833	0.2088080687
0.4	0.4061358315	0.4062405393	0.4062404869
0.6	0.7624762220	0.7644238890	0.7644237458
0.8	1.398082863	1.414046831	1.4140464860
1.0	2.507874151	2.591594802	2.5915940280

Table 2: Numerical comparison of $I(t)$ for the first set of data.

t	LADM	RK4	MVIM
0.0	0.0	0.0	0.0
0.2	$0.6032706956e-5$	$0.6032702150e-5$	$0.6032701792e-5$
0.4	$0.1315891002e-4$	$0.1315834073e-4$	$0.1315833962e-4$
0.6	$0.2123298178e-4$	$0.2122378506e-4$	$0.2122378281e-4$
0.8	$0.3024270157e-4$	$0.3017741955e-4$	$0.3017741544e-4$
1.0	$0.4033321858e-4$	$0.4003781468e-4$	$0.4003781139e-4$

Table 3: Numerical comparison of $V(t)$ for the first set of data.

t	LADM	RK4	MVIM
0.0	0.1	0.1	0.1
0.2	0.06187995305	0.06187984331	0.06187984461
0.4	0.03830818047	0.03829488788	0.03829488952
0.6	0.02391981608	0.02370455014	0.02370455148
0.8	0.01621234343	0.01468036377	0.01468036471
1.0	0.01605502238	0.00910084504	0.00910084552

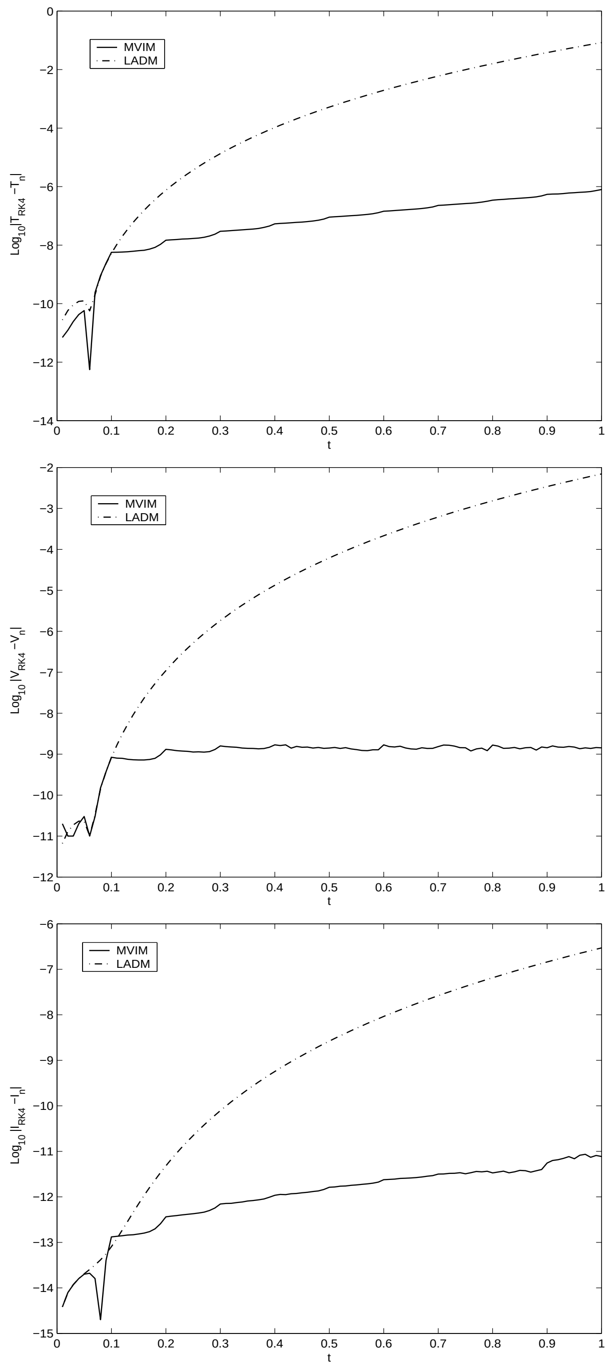

Figure 1: $\log_{10}$ of errors for the MVIM and LADM methods for the first set of data : top for $T(t)$; middle for $V(t)$ and down for $I(t)$.

Table 4: Numerical comparison of $T(t)$ for the second set of data.

t	LADM	RK4	MVIM
0.0	0.1	0.1	0.1
0.2	0.5285174083	0.5285174234	0.5285174236
0.4	0.9418326327	0.9418344479	0.9418344479
0.6	1.3405204714	1.3405495034	1.3405495030
0.8	1.7249888432	1.7251930023	1.7251930000
1.0	2.0953428921	2.0962593326	2.0962593280

Table 5: Numerical comparison of $I(t)$ for the second set of data.

t	LADM	RK4	MVIM
0.0	0.0	0.0	0.0
0.2	$1.172523849e-4$	$1.1723542076e-4$	$1.172354059e-4$
0.4	$2.701675420e-4$	$2.6814398247e-4$	$2.681439603e-4$
0.6	$4.222316707e-4$	$3.8989790302e-4$	$3.898978830e-4$
0.8	$6.981793786e-4$	$4.7092758249e-4$	$4.709275709e-4$
1.0	$1.537127434e-3$	$5.1749118375e-4$	$5.174911792e-4$

Table 6: Numerical comparison of $V(t)$ for the second set of data.

t	LADM	RK4	MVIM
0.0	0.1	0.1	0.1
0.2	0.0572073701	0.0572071940	0.0572071976
0.4	0.0330062295	0.0329846473	0.0329846514
0.6	0.0197116921	0.0193587664	0.0193587698
0.8	0.0142603131	0.0117277521	0.0117277547
1.0	0.0190408847	0.0074628631	0.0074628647

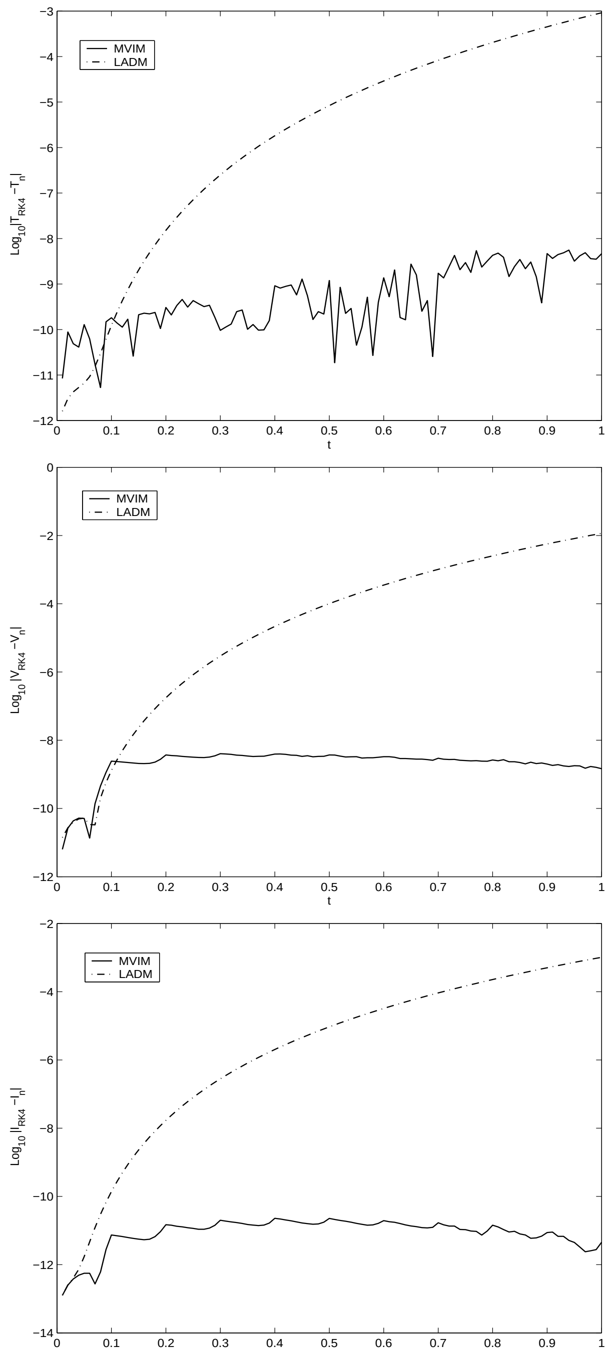

Figure 2: $\log_{10}$ of errors for the MVIM and LADM methods for the second set of data: top for $T(t)$; middle for $V(t)$ and down for $I(t)$.

6 Conclusion

We have applied the multistage variational iteration method (MVIM) for solving a general form of the system of first-order differential equations. A theorem for the convergence of the method has been presented. Then, the method has been applied to solve a model for HIV infection of $CD4^+$ T cells. Numerical results show that the MVIM is very effective and is superior to the LADM method.

Acknowledgements

Thanks are expressed to Dr. Davod Khojasteh Salkuyeh, editor in chief of the journal, for his valuable comments and suggestions. The author also would like to thank the Young Researchers and Elite Club, Islamic Azad University, Ardabil Branch for its financial support.

References

[1] I. Ateş and A. Yildirim, *Application of variational iteration method to fractional initial-value problems*, Int. J. Nonlin. Sci. Num. **10** (2009) 877–883.

[2] J. Biazar, *Solution of the epidemic model by Adomian decomposition method*, Appl. Math. Comput. **173** (2006) 1101–1106.

[3] J.C. Butcher, *Numerical Methods for Ordinary Differential Equations*, John Wiley & Sons, 2003.

[4] C. Chun, *Variational iteration method for a reliable treatment of heat equations with ill-defined initial data*, Int. J. Nonlin. Sci. Num. **9** (2008) 435–440.

[5] R.V. Culshaw and S. Ruan, *A delay-differential equation model of HIV infection of $CD4^+$ T-cells*, Math Biosci. **165** (2000) 27–39.

[6] Y. Ding and H. Ye, *A fractional-order differential equation model of HIV infection of $CD4^+$ T-cells*, Math. Comput. Model. **50** (2009) 386–392.

[7] F.Z. Geng and M.G. Cui, *Solving nonlinear multi-point boundary value problems by combining homotopy perturbation and variational iteration methods*, Int. J. Nonlin. Sci. Num. **10** (2009) 597–600.

[8] S.M. Goh, M.S.M. Noorani and I. Hashim, *Introducing variational iteration method to a biochemical reaction model*, Nonlinear Anal. Real World Appl. **11** (2010) 2264–2272.

[9] J.H. He, *A new approach to nonlinear partial differential equations*, Commun. Nonlinear Sci. Num. Simul. **2** (1997) 230–235.

[10] J.H. He, *Approximate analytical solution for seepage flow with fractional derivatives in porous media*, Comput. Meth. Appl. Mech. Eng. **167** (1998) 27–68.

[11] J.H. He, *Approximate solution of nonlinear differential equations with convolution product nonlinearities*, Comput. Meth. Appl. Mech. Eng. **167** (1998) 69–73.

[12] J.H. He, *Variational iteration method - a kind of non-linear analytical technique: Some examples*, Int. J. Non-Linear Mech. **34** (1999) 699–708.

[13] J.H. He, *Variational iteration method for autonomous ordinary differential systems*, Appl. Math. Comput. **114** (2000) 115–123.

[14] J.H. He, *Variational iteration method for delay differential equations*, Commun. Nonlinear Sci. Num. Simul. **2** (1997) 235–236.

[15] J.H. He, *Notes on the optimal variational iteration method*, Appl. Math. Lett. **25** (2012) 1579–1581.

[16] D. Ho, A. Neumann, A. Perelson, W. Chen, J. Leonard and M. Markowitz, *Rapid turnover of plasma virions and $CD4^+$ lymphocytes in HIV-1 infection*, Nature, **373** (1995) 123–126.

[17] H. Jafari and V. Daftardar-Gejji, *Revised Adomian decomposition method for solving systems of ordinary and fractional differential equations*, Appl. Math. Comput. **181** (2006) 598–608.

[18] X. Jiang, X. Zhou, X. Shi and X. Song, *Analysis of stability and Hopf bifurcation for a delay-differential equation model of HIV infection of $CD4^+$ T-cells*, Chaos Soliton. Fract. **38** (2008) 447–460.

[19] D.W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations*, third edition, Oxford University Press, 1999.

[20] A.S. Perelson, D.E. Kirschner and R. De Boer, *Dynamics of HIV infection of $CD4^+$ T cells, Mathematical Biosciences*, **114** (1993) 81–125.

[21] A.S. Perelson and P.W. Nelson, *Mathematical analysis of HIV-1 dynamics in vivo*, SIAM Rev. **41** (1999) 3–44.

[22] L. Rong, M.A. Gilchrist, Z. Feng and A.S. Perelson, *Modeling within host HIV-1 dynamics and the evolution of drug resistance: Trade offs between viral enzyme function and drug susceptibility*, J. Theor. Biol. **247** (2007) 804–818.

[23] R.J. LeVeque, *Finite difference methods for ordinary and partial differential equations: steady-state and time-dependent problems*, SIAM, Philadelphia, 2007.

[24] S. Liao, *On the homotopy analysis method for nonlinear problems*, Appl. Math. Comput. **147** (2004) 499–513.

[25] S. Liao, *An optimal homotopy-analysis approach for strongly nonlinear differential equations*, Commun. Nonlinear Sci. Num. Simul. **15** (2010) 2003–2016.

[26] M.A. Noor and S.T. Mohyud-Din, *Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials*, Int. J. Nonlin. Sci. Num. **9** (2008) 141–156.

[27] M.S.M. Noorani, I. Hashim, R. Ahmad, S.A. Bakar, E.S. Ismail and A.M. Zakaria, *Comparing numerical methods for the solutions of the Chen system*, Chaos Soliton. Fract. **32** (2007) 1296–1304.

[28] Z.M. Odibat, *A study on the convergence of variational iteration method*, Math. Comput. Model. **51** (2010) 1181–1192.

[29] Z.M. Odibat, C. Bertelle, M.A. Aziz-Alaoui and G.H.E. Duchamp, *A multi-step differential transform method and application to non-chaotic and chaotic systems*, Computt. Math. Appl. **59** (2010) 1462–1472.

[30] M.Y. Ongun, *Laplace Adomian decomposition method for solving a model for HIV Infection of* $CD4^+$ *T cells*, Math. Comput. Model. **53** (2011) 597–603.

[31] H. Ozer, *Application of the variational iteration method to the boundary value problems with jump discontinuities arising in solid mechanics*, Int. J. Nonlin. Sci. Num. **8** (2007) 513–518.

[32] D.K. Salkuyeh, *Convergence of the variational iteration method for solving linear systems of ODEs with constant coefficients*, Comput. Math. Appl. *56* (2008) 2027–2033.

[33] D.K. Salkuyeh and H.R. Ghehsareh, *Convergence of the variational iteration method for the telegraph equation with integral conditions*, Numer, Meth. Part. D. E. **27** (2011) 1442–1455.

[34] A.K. Sen, *An approximate solution for the transient kinetic behavior of enzyme-catalyzed reactions: the irreversible Michaelis-Menten kinetics*, J. Math. Biosci. **85** (1987) 145–151.

[35] A.K. Sen, *An application of the Adomian decomposition method to the transient behavior of a model biochemical reaction*, J. Math. Anal. Appl. **131** (1988) 232–245.

[36] P.K. Srivastava and P. Chandra, *Modeling the dynamics of HIV and* $CD4^+$ *T cells during primary infection*, Nonlinear Anal. Real World Appl. **11** (2010) 612–618.

[37] M. Tatari and M. Dehghan, *On the convergence of He's variational iteration method*, J. Comput. Appl. Math. **207** (2007) 121–128.

[38] L. Wang and M.Y. Li, *Mathematical analysis of the global dynamics of a model for HIV infection of* $CD4^+$ *T cells*, Math. Biosci. **200** (2006) 44–57.

[39] X. Wang and X. Song, *Global stability and periodic solution of a model for HIV infection of* $CD4^+$ *T cells*, Appl. Math. Comput. **189** (2007) 1331–1340.