Journal of Mathematical Modeling
Vol. 2, No. 1, 2014, pp. 74-89

JMM

Arrival probability in the stochastic networks with an established discrete time Markov chain

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Abstract. The probable lack of some arcs and nodes in the stochastic networks is considered in this paper, and its effect is shown as the arrival probability from a given source node to a given sink node. A discrete time Markov chain with an absorbing state is established in a directed acyclic network. Then, the probability of transition from the initial state to the absorbing state is computed. It is assumed to have some wait states, if there is a physical connection but not any immediate communication between two nodes. The Numerical results show, the critical nodes and arcs are detected by the proposed method and it can be used to anticipate probable congestion in communication and transportation networks.

Keywords: Stochastic networks, unstable networks, stochastic shortest path, discrete time Markov chain.
AMS Subject Classification: 90B15, 90C40.

1 Introduction

The shortest path problem is one of the fundamental network optimization problems. The deterministic shortest path problem has been studied

[*]Corresponding author.
Received: 3 February 2014 / Revised: 24 February 2014 / Accepted: 26 February 2014

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extensively and applied in many fields of optimization. There are polynomial time algorithms to solve the deterministic shortest path problem [5, 3, 11]. However, the stochasticity nature of real world problems especially telecommunication networks and transportation networks caused the new stochastic version of problem appears [4]. The stochastic shortest path problem (SSP) is defined in various aspects of finding the best path with stochastic optimality conditions. However, most of researchers considered stochastic changes of the arc lengths in a network or the cost variables was stochastic variables. In their works, it was supposed continuous or discrete probability distribution functions for the arc lengths and costs, then tried to find the shortest path with the most probability to arrive from a given source node to a given sink node. In this paper, the arrival probability is computed in a network with discrete distribution probability of the arcs existence.

Wu et al. [14] considered a public transportation network that arcs can only be traversed at certain points in time and weights of these arcs change in a day. They modeled such a public transportation network as a stochastic and time-dependent network. To find optimal routs through a network with probability distributed arc weights, they applied k-shortest path algorithms. Fan et al. [6] proposed a procedure for identifying dynamic routing policies in stochastic transportation networks. The goal was to identify the next node to visit such that the probability of arriving at the destination is maximized which is called arriving on time problem. The Bellman principle of optimality was applied to formulate the mathematical model of this problem.

Numerous researchers attempted to model and solve SSP using Markov decision process. Kulkarni [10] developed an exact method to compute the distribution function of the length of the shortest path from a given source node to a given sink node in a directed network with exponentially distributed arc lengths. A Continuous Time Markov Chain (CTMC) is constructed and the time until absorption into the absorbing state starting from the initial state is equal to the length of the shortest path in the network. There are some basic assumptions in the Kulkarni's paper; the network is modeled as a communication network which is an acyclic directed network. In each transition possibly there are some useless nodes and arcs those are unable to transmit message; so in such a transition from one state to another more than one node or one arc can be added to the new state. Our assumptions are similar to Kulkarni's work, but it is a Discrete Time Markov Chain (DTMC) and in any state transition only one node can be added to the new state because there some wait states are assumed. Also, in the proposed model of this paper there is not useless nodes and a unit flow is sent from the source node to the sink node.

Azaron and Modarres [2] introduced a method to solve a queuing network problem by solving equivalent stochastic shortest path problem. They developed Kulkarni's method to queuing networks. The service time in each station is either exponentially distributed or belongs to a special class of Coxian distribution. The transport times between every pair of service stations are independent random variables with exponential distributions. In their proposed method the network of queues is transformed into an equivalent stochastic network and then the distribution function of the shortest path of this stochastic network is determined. The wait states are extended in our work; it is possible to wait in the nodes like queueing networks and further there are some wait states that by traversing some arcs, it is not created a new state.

Thomas and White [13] discussed the congestion control in communication networks. By using a Markov decision process, they modeled and analyzed the problem of constructing a minimum expected total cost route from an origin to a destination. They considered the congestion dissipates over time according to some known probability distribution. The costs of arcs are stochastic but there is an option to wait with constant cost. The problem objective is to determine a policy for selecting a path that minimizes the expected total cost of the trip. Fan et al. [7] considered a congestible network with correlated link costs and wanted to minimize the expected travel time. Each link is assumed to be in one of two possible conditions congested or uncongested. Conditional probability density functions for link travel times are assumed known for each condition. We assume the lack of arcs and nodes to transmit messages because of congestion or other accidents. Also, there are the physical state to make decision according to current state.

The computation of minimal length path in incomplete stochastic networks, when travel times between nodes are allowed to be exponentially distributed random variables, was formulated as a linear programming problem by Peer and Sharma [12]. There are two sets of nodes in their assumption, with possible failure and without failure or always working. They use random number generation to read the message travel times and to obtain the objective function in order to solve it by simplex method. For the different set of random numbers generated, the various streams of travel times are obtained and applied in the objective function in order to obtain the set of expected shortest paths in determining distribution. Ji in [9] proposed the concepts of expected shortest path, α-shortest path and the most shortest path, and presented three related new types of models, expected value model, chance-constrained programming and dependent-chance programming. In order to solve these models, a hybrid intelligent algorithm integrating stochastic simulation and genetic algorithm is developed.

In this paper, the routing problem is considered under uncertainty conditions related to the physical topology of a network. There are two options at any node toward the destination node, departing from the current state to a new state, when a larger labeled node is visited or wait in the current state expecting better conditions. A DTMC with an absorbing state is established and the objective is to find the probability of arrival from a given source node to a given sink node. The uncertainty of the topology causes several unstable connections between nodes, however the original topology of the network determine likely and unlikely connections between pairs of nodes. The conditional probability of departure from one node to other node is known. The transition matrix of DTMC is obtained and the probability of arrival to the sink node from the source node is computed.

This paper is organized as follow. In Section 2 some definitions and assumptions of networks with unstable topology is introduced. Section 3 reviews the concept of the stochastic process DTMC. In Section 4 the established DTMC in the network is described. Section 5 contains some numerical results of accomplishment of the proposed method on some networks with various topologies. Finally Section 6 is conclusion and the direction of future works is proposed.

2 Unstable network modeling

In this section we introduce some definitions and assumptions of networks with unstable topology. Consider network $G = (N, A)$ with node set N and arc set A. There exists a stable physical topology that shows the connection of any pair nodes $i,j \in N$ if $(i, j) \in A$. So, the communication is possible between those nodes that an arc connects them. G is a directed network and does not have any directed cycle. Then we can label nodes in topological order such that for any $(i, j) \in A$, $i < j$ (see [1]).

To model an unstable topology for network G think about the communication networks, where there are some physical connections between nodes but we can not traverse any more toward destination node because of unanticipated congestion. There are two options; first, wait at particular node expecting presence of some facilities to release from the current state. To model such states we consider some artificial loops (shown with dash arcs in Figure 1) at any nodes except the sink node. Second, although it dose

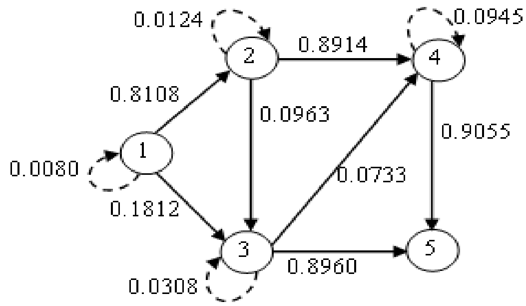

Figure 1: Example network with 5 nodes and 7 arcs

not create a new state, try to use some arcs. We define these two options as wait states because they do not create a new state. In our assumptions a new state is created whenever we can traverse to a new node with larger label. Figure 1 shows example network with its topological ordered nodes; this initial topology of network is stable topology. Node 1 is the source node and node 5 is the sink node. It is impossible to traverse arc $(1, 4)$ because it does not exist in the stable topology of example network. However, existence of arc $(3, 4)$ in the stable topology does not mean it is possible to communicate between node 3 and node 4 all the time. So, the existence probability is defined for all the arcs, those can be probably existed in the current unstable topology. In this paper, network $G(t) = (N(t), A(t))$ describes the network state in topology time t. We consider stable topology of network for $t = 0$ and unstable topologies for other times created from initial stable topology. So, the current topology of time t is an unstable topology if and only if there is an arc $(i, j) \in A \setminus A(t)$. Notice, time t is a nonnegative integer variable. The numbers on arcs in Figure 1 show the initial wait and leaving probabilities for any node.

The decision variable of arc (i, j) according to the topology of time t is shown by $x_{ij}(t)$. The probability of existence of arc (i, j) at current topology of time t is defined as follow

$$q_{ij}(t) = P[x_{ij}(t) = 1], \qquad (i, j) \in A(t) .$$

The existence of artificial arc (i, i) means we have decided to wait at node i either there are any departure arc from node i or not. The existence probability of arc (i, i) is

$$q_{ii}(t) = P[x_{ii}(t) = 1] = 1 - \textstyle\sum_{\{j:(i,j)\in A(t)\}} q_{ij}(t).$$

So, the wait probability in node i depends on leaving arcs of node i in time t. We compute the wait probability of node i for unstable topologies in two ways. In first method, if arc $(i, j) \in A \setminus A(t)$ then all leaving probabilities and the wait probability of node i are changed by reproducing random numbers. It is assumed in the first method some facilities can be provided instead of disabled arcs thus both the leaving and the wait probabilities are changed in this method. In the second method, if arc $(i, j) \in A \setminus A(t)$ then $q_{ii}(t) = q_{ii}(0) + q_{ij}(0)$.

3 Discrete time Markov chain (DTMC)

A stochastic process is called Markov chain if it satisfies the Markov property; the conditional probability of the next state is only depends on the current state and independent of the previous states

$$P[S_{r+1} = k \mid S_r = l, S_{r-1} = n, \dots, S_1 = m] = P[S_{r+1} = k \mid S_r = l].$$

Then the discrete time process $S = \{S_r, r = 1, 2, 3, \dots\}$ is a discrete time Markov chain. Although it is not mentioned, the process is the homogenous Markov chain (see [8]). The transition probability p_{kl} satisfies the following conditions

I. $0 \leq p_{kl} \leq 1$

II. $\sum_l p_{kl} = 1$, for $k = 1, 2, \dots, |S|$.

The state transition probabilities as the entries of an $|S| \times |S|$ matrix P, where p_{kl} is the kth row and the lth column, is called the state transition probability matrix

$$P = [p_{kl}]_{|S| \times |S|}.$$

The transition probability matrix P is a stochastic matrix because for any row k, $\sum_{l=1}^{|S|} p_{kl} = 1$. The probability that the process is in state k before $P[S_1 = k]$ makes the first transition, then the set $\{P[S_1 = k]\}$ defines the initial conditions for the process and $\sum_k P[S_1 = k] = 1$. In this paper, the initial state of DTMC contains the single source node $S_1 = \{1\}$ and the probability of the transition to the next states is equal to the probability of the existence of the departure arcs plus the the probability of existence the artificial wait arc.

Let $P[X(n) = l]$ denotes the probability that the process is in state l at the end of the first n transition, then for a finite process the probability of arrival to the sink node by n transition is $\sum_n P[X(n) = S_{|S|}]$. Then, the arrival probability from the source node to the sink node by finite transitions in the network with established DTMC is defined as the summation probability of all paths from the initial state S_1 to the absorbing state $S_{|S|}$.

Table 1: The state space of the example network

State space	Current nodes
S_1	$\{1\}$
S_2	$\{1,2\}$
S_3	$\{1,3\}$
S_4	$\{1,2,3\}$
S_5	$\{1,2,4\}$
S_6	$\{1,3,4\}$
S_7	$\{1,2,3,4\}$
S_8	$\{1,2,3,4,5\}$

The state space S of the example network is shown in Table 1. State S_4 contains nodes $\{1,2,3\}$ and all connected components of network G constructed by nodes 1, 2 and 3. Figure 2 shows constructed connected components of state S_3 of DTMC for the example network.

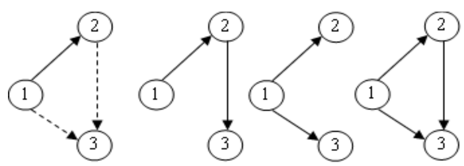

Figure 2: Constructed connected components of state S_4

4 DTMC for unstable topology

The states of DTMC contains those nodes are unable to receive message at the current topology of time t. In the final state the given sink node is reached and the process does not progress any more. So, the absorbing state contains all nodes of the network. It is not allowed to return from a reached node but could be waited. Following assumptions describes the creation of the state space of stochastic process DTMC

i. By arriving the sink node the process can traverse neither any node nor any arc (the absorbing state)

ii. According to the current state, it is allowed to traverse just one arc (not necessarily the departure arc of the current node with maximum labeled number amongst the current state nodes) *iii.* The next state is created if a new node is added to the set of previous nodes.

If in a directed acyclic network that $out - degree$ of any node is at least one except the given sink node, then for any node i in the network there is a path from the given source node to the given sink node which traverses the node i [1]. Therefore, any node could be in a path from the source node to the sink node. We suppose a message has been sent from the source node in all directions and want to determine when the sink node will be reached under the topology uncertainty conditions of the network. The wait probabilities which are the diagonal arrays of transition matrix P are obtained by Lemma 1. We fix time t and for convenience variable t is omitted in the following lemmas, however they are accurate for all the times and do not depend on variable t. The determined factors are q_{ij} and q_{ii} probabilities whether they are varying according to topology of time t.

Lemma 1. *Suppose P_{kk} is the (k,k) array of the matrix P (the transition probability matrix of DTMC for network G with unstable topology) and $S_k = \{v_0 = 1, \ldots, v_m\}$ is the current state then the wait probability of state S_k is*

$$P_{kk} = \begin{cases} 1 - \sum_{j=k+1}^{|S|} P_{kj}, & \text{if } k < |S| \\ 1, & \text{if } k = |S|. \end{cases}$$

During transition from the current state to the new state it is necessary to reach a node other than the current state nodes. The other arrays of transition matrix P are obtained by Lemma 2. To clarify the formula of transition probabilities P_{kl}, we define events $E_{v_i v_j}$ for traversing arc (v_i, v_j) during transition from state S_k to state S_l such that $|S_l - S_k| = 1$. The formula of P_{kl} contains two elements and should be computed. There are two options in node v_m of the current state S_k. To wait at v_m by traversing artificial arc (v_m, v_m) and to depart from v_m toward node $v_j \in S_l - S_k$. Notice it is impossible to wait at nodes $v_i \in S_k - \{v_m\}$ because it should be departed to reach the current state nodes however it is not necessary for node v_m with largest label.

If $v_j \in S_l - S_k$ then one or all of events $E_{v_i v_j}$ can be happen for $v_i \in S_k - \{v_m\}$ and $(v_i, v_j) \in A(t)$. The arrival probability of node v_j from the current state S_l is equal to $P[\bigcup_{\{v_i \in S_k - \{v_m\}, (v_i, v_j) \in A(t)\}} E_{v_i v_j}]$. It should be prevented the current state nodes to reach other nodes than $v_j \in S_l - S_k$ and it is equal to $\prod_{v_i \in S_k - \{v_m\}} (1 - \sum_{\{v_r \neq v_j, v_r \notin S_k, (v_i, v_r) \in A(t)\}} q_{v_i v_r})$.

Lemma 2. *The transition probability P_{kl} is the (k, l) array of the matrix P, and it shows the transition from state S_k to state S_l such that $S_k =$*

$\{v_0 = 1, \ldots, v_m\}$, $|S_l - S_k| = 1$ *and* $v_j \in S_l - S_k$, $k \neq l$, $l < |S|$. *If* $l < k$ *then* $P_{kl} = 0$, *and if* $l > k$ *we have*

$P_{kl} =$
$P[\bigcup_{\left\{\substack{v_i \in S_k - \{v_m\} \\ (v_i,v_j) \in A(t)}\right\}} E_{v_i v_j}] \times (\prod_{v_i \in S_k - \{v_m\}} (1 - \sum_{\left\{\substack{v_r \neq v_j, v_r \notin S_k \\ (v_i,v_r) \in A(t)}\right\}} q_{v_i v_r})) \times q_{v_m v_m} +$
$q_{v_m v_j}$.

To compute the transition probabilities $P_{k|S|}$ for $k = 1, 2, \ldots, |S| - 1$ it should be noticed the final state is the absorbing state containing all nodes and the stochastic process does not progress any more. So, it is sufficient to consider leaving arcs toward the sink node from the current state nodes which is stated as Lemma 3.

Lemma 3. *To compute the transition probability from state* $S_k = \{v_0 = 1, \ldots, v_m\}$ *to the absorbing state* $S_{|S|}$ *suppose* $v_n \in S_{|S|}$ *is the given sink node of network* G, *then* $(k, |S|)$ *array of matrix* P, $k = 1, 2, \ldots, |S| - 1$ *is*

$$P_{k|S|} = P[\bigcup_{\left\{\substack{v_i \in S_k, \\ (v_i,v_n) \in A(t)}\right\}} E_{v_i v_n}].$$

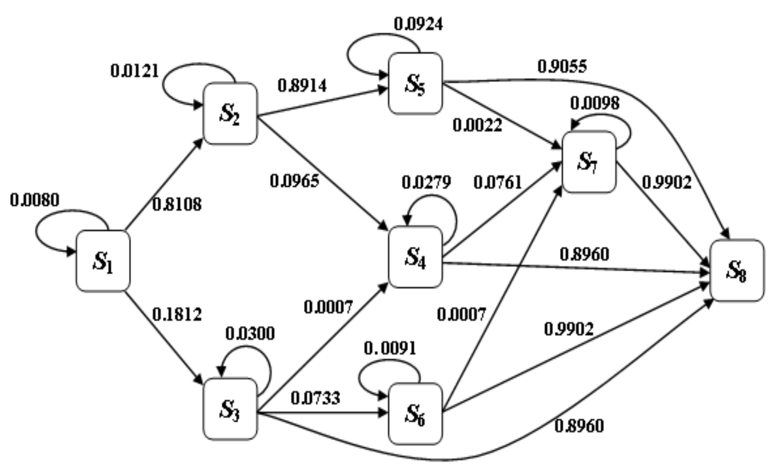

Figure 3: The state space diagram of the established DTMC

The state space diagram of the established DTMC for the example network when $t = 0$ is created as Figure 3; the values on arcs show waiting and transition for any state of the initial topology of the example network. We want to obtain the probability of arrival to node 5 from node 1 which is defined as the summation probability of all paths from the initial state S_1 to the absorbing state S_8. For the example network three unstable topologies are considered. The disabled arcs in various topologies and the effects on the wait probabilities are shown in Table 2.

Table 2: The disabled arcs during topology of time t

Time t	Disabled Arcs	Changed Wait Probabilities	
		Method 1	Method 2
1	$(2,3)$	$q_{22} = 0.4129$	$q_{22} = 0.7747$
2	$(1,3),(3,5)$	$q_{11} = 0.2692, q_{33} = 0.4981$	$q_{11} = 0.6730, q_{33} = 0.5433$
3	$(2,3),(2,4),(3,5)$	$q_{22} = 1, q_{33} = 0.0391$	$q_{22} = 1, q_{33} = 0.5433$

Transition probability P_{47} computed from constructed states as shown in Figure 4 and it is equal to $q_{33}q_{24} + q_{34}$. It is possible to wait at node 3 but not other nodes of state S_4.

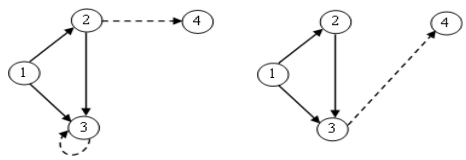

Figure 4: The constructed states during transition from S_4 to S_7

We apply two different methods to recompute the wait and leaving probabilities of any node when one of its leaving arcs is disabled during unstable topologies. In first method, we reproduce random numbers for current existent arcs which the arrival probability and its changes by this method is shown as solid line in Figure 5. In second method, we just add the leaving probability of omitted arc on the wait probability of the head node which the arrival probability and its changes by this method is shown as dashed line in Figure 5. It is seen, when arcs $(1,3)$ and $(3,5)$ are disabled, the arrival probability is reduced rapidly; however the omission of arc $(2,3)$ dose not affect the arrival probability.

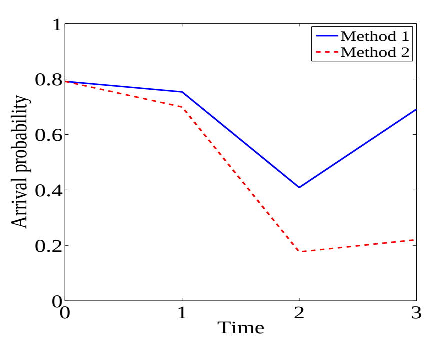

Figure 5: Arrival probabilities of the example network

Table 3: The disabled arcs of networks 1, 2 and 3

	Disabled Arcs			
Time t	1	2	3	4
Network 1	$(2,3)$ $(3,10)$ $(5,8)$ $(7,8)$ $(10,11)$	$(1,3)$ $(1,6)$ $(4,8)$ $(5,8)$ $(7,8)$	$(1,2)$ $(1,6)$ $(2,6)$ $(2,8)$ $(6,7)$ $(10,11)$	$(6,10)$
Network 2	$(11,15)$	$(2,7)$ $(5,6)$ $(10,14)$ $(11,16)$	$(5,9)$ $(6,7)$ $(10,13)$ $(10,14)$	$(1,2)$ $(1,6)$ $(2,3)$ $(2,5)$ $(2,7)$ $(11,14)$
Network 3	$(4,6)$	$(2,5)$ $(5,9)$ $(2,11)$ $(6,7)$ $(6,11)$ $(8,10)$	$(3,4)$ $(4,5)$ $(4,10)$	$(5,10)$ $(8,9)$ $(8,10)$ $(8,11)$ $(9,10)$

5 Numerical examples

In this section some executions of the proposed method on various networks are presented. All topologies (stable and unstable) for all networks are created randomly. The leaving and waiting probabilities of nodes are stochastic numbers produced by uniform distribution function. Three networks with different topologies are considered. For all networks 4 unstable topologies are assumed. The disabled arcs during four unstable topologies is determined in Table 3.

The first network is created randomly with 11 nodes and 28 arcs shown in Figure 6. We use two propositions 1 and 2 inductively to be sure there will be a path from the source node to the sink node in its beginning topology and the created network is a cycle-free network (see [1]).

Proposition 1. *If node k is the first node with larger index than source node 1 and in-degree$(k) = 0$; let $1 \leq l < k$ is an arbitrary node then adding arc (l, k) there exists a path from source node 1 to node k.*

Proposition 2. *If node k is the first node with smaller index than sink node n and out-degree$(k) = 0$; let $k < l \leq n$ is an arbitrary node then adding arc (k, l) there exists a path from node k to sink node n.*

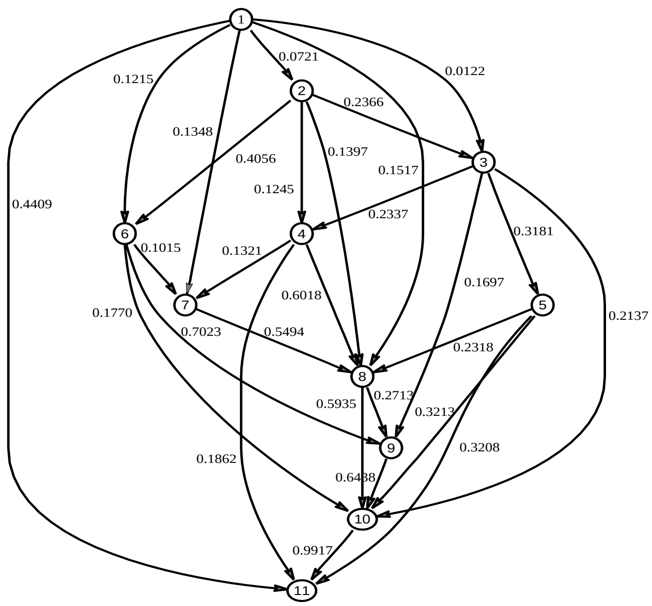

Figure 6: Network 1 with 11 nodes and 28 arcs

The probabilities of arrival to the sink node in all topologies of network 1 are shown in Figure 7.

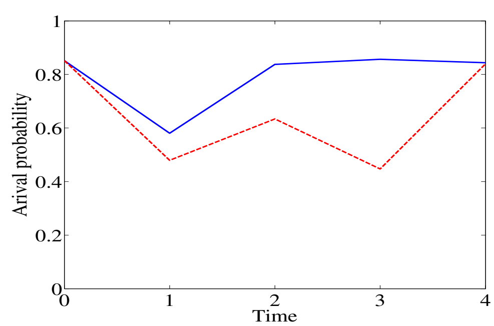

Figure 7: Arrival probabilities of network 1

The second network is created with 16 nodes and 42 arcs shown in Figure 8 and it is a gridded network.

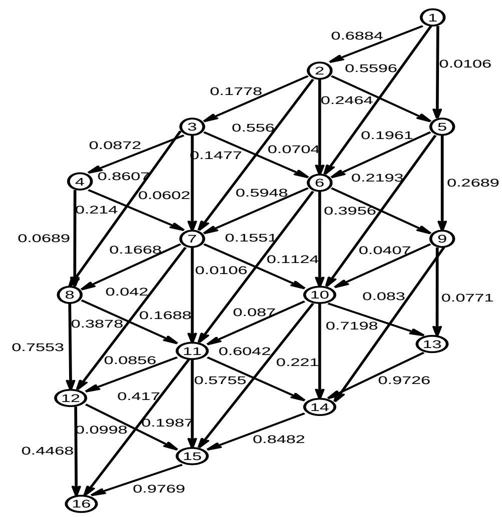

Figure 8: Network 2 with 16 nodes and 42 arcs

The probabilities of arrival to the sink node in all topologies of network 2 are shown in Figure 9.

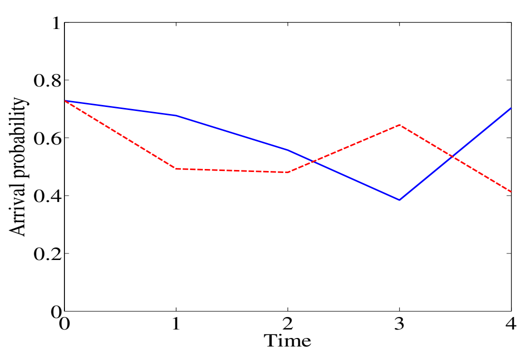

Figure 9: Arrival probabilities of the example network 2

The third network is created with 11 nodes and 55 arcs and it is a complete network. The traverse probabilities of edges for complete network 3 is given as follows

$$\begin{bmatrix}
0.0031 & 0.0128 & 0.0089 & 0.6263 & 0.0132 & 0.0485 & 0.0741 & 0.0051 & 0.0338 & 0.0943 & 0.0799 \\
0 & 0.0402 & 0.0131 & 0.0407 & 0.0817 & 0.0761 & 0.0822 & 0.0832 & 0.0365 & 0.4500 & 0.0963 \\
0 & 0 & 0.0374 & 0.0802 & 0.0108 & 0.0690 & 0.6994 & 0.0190 & 0.0257 & 0.0460 & 0.0126 \\
0 & 0 & 0 & 0.0210 & 0.0579 & 0.0758 & 0.0160 & 0.0993 & 0.1378 & 0.5505 & 0.0417 \\
0 & 0 & 0 & 0 & 0.0632 & 0.1556 & 0.0444 & 0.1092 & 0.4803 & 0.0660 & 0.0813 \\
0 & 0 & 0 & 0 & 0 & 0.0273 & 0.1347 & 0.1545 & 0.0903 & 0.0119 & 0.5813 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0871 & 0.0325 & 0.0020 & 0.1639 & 0.7146 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1205 & 0.8011 & 0.0363 & 0.0422 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0336 & 0.0710 & 0.8954 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0841 & 0.9159 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000
\end{bmatrix}$$

The probabilities of arrival to the sink node in all topologies are shown in Figure 10.

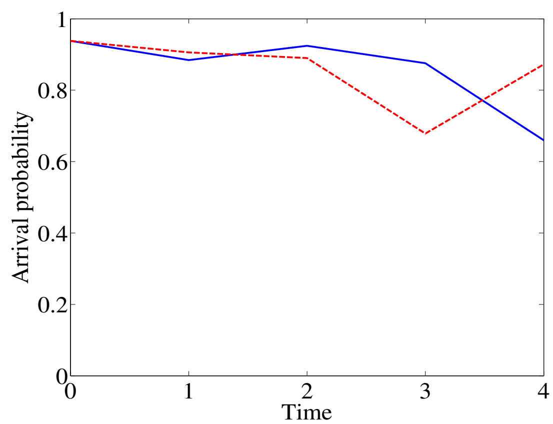

Figure 10: Arrival probabilities of the example network 3

6 Conclusion

We modeled the unstable topology of networks as discrete time Markov chain (DTMC) stochastic process. The arrival probability from a given source node to a given sink node is computed in various networks. Some arcs are disabled during the considered unstable topologies and the effects on the arrival probability were illustrated. Accomplishment of proposed method on various networks shows the omission of some arcs can hit the arrival probability and some other arcs if are omitted their effects are not considerable. Although there is not any restriction, arcs selection to be disabled and their number are randomly done in this paper. Extension of described model to the continuous time varying networks and using the meta-heuristic methods to apply the proposed method on the larger scale networks can be considered as feature works guidelines.

Acknowledgements

The authors would like to thank the editor and the anonymous referees for their helpful comments.

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