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Basic results on distributed order fractional hybrid differential equations with linear perturbations

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Abstract. In this article, we develop the distributed order fractional hybrid differential equations (DOFHDEs) with linear perturbations involving the fractional Riemann-Liouville derivative of order $0 < q < 1$ with respect to a nonnegative density function. Furthermore, an existence theorem for the fractional hybrid differential equations of distributed order is proved under the mixed φ-Lipschitz and Caratheodory conditions. Some basic fractional differential inequalities of distributed order are utilized to prove the existence of extremal solutions and comparison principle.

Keywords: Fractional hybrid differential equations, distributed order, extremal solutions, Banach algebra.
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1 Introduction

The fractional differential equations have received increasing attention, because the behavior of many physical systems can be properly described by using the fractional order system theory [12, 19]. In recent years, quadratic and linear perturbations of nonlinear differential equations in the Banach algebras, have attracted much attention to researchers. A perturbation of a nonlinear equation which involves the addition or subtraction of a term

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is called a linear perturbation and a perturbation which involves the multiplication or division by a term is called a quadratic perturbation of the equations. The details of different types of perturbations for a nonlinear differential and integral equations are given in Dhage [11]. These type of equations have been called the hybrid differential equations [6, 7, 8, 18]. Dhage and Lakshmikantham [10] established existence, uniqueness and some fundamental differential inequalities for the first order hybrid differential equations with quadratic perturbations of second type. Later, Zhao et al. [21] developed the following fractional hybrid differential equations (FHDE) involving the Riemann-Liouville differential operators of order $0 < q < 1$, with quadratic perturbations of second type

$$\begin{cases} D^q[\frac{x(t)}{f(t,x(t))}] = g(t, x(t)), \quad t \in J, \\ x(0) = 0, \end{cases} \tag{1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R})$. Dhage and Jadhav [9] discussed the first-order hybrid differential equations with linear perturbations of second type. Next, Lu et al. [14] developed the following FHDE involving the Riemann-Liouville differential operators of order $0 < q < 1$, with linear perturbations of second type:

$$\begin{cases} D^q[x(t) - f(t, x(t))] = g(t, x(t)), \quad t \in J, \\ x(t_0) = x_0, \end{cases} \tag{2}$$

where $f, g \in C(J \times \mathbb{R}, \mathbb{R})$. They established the existence, uniqueness and some fundamental fractional differential inequalities to prove existence of the extremal solutions of Eq. (2). Also, they considered necessary tools under the φ-Lipschitz conditions to prove the comparison principle.

Now, in this article in view of the distributed order fractional derivative [2, 3, 4], we develop the distributed order fractional hybrid differential equations (DOFHDEs) involving linear perturbations of second type with respect to a nonnegative density function.

In this regard, in Section 2 we introduce the distributed order fractional hybrid differential equations with linear perturbations of second type. Section 3 is about existence results for these equations. In Section 4, we express some fundamental fractional differential inequalities of distributed order. Next, we prove the existence theorem for this class and express the existence of extremal solution theorem and comparison theorem in sections 5 and 6. Finally, the main conclusions are set.

2 Distributed order fractional hybrid differential equation

In this section, we recall some definitions which are used throughout this paper. Let $\mathbb{R}$ be the real line and $J = [0, T]$ be a bounded interval in $\mathbb{R}$ for some $T \in \mathbb{R}$. Also, let $C(J \times \mathbb{R}, \mathbb{R})$ denotes the class of continuous functions $f : J \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{C}(J \times \mathbb{R})$ denote the Caratheodory class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ which are Lebesgue integrable bounded by a Lebesgue integrable function on J. Moreover

(i) the map $t \mapsto g(t, x)$ is measurable for each $x \in \mathbb{R}$,

(ii) the map $x \mapsto g(t, x)$ is continuous for each $t \in J$.

Definition 1. ([12, 19]) The fractional integral of order q with the lower limit t_0 for the function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{1-q}} ds, \quad t > t_0,\ q > 0. \tag{3}$$

Definition 2. ([12, 19]) The Riemann-Liouville derivative of order q with the lower limit t_0 for the function $f : [t_0, \infty) \to \mathbb{R}$ is written as

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{q+1-n}} ds, \quad t > t_0,\ n-1 < q < n. \tag{4}$$

We consider the DOFHDEs, involving linear perturbations of second type and the Riemann-Liouville differential operator of order $0 < q < 1$ with respect to the nonnegative density function $b(q) > 0$,

$$\begin{cases} \displaystyle\int_0^1 b(q) D^q [x(t) - f(t, x(t))] dq = g(t, x(t)), \quad t \in J, \quad \int_0^1 b(q) dq = 1, \\ \qquad\qquad\qquad\qquad x(0) = 0, \end{cases} \tag{5}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R})$. By a solution of DOFHDEs (5), we mean a function $x \in C(J, \mathbb{R})$ such that

(i) the function $t \mapsto x - f(t, x)$ is continuous for each $x \in \mathbb{R}$,

(ii) x satisfies Eq. (5).

Remark 1. Suppose that

$$b(q) = a_0 \delta(q - q_0) + a_1 \delta(q - q_1) + a_2 \delta(q - q_2) + \cdots + a_n \delta(q - q_n), \tag{6}$$

which $1 > q_n > q_{n-1} > \cdots > q_0 > 0$ and a_i for $i = 0, 1, 2, \dots, n$ is nonnegative constant coefficients and δ is the Dirac delta function. Also let $Y(t) = x(t) - f(t, x(t))$. For this case, the DOFHDE (5) is

$$\begin{aligned} a_0 D^{q_0}[Y(t)] + a_1 D^{q_1}[Y(t)] + \cdots + a_n D^{q_n}[Y(t)] = g(t, x(t)), \\ x(0) = 0, \end{aligned}$$

where $t \in J$. The details of special cases for density function $b(q)$ are given in Noroozi et al. [15, 16, 17].

3 Existence Result

In this section, we prove the existence results for the DOFHDE (5) on the closed and bounded interval $J = [0, T]$ under mixed φ-Lipschitz and Caratheodory conditions on the nonlinearities involved in it. We place the DOFHDE (5) in the space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. Define a supremum norm $\|.\|$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)|, \tag{7}$$

and a multiplication in $C(J, \mathbb{R})$ by

$$(xy)(t) = x(t)y(t), \tag{8}$$

for $x, y \in C(J, \mathbb{R})$. Clearly $C(J, \mathbb{R})$ is a Banach algebra with respect to above norm and multiplication in it. By $L^1(J, \mathbb{R})$ denote the space of Lebesgue integrable real-valued function on J equipped with the norm $\|.\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^T |x(s)| \, ds. \tag{9}$$

We prove the existence of a solution for DOFHDE (5) by a fixed point theorem in the Banach algebra due to Dhage [6].

Definition 3. Let X be a Banach space. A mapping $T : X \to X$ is called φ-Lipschitzian if there exists a continuous and nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|Tx - Ty\| \leq \varphi(\|x - y\|),$$

for all $x, y \in X$, where $\varphi(0) = 0$. Further, if φ satisfies the condition $\varphi(r) < r$, $r > 0$, then T is called a nonlinear contraction with a control function φ.

Theorem 1. *([6]) Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $A : X \to X$ and $B : S \to X$ be two operators such that*

(a) *A is nonlinear contraction;*

(b) *B is completely continuous;*

(c) *$Ax + Bx \in S$ for all $x \in S$.*

Then the operator equation $Ax + Bx = x$ has a solution in S.

At this point, we consider some hypotheses as follows.

(A_0) The function $x \mapsto x - f(t,x)$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.

(A_1) There exists a constants $P \geq L > 0$ such that

$$| f(t,x) - f(t,y) | \leq \frac{L \mid x-y \mid}{P + \mid x-y \mid}, \tag{10}$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

(A_2) There exists a function $h \in L^1(J, \mathbb{R})$ such that

$$| g(t,x) | \leq h(t), \tag{11}$$

for all $t \in J$ and $x \in \mathbb{R}$.

(A_3) $f(0,0) = 0$.

Theorem 2. *(Titchmarsh Theorem [5]) Let $F(s)$ be an analytic function which has a branch cut on the real negative semiaxis. Furthermore, $F(s)$ has the following properties*

$$F(s) = O(1), \quad |s| \to \infty, \tag{12}$$

$$F(s) = O(\frac{1}{|s|}), \quad |s| \to 0, \tag{13}$$

for any sector $|arg(s)| < \pi - \eta$, where $0 < \eta < \pi$. Then, the Laplace transform inversion $f(t)$ can be written as the Laplace transform of the imaginary part of the function $F(re^{-i\pi})$ as follows

$$f(t) = \mathcal{L}^{-1}\{F(s); t\} = \frac{1}{\pi} \int_0^{\infty} e^{-rt} \Im(F(re^{-i\pi}))dr. \tag{14}$$

We apply the following lemma to prove the main existence theorem of this section.

Lemma 1. *Assume that hypothesis (A_0) and (A_3) hold. Then, for any $h \in L^1(J, \mathbb{R})$ and $0 < q < 1$, the function $x \in C(J, \mathbb{R})$ is a solution of the DOFHDE (5) if and only if x satisfies the following equation*

$$x(t) = f(t, x(t)) + \frac{1}{\pi}\int_0^t \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t-\tau\} g(\tau, x(\tau)) d\tau, \tag{15}$$

such that $0 \leq \tau \leq t \leq T$ *and*

$$B(s) = \int_0^1 b(q) s^q dq. \tag{16}$$

Proof. Applying the Laplace transform on both sides of (5) and letting

$$Y(t) = x(t) - f(t, x(t)), \tag{17}$$

we have

$$\begin{aligned} \mathcal{L}\{\int_0^1 b(q) D^q Y(t) dq; s\} &= \mathcal{L}\{g(t, x(t)); s\} \\ &= \int_0^1 b(q)[s^q Y(s) - D_t^{q-1} Y(0)] dq \\ &= G(s). \end{aligned} \tag{18}$$

Since $Y(0) = 0$, we have

$$Y(s)(\int_0^1 b(q) s^q dq) = G(s),$$

and hence,

$$Y(s) = \frac{1}{B(s)} G(s), \tag{19}$$

such that

$$B(s) = \int_0^1 b(q) s^q dq. \tag{20}$$

Now, using the inverse Laplace transform on both sides of (19) and applying the convolution product, we get

$$\begin{aligned} \mathcal{L}^{-1}\{Y(s)\} &= x(t) - f(t, x(t)) = \mathcal{L}^{-1}\{\frac{1}{B(s)} G(s)\} \\ &= \int_0^t \mathcal{L}^{-1}\{\frac{1}{B(s)}; t-\tau\} g(\tau, x(\tau)) d\tau, \end{aligned}$$

or equivalently

$$x(t) = f(t, x(t)) + \int_0^t \mathcal{L}^{-1}\{\frac{1}{B(s)}; t-\tau\} g(\tau, x(\tau)) d\tau. \tag{21}$$

Since $B(s)$ is an analytic function which has a branch cut on the real negative semiaxis, according to the Titchmarsh theorem (2) we get

$$x(t) = f(t, x(t)) + \frac{1}{\pi}\int_0^t \int_0^\infty e^{-r(t-\tau)} \Im\{\frac{1}{B(re^{-i\pi})}\} g(\tau, x(\tau)) dr d\tau, \tag{22}$$

which by the Laplace transform definition, Eq. (15) is held. Conversely, assume x satisfies Eq. (15), therefore, x satisfies the equivalent equation (21). Hence, we have

$$x(t) - f(t, x(t)) = \int_0^t \mathcal{L}^{-1}\{\frac{1}{B(s)}; t-\tau\} g(\tau, x(\tau)) d\tau. \tag{23}$$

Using the Laplace transform operator on both sides of Eq. (23), Eq. (19) also holds. Since $Y(0) = 0$, we obtain Eq. (18) and by applying the inverse Laplace transform, (5) also holds. By $t = 0$ in Eq. (15), we have

$$x(0) - f(0, x(0)) = 0 = 0 - f(0, 0).$$

According to hypothesis (A_0), the map $x \mapsto x - f(0, x)$ is injective in $\mathbb{R}$ and hence $x(0) = 0$. □

Theorem 3. *Suppose that hypothesis (A_0)-(A_2) hold. Then, the DOFHDE* (5) *has a solution defined on J.*

Proof. We set $X = C(J, \mathbb{R})$ as a Banach algebra and define a subset S of X by

$$S = \{x \in X | \; \|x\| \le N\}, \tag{24}$$

where $N = L + F_0 + \dfrac{M \|h\|_{L^1}}{\pi}$ and $F_0 = \sup_{t \in J} |f(t, 0)|$.

Clearly, S is a closed, convex and bounded subset of the Banach algebra X. Now, using the hypotheses (A_0)-(A_2), it can be shown by an application of Lemma 1, DOFHDE (5) is equivalent to the nonlinear equation (15). Define two operators $A : X \longrightarrow X$ and $B : S \longrightarrow X$ by

$$Ax(t) = f(t, x(t)), \quad t \in J, \tag{25}$$

and

$$Bx(t) = \frac{1}{\pi}\int_0^t \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t-\tau\} g(\tau, x(\tau)) d\tau, \quad t \in J, \tag{26}$$

thus, from Eq. (15), we obtain an operator equation as follows:

$$Ax(t) + Bx(t) = x(t), \quad t \in J. \tag{27}$$

We will show that the operators A and B satisfy all the conditions of Theorem 1. First, we show that A is a Lipschitz operator on X with the Lipschitz constant L. let $x, y \in X$ which by hypothesis (A_1) we have

$$|Ax(t) - Ay(t)| = |f(t, x(t)) - f(t, y(t))| \leq \frac{L|x(t) - y(t)|}{P + |x(t) - y(t)|} \leq \frac{L\|x - y\|}{P + \|x - y\|},$$

and if for all $x, y \in X$ take a supremum over t, then we obtain

$$\|Ax - Ay\| \leq \frac{L\|x - y\|}{P + \|x - y\|}. \tag{28}$$

for all $x, y \in X$. This shows that A is a nonlinear contraction on X with a control function φ defined by $\varphi = \dfrac{Lr}{P + r}$.

We refer the readers to [15, 16] for showing that $B(S)$ is a completely continuous operator on S and

$$|Bx(t)| \leq \frac{M\,\|h\|_{L^1}}{\pi}, \tag{29}$$

for all $x \in S$. Thus, the condition (b) from Theorem 1 is held.

For checking the condition (c) of Theorem 1, let $x \in S$, then, by hypothesis (A_1) we get

$$\begin{aligned}
|Ax(t) + Bx(t)| &\leq |Ax(t)| + |Bx(t)| \\
&= |f(t, x(t))| + \left|\frac{1}{\pi}\int_0^t \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t - \tau\} g(\tau, x(\tau)) d\tau\right| \\
&\leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| + \frac{M\,\|h\|_{L^1}}{\pi} \\
&\leq L + F_0 + \frac{M\,\|h\|_{L^1}}{\pi},
\end{aligned}$$

which by taking a supremum over t, we obtain

$$\|Ax(t) + Bx(t)\| \leq L + F_0 + \frac{M\,\|h\|_{L^1}}{\pi} = N, \tag{30}$$

and the condition (c) of Theorem 1 is satisfied. Thus, all the conditions of Theorem 1 are satisfied and hence the operator equation $x = Ax + Bx$ has a solution in S. As a result, DOFHDE (5) has a solution on J. This completes the proof. $\square$

4 Distributed order fractional hybrid differential inequalities

In this section, we prove the fundamental results related to strict and non-strict inequalities for the DOFHDE (5). We begin with a result of strict inequalities. The following lemma may be useful in next sections.

Lemma 2. *(Lakshmikantham and Vatsala [13])*
Let $m : \mathbb{R}^+ \to \mathbb{R}$ be locally Hölder continuous such that for any $t_1 \in (0, \infty)$, we have

$$m(t_1) = 0, \quad m(t) \leq 0, \quad 0 \leq t \leq t_1, \tag{31}$$

then

$$D^q m(t_1) \geq 0. \tag{32}$$

Theorem 4. *Suppose that the hypothesis (A_0) holds and there exist two functions $u, v : [0, T] \to \mathbb{R}$, which are locally Hölder continuous such that*

$$\int_0^1 b(q) D^q [u(t) - f(t, u(t))] dq \leq g(t, u(t)), \tag{33}$$

$$\int_0^1 b(q) D^q [v(t) - f(t, v(t))] dq \geq g(t, v(t)), \tag{34}$$

where $b(q) > 0$ is the density function and $\int_0^1 b(q) dq = 1$. Then

$$u(0) < v(0), \tag{35}$$

implies

$$u(t) < v(t), \tag{36}$$

for all $t \in J$.

Proof. Assume that the inequality (34) is strict and the inequality (36) is false. Then the set Z^* defined by

$$Z^* = \{t \in J \ : \ u(t) \geq v(t), t \in J\}, \tag{37}$$

is non-empty. By denoting $t_1 = \inf Z^*$ and without loss of generality, we may suppose that $u(t_1) = v(t_1)$ and $u(t) < v(t)$ for all $t < t_1$. Define the function U and V on J as

$$U(t) = u(t) - f(t, u(t)), \quad V(t) = v(t) - f(t, v(t)),$$

then, we have

$$U(t_1) = V(t_1), \tag{38}$$

and in view of the hypothesis (A_0) for all $t < t_1$, we get

$$U(t_1) = V(t_1). \tag{39}$$

Now, by setting

$$m(t) = U(t) - V(t), \quad 0 \leq t \leq t_1, \tag{40}$$

we have

$$m(t) \leq 0, \;\; 0 \leq t \leq t_1, \;\; m(t_1) = 0, \tag{41}$$

which by Lemma 2 we obtain $D^q m(t_1) \geq 0$ and for $b(q) > 0$, we get

$$\int_0^1 b(q) D^q [m(t_1)] dq \geq 0.$$

Also, by the inequalities (33) and (34), we find that

$$g(t_1, u(t_1)) \geq \int_0^1 b(q) D^q [U(t_1)] dq \geq \int_0^1 b(q) D^q [V(t_1)] > g(t_1, v(t_1)). \tag{42}$$

This is a contradiction with $u(t_1) = v(t_1)$ and hence the set Z^* is empty. Finally, the inequality (36) holds for all $t \in J$. □

Theorem 5. *Suppose that the conditions of Theorem 4 and the inequalities (33) and (34) hold. Also, for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \geq x_2$, assume that there exists a real number $M > 0$, such that*

$$g(t, x_1) - g(t, x_2) \leq \frac{M}{1 + t^q} [(x_1 - f(t, x_1)) - (x_2 - f(t, x_2))], \qquad t \in J, \tag{43}$$

and

$$M \leq \int_0^1 \frac{b(q)}{T^q \Gamma(1-q)} dq, \qquad \int_0^1 b(q) dq = 1.$$

Then

$$u(0) \leq v(0), \tag{44}$$

which implies for all $t \in J$,

$$u(t) \leq v(t). \tag{45}$$

Proof. Let $\epsilon > 0$ be given. Setting

$$v_\epsilon(t) - f(t, v_{\epsilon(t)}) = v(t) - f(t, v(t)) + \epsilon(1 + t^q), \tag{46}$$

we find that

$$v_\epsilon(t) - f(t, v_\epsilon(t)) > v(t) - f(t, v(t)),$$

and by hypothesis (A_0), we get

$$v_\epsilon(t) > v(t). \tag{47}$$

Now, for all $t \in J$ we define

$$V_\epsilon(t) = v_\epsilon(t) - f(t, v_\epsilon(t)), \quad V(t) = v(t) - f(t, v(t)),$$

which by the relation (43), we get

$$g(t, v) \geq g(t, v_\epsilon) - \frac{M}{1 + t^q}(V_\epsilon - V).$$

Since

$$V_\epsilon - V = \epsilon(1 + t^q),$$

we obtain

$$g(t, v) \geq g(t, v_\epsilon) - \epsilon M. \tag{48}$$

Applying the fractional differential of distributed operator $\int_0^1 b(q) D^q dq$, on the both sides Eq. (46), we have

$$\int_0^1 b(q) D^q [V_\epsilon(t)] dq = \int_0^1 b(q) D^q [V(t)] dq + \epsilon \int_0^1 b(q) D^q [1 + t^q] dq. \tag{49}$$

Hence by using the relations (34) and (49) and $M \leq \int_0^1 \frac{b(q)}{T^q \Gamma(1-q)} dq$, we find that

$$\int_0^1 b(q) D^q [V_\epsilon(t)] dq \geq g(t, v(t)) + \epsilon \int_0^1 b(q) \left(\frac{1}{t^q \Gamma(1-q)} + \Gamma(1+q)\right) dq, \tag{50}$$

$$> g(t, v_\epsilon(t)) - M\epsilon + \epsilon \int_0^1 \frac{b(q)}{t^q \Gamma(1-q)} dq$$

$$> g(t, v_\epsilon(t)) - M\epsilon + M\epsilon = g(t, v_\epsilon(t)).$$

Also, we get $v_\epsilon(0) > v(0) \geq u(0)$ which by setting $v = v_\epsilon$ for all $t \in J$, we obtain $u(t) < v_\epsilon(t)$. Since $\epsilon > 0$ is arbitrary, by taking the limit as $\epsilon \to 0$, we deduce that $u(t) \leq v(t)$. □

5 Existence of maximal and minimal solutions

In this section, we prove the existence of maximal and minimal solutions for DOFHDE (5) on $J = [0, T]$. We need the following definition in what follows.

Definition 4. *A solution y of DOFHDE (5) is maximal if for all $t \in J$ and solution x of this system, $x(t) \leq y(t)$. Similarly, a solution z of the DOFHDE (5) is minimal if for all $t \in J$, one has $z(t) \leq x(t)$, such that x is the solution of the DOFHDE (5).*

We discuss the case of a maximal solution only, as the case of a minimal solution is similar and can be obtained with the same arguments with appropriate modifications. Given an arbitrary small real number $\epsilon > 0$, consider the following initial value problem of DOFHDE of order $0 < q < 1$,

$$\begin{cases} \int_0^1 b(q) D^q[x(t) - f(t, x(t))] dq = g(t, x(t)) + \epsilon, \quad t \in J, \quad \int_0^1 b(q) dq = 1, \\ x(0) = 0, \end{cases} \tag{51}$$

Theorem 6. *Suppose that the hypotheses (A_0)-(A_2) hold, then for every small number $\epsilon > 0$, DOFHDE (51) has a solution defined on $J = [0, T]$.*

Proof. The proof is similar to Theorem 3 and we omit the details. □

Our main existence theorem for a maximal solution for DOFHDE (5) is as follows.

Theorem 7. *Suppose that the hypotheses (A_0)-(A_2) hold, then the DOFHDE (5) has a maximal solution on $J = [0, T]$.*

Proof. We set $\{\epsilon_n\}_0^\infty$ as a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. By Theorem 6, then there exists a solution $y(t, \epsilon_n)$ of the DOFHDE defined on J

$$\begin{cases} \int_0^1 b(q) D^q[x(t) - f(t, x(t))] dq = g(t, x(t)) + \epsilon_n, \quad t \in J, \quad \int_0^1 b(q) dq = 1, \\ x(0) = 0, \end{cases} \tag{52}$$

where $b(q)$ is a nonnegative density function. Then, for any solution w of DOFHDE (5), any solution of auxiliary problem (52) satisfies

$$\int_0^1 b(q) D^q[y(t, \epsilon_n) - f(t, y(t, \epsilon_n))] dq = g(t, y(t, \epsilon_n)) + \epsilon_n > g(t, y(t, \epsilon_n)). \tag{53}$$

Also, for any solution w of the DOFHDE (5) we get

$$\int_0^1 b(q)D^q[w(t) - f(t,w(t))]dq \leq g(t,w(t)), \tag{54}$$

such that $w(0) = 0 \leq y(0,\epsilon_n) = \epsilon_n$. Thus, by applying Theorem 5, we have

$$w(t) \leq y(t,\epsilon_n), \qquad t \in J, \qquad n = 0,1,2,\dots \tag{55}$$

Also, since $\epsilon_2 = y(0,\epsilon_2) \leq y(0,\epsilon_1) = \epsilon_1$, in view of Theorem 5, we obtain

$$y(t,\epsilon_2) \leq y(t,\epsilon_1).$$

Then, $\{y(t,\epsilon_n)\}$ is decreasing sequence of positive real numbers and the limit

$$y(t) = \lim_{n\to\infty} y(t,\epsilon_n), \tag{56}$$

exists. We shall show that the limit (56) is uniform on $J = [0,T]$. To see this, we prove the sequence $y(t,\epsilon_n)$ is equicontinuous. Suppose that $t_1, t_2 \in J$ such that $t_1 < t_2$. Since $y(t,\epsilon_n)$ is the solution of DOFHDE (52), then by Lemma 1, $y(t,\epsilon_n)$ satisfies the equation

$$y(t,\epsilon_n) = f(t,y(t,\epsilon_n)) + \frac{1}{\pi}\int_0^t \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t-\tau\}(g(\tau,y(\tau,\epsilon_n)) + \epsilon_n)d\tau. \tag{57}$$

Therefore, by [15] (the relations (4.30) and (4.31)) we have

$$\begin{aligned}
&|y(t_1,\epsilon_n) - y(t_2,\epsilon_n)| \\
&= |\ (f(t_1,y(t_1,\epsilon_n)) + \frac{1}{\pi}\int_0^{t_1} \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t_1-\tau\}(g(\tau,y(\tau,\epsilon_n)) + \epsilon_n)d\tau) \\
&\quad -(f(t_2,y(t_2,\epsilon_n)) + \frac{1}{\pi}\int_0^{t_2} \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t_2-\tau\}(g(\tau,y(\tau,\epsilon_n)) + \epsilon_n)d\tau)| \\
&\leq |f(t_1,y(t_1,\epsilon_n)) - f(t_2,y(t_2,\epsilon_n))| + \frac{M'(\|h\|_{L^1} + \epsilon_n)}{\pi}\ln(\frac{(c+t_1-t_2)(c-t_1)}{c(c-t_2)}) \\
&\quad + \frac{M'(\|h\|_{L^1} + \epsilon_n)}{\pi}\ln(\frac{c}{c+t_1-t_2}).
\end{aligned} \tag{58}$$

Since f is a continuous on a compact set $J \times [-N,N]$, it is uniformly continuous. Hence,

$$|f(t_1,y(t_1,\epsilon_n)) - f(t_2,y(t_2,\epsilon_n))| \to 0 \quad as \quad t_1 \to t_2,$$

uniformly converges for all $n \in \mathbb{N}$. Thus, for $\epsilon > 0$ there exists $\delta > 0$ such that for $|t_1 - t_2| < \delta$, we have

$$|y(t_1,\epsilon_n) - y(t_2,\epsilon_n)| < \epsilon, \quad n \in \mathbb{N},$$

which implies that for all $t \in J$, $y(t, \epsilon_n) \to y(t)$. Now, taking the limits Eq. (57) when $n \to \infty$, we get

$$y(t) = f(t,y(t)) + \frac{1}{\pi}\int_0^t \mathcal{L}\{\Im\{\frac{1}{B(re^{-i\pi})}\}; t-\tau\} g(\tau, y(\tau) d\tau, \qquad t \in J.$$

Therefore, y is a solution of the DOFHDE (5) on J and from inequality (55), we deduce $w(t) \leq y(t)$. Hence, the DOFHDE (5) has a maximal solution on $J = [0, T]$. □

6 Comparison theorems

The main problem of differential inequalities is to estimate a bound for the solution set for the differential inequality related to DOFHDE (5). In this section, we prove that the maximal and minimal solutions serve as bounds for the solutions of the related differential inequality to DOFHDE (5) on $J = [0, T]$.

Theorem 8. *Suppose that the hypotheses* (A_0)*-*(A_2) *hold. Also, assume that there exists a real number* $M > 0$*, such that for all* $t \in J$

$$g(t, x_1) - g(t, x_2) \leq \frac{M}{1+t^q}[(x_1 - f(t, x_1)) - (x_2 - f(t, x_2))], \tag{59}$$

for all $x_1, x_2 \in \mathbb{R}$ *with* $x_1 \geq x_2$*, where*

$$M \leq \int_0^1 \frac{b(q)}{T^q\Gamma(1-q)} dq, \qquad \int_0^1 b(q)dq = 1. \tag{60}$$

Furthermore, if there exists a function $w \in C(J, \mathbb{R})$*, such that*

$$\begin{cases} \displaystyle\int_0^1 b(q) D^q [w(t) - f(t, w(t))] dq \leq g(t, w(t)), \quad t \in J, \\ w(0) \leq 0, \end{cases} \tag{61}$$

then, for all $t \in J$

$$w(t) \leq y(t), \tag{62}$$

where y *is a maximal solution of the DOFHDE* (5)*.*

Proof. Setting $\epsilon > 0$ and using Theorem 7, $y(t, \epsilon)$ is a maximal solution of the DOFHDE (51) such that

$$y(t) = \lim_{\epsilon \to 0} y(t, \epsilon), \tag{63}$$

is uniform on $J = [0, T]$. Therefore, for nonnegative density function $b(q)$, we have

$$\begin{cases} \displaystyle\int_0^1 b(q)D^q[y(t,\epsilon) - f(t, y(t,\epsilon))]dq = g(t, y(t,\epsilon)) + \epsilon, \quad t \in J, \quad \int_0^1 b(q)dq = 1, \\ y(0,\epsilon) = 0. \end{cases} \tag{64}$$

Hence

$$\begin{cases} \displaystyle\int_0^1 b(q)D^q[y(t,\epsilon) - f(t, y(t,\epsilon))]dq > g(t, y(t,\epsilon)), \quad t \in J, \quad \int_0^1 b(q)dq = 1, \\ y(0,\epsilon) = 0. \end{cases} \tag{65}$$

Now, by Theorem 5 for the inequalities (1) and (65) we obtain $w(t) < y(t,\epsilon)$. Finally, the limit (63) implies that $w(t) \leq y(t)$. □

Corollary 1. *Suppose that the hypotheses (A_0)-(A_2) and the conditions (59) and (60) hold. If there exists a function $u \in C(J, \mathbb{R})$ such that*

$$\begin{cases} \displaystyle\int_0^1 b(q)D^q[u(t) - f(t, u(t))]dq \geq g(t, u(t)), \quad t \in J, \\ u(0) > 0, \end{cases}$$

then

$$z(t) \leq u(t),$$

where z is a minimal solution of the DOFHDE (5).

Next theorem is a result about the uniqueness of solutions of DOFHDE (5).

Theorem 9. *Suppose that the hypotheses (A_0)-(A_2) and the conditions (59) and (60) hold. If identically zero function is the only solution of the differential equation*

$$\int_0^1 b(q)D^q[p(t)]dq = \frac{M}{1+t^q}p(t), \qquad p(0) = 0, \qquad \int_0^1 b(q)dq = 1, \tag{66}$$

then, the DOFHDE (5) has a unique solution on $J = [0, T]$.

Proof. According to Theorem 3, the DOFHDE (5) has a solution on $J = [0, T]$. Let v_1 and v_2 be two solution of the DOFHDE (5) with $v_1 > v_2$ and set the function $p : J \to \mathbb{R}$

$$p(t) = (v_1(t) - f(t, v_1(t))) - (v_2(t) - f(t, v_2(t))). \tag{67}$$

Since $v_1 > v_2$, by the hypothesis (A_0) we obtain $p(t) > 0$. Therefore, for the nonnegative density function $b(q)$, we get

$$\begin{aligned}
\int_0^1 b(q)D^q[p(t)]dq &\leq \int_0^1 b(q)D^q[v_1(t) - f(t, v_1(t))]dq \\
&\quad - \int_0^1 b(q)D^q[v_2(t) - f(t, v_2(t))]dq \\
&\leq g(t, v_1) - g(t, v_2) \\
&\leq \frac{M}{1+t^q}[(v_1 - f(t, v_1)) - (v_2 - f(t, v_2))] \\
&= \frac{M}{1+t^q}p(t), \qquad t \in J, \qquad p(0) = 0.
\end{aligned}$$

Since identically zero function is the only solution of the differential equation (66), applying Theorem 8 with $f(t, x) \equiv 0$, implies that $p(t) \leq 0$, which is a contradiction with $p(t) > 0$. Finally, $v_1 = v_2$. □

7 Numerical example

In this section, as showing the constructive theorems and lemmas in previous sections, we state the following example in terms of the complementary error function.

Example 1. We consider the following initial value problem in the fractional Riemann-Liouville derivative on $[0, 1]$

$$D_t^\alpha(x + x^3) = Erfc(x), \quad x(0) = 0, \tag{68}$$

which we set $f(t, x(t)) = -x^3(t)$ and $g(t, x(t)) = Erfc(x(t))$. Now, we consider the approximated solution of (68) by the shifted Legendre polynomials $x(t) = \sum_{i=0}^n c_i P_i(x)$ and use the following relation for the fractional derivative of $x(t)$

$$D_t^\alpha x(t) = \sum_{i=\lceil\alpha\rceil}^{n} \sum_{k=\lceil\alpha\rceil}^{i} c_i b_{i,k}^{(\alpha)} x^{k-\alpha}, \tag{69}$$

$$b_{i,k}^{(\alpha)} = \frac{(-1)^{k+i}(i+k)!}{k!(i-k)!\Gamma(k+1-\alpha)}, \tag{70}$$

where $\lceil\alpha\rceil$ is the largest integer less than or equal to α. If we collocate Eq. (68) at n points on $[0, 1]$, then we can get the solution with respect to n

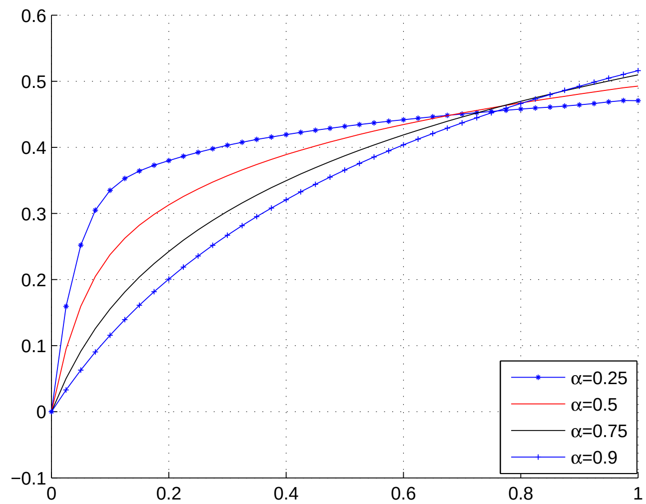

Figure 1: The Solution of FDE (68) for $\alpha = 0.25, 0.5, 0.75, 0.9$.

unknown coefficients $c_i, i = 1, 2, \cdots, n$. We show this for $n = 10$ in Figure 1 for different values of α. For more details of this method see for example [1, 20].

8 Conclusions

In this paper, we introduced a new class of the fractional hybrid differential equations with linear perturbations of second type. We pointed out a fixed point theorem in the Banach algebra for the existence of solution. Also, by fractional hybrid differential inequalities, we established the existence of extremal solution and proved some comparison theorems for this class. These results enable us to find the extremal solutions of many fractional differential equations with respect to the various order density function.

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