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Exact and numerical solutions of linear and non-linear systems of fractional partial differential equations

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Abstract. The present study introduces a new technique of homotopy perturbation method for the solution of systems of fractional partial differential equations. The proposed scheme is based on Laplace transform and new homotopy perturbation methods. The fractional derivatives are considered in Caputo sense. To illustrate the ability and reliability of the method some examples are provided. The results obtained by the proposed method show that the approach is very efficient, less computational and can be applied to other linear and nonlinear partial differential equations.

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1 Introduction

In recent decades, fractional differential equations have found considerable importance due to their application in various branches of engineering

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and applied sciences. Studies show that the behavior of many physical systems is based on the theory of fractional derivatives. Fractional differential equations are increasingly used to model problems in acoustics and thermal systems, rheology and mechanical systems, signal processing and systems identification, control and robotics and other areas of applications [6, 19, 4, 5]. It is worth noting that, recently much attention has been paid to the distributed-order differential equations and their applications in engineering fields that both integer-order systems and fractional-order systems are special cases of distributed-order systems. The reader may refer to [17, 16, 3]. Partial differential systems are very important in various fields of science and technology, especially in biology, solid state physics and bioengineering. Many researchers have applied various methods to study the solutions of systems of fractional partial differential systems such as the Adomian decomposition method [9], the variational iteration method [7], the perturbation analysis method [14, 10], and the differential transform method [11, 18], etc.

In this paper we introduce a new form of homotopy perturbation and Laplace transform methods by extending the idea of [2, 8], that we called LTNHPM to solve the time-fractional linear and nonlinear partial differential. This method leads to computable and efficient solutions to linear and nonlinear operator equations. The corresponding solutions of the integer-order equations are found to follow as special cases of those of fractional-order equations. We consider the system of fractional-order equations of the form

$$D_t^{\mu_i} u_i + N_i(u_1, \ldots, u_n) = f_i(x_1, \ldots, x_{n-1}, t), \quad i = 1, 2, \ldots, n, \quad 0 < \mu_i \leq 1, \tag{1}$$

with initial data

$$u_i(x_1, x_2, \ldots, x_{n-1}, 0) = g_i(x_1, x_2, \ldots, x_{n-1}),$$

where u_i, $i = 1, 2, \ldots, n$, are unknown functions and $N_1, N_2, \ldots, N_n$ are nonlinear operators, which usually depend on the functions u_i and their derivatives. For $i = 1, 2, \ldots, n$, f_i are known analytical functions.

This paper is organized as follows. In Section 2, we recall some basic definitions and results dealing with the fractional calculus and Laplace transform which are later used in this paper. The basic idea behind the new method is illustrated in Section 3. Finally, to give a clear overview of our main result, four illustrative systems of fractional partial differential equations are investigated in Section 4.

2 Preliminaries and notations

Some basic definitions and properties of the fractional calculus theory which are used in this paper are given as follows.

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$ and it is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu$, $n \in N$. Clearly $C_\mu \in C_\beta$ if $\beta > \mu$.

Definition 2. The left sided Riemann-Liouville fractional integral operator of order $\mu > 0$, of a function $f \in C_\mu$, $\mu \geq -1$ is defined as follows

$$J^\mu f(t) = \begin{cases} \frac{1}{\Gamma(\mu)} \displaystyle\int_0^t (t-\tau)^{\mu-1} f(\tau)\, d\tau, & \mu > 0, \\ f(t), & \mu = 0, \end{cases} \tag{2}$$

where $\Gamma(.)$ is the well-known Gamma function.

Some of the most important properties of operator J^μ for $f(t) \in C_\mu$, $\mu, \beta \geq 0$ and $\gamma > -1$ are as follows

$$(1)\ \ J^\mu J^\beta f(t) = J^{(\mu+\beta)} f(t),$$

$$(2)\ \ J^\mu J^\beta f(t) = J^\beta J^\mu f(t),$$

$$(3)\ \ J^\mu t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\mu+\gamma+1)} t^{\mu+\gamma}.$$

Definition 3. Amongst a variety of definitions for fractional order derivatives, Caputo fractional derivative has been used [12, 15] as it is suitable for describing various phenomena, since the initial values of the function and its integer order derivatives have to be specified, so Caputo fractional derivative of function $f(t)$ is defined as

$$D^\mu f(t) = J^{n-\mu} f(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\mu-n+1}} d\tau, \tag{3}$$

where $n - 1 < \mu \leq n$, $n \in N$, $t \geq 0$ and $f(t) \in C_{-1}^n$.

The Caputo fractional derivative is considered because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

In this paper, we consider fractional systems of partial differential equations, where the unknown function $u(x,t)$ is a assumed to be a causal function of space and time, respectively, and the fractional derivatives are taken in Caputo sense as follows.

Definition 4. The Caputo time-fractional derivative operator of order $\mu > 0$ is defined as

$$\begin{aligned} D_t^{\mu} u(x,t) &= J_t^{n-\mu} u(x,t) \\ &= \frac{1}{\Gamma(n-\mu)} \int_0^t (t-\tau)^{n-\mu-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, \qquad n-1 < \mu \leq n. \end{aligned} \tag{4}$$

Definition 5. The Laplace transform of a function $u(x,t)$, $t > 0$ is defined as:

$$\mathcal{L}[u(x,t)] = \int_0^{\infty} e^{-st}\, u(x,t) dt,$$

where s can be either real or complex. The Laplace transform $\mathcal{L}[u(x,t)]$ of the Caputo derivative is defined as:

$$\mathcal{L}\{D_t^{\mu} u(x,t)\} = s^{\mu} \mathcal{L}\{u(x,t)\} - \sum_{k=0}^{n-1} u^{(k)}(x,0^+) s^{\mu-1-k}, \quad n-1 < \mu \leq n. \tag{5}$$

Lemma 1. *If $n-1 < \mu \leq n$, $n \in N$ and $k \geq 0$, then we have*

$$\mathcal{L}^{-1}\{\frac{1}{s^{\mu}} \mathcal{L}\{\frac{t^{k\mu}}{\Gamma(k\mu+1)}\}\} = \frac{t^{k\mu+\mu}}{\Gamma(k\mu+\mu+1)}, \tag{6}$$

where $\mathcal{L}^{-1}$ is inverse Laplace transform.

The Mittag-Leffler function plays a very important role in the fractional differential equations, was in fact introduced by Mittag-Leffler in 1903 [13]. The Mittag-Leffler function $E_{\mu}(z)$ with $\mu > 0$ is defined by the following series representation,

$$E_{\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + 1)}, \tag{7}$$

where $z \in C$. The following identity result from the definition

$$E_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z.$$

3 Basic ideas of the LTNHPM

In this Section, to illustrate the basic concepts NHPM [1], we can construct the following homotopies:

$$\begin{aligned} H(V_i, p) = (1-p)[D_t^{\mu_i} V_i - u_{i,0}] + p[D^{\mu_i} V_i + N_i(V_1, \ldots, V_n) \\ -f_i(x_1, \ldots, x_{n-1}, t)] = 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{8}$$

or equivalently

$$\begin{aligned} H(V_i, p) = D_t^{\mu_i} V_i - u_{i,0} + p[u_{i,0} + N_i(V_1, \ldots, V_n) - f_i(x_1, \ldots, x_{n-1}, t))] \\ = 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{9}$$

where $p \in [0, 1]$ is an embedding parameter and $u_{i,0}$ are the initial approximation for the solution of (1).
Clearly, the homotopy equations $H(V_i, 0) = 0$ and $H(V_i, 1) = 1$ are equivalent to the equations $D_t^{\mu_i} V_i - u_{i,0} = 0$ and $D^{\mu_i} V_i + N_i(V_1, \ldots, V_n) - f_i(x_1, \ldots, x_{n-1}, t) = 0$ respectively, for $i = 1, 2, \ldots, n$. Thus, a monotonous change of parameter p from zero to one corresponds to a continuous change of the trivial problem $D_t^{\mu_i} V_i - u_{i,0} = 0$ to the original problem.
By applying Laplace transform on both sides of (9), we have

$$\begin{aligned} \mathcal{L}\{D_t^{\mu_i} V_i - u_{i,0} + p[u_{i,0} + N_i(V_1, \ldots, V_n) - f_i(x_1, \ldots, x_{n-1}, t)]\} = 0, \\ i = 1, 2, \ldots, n. \end{aligned} \tag{10}$$

Using of (5), we derive

$$\begin{aligned} s^{\mu_i}\, \mathcal{L}\{V_i\} - s^{\mu_i - 1} V_i(0) = \mathcal{L}\{u_{i,0} - p[u_{i,0} + N_i(V_1, \ldots, V_n) \\ -f_i(x_1, \ldots, x_{n-1}, t)]\}, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{11}$$

or

$$\begin{aligned} \mathcal{L}\{V_i\} = \tfrac{1}{s} V_i(0) + \tfrac{1}{s^{\mu_i}}\{\mathcal{L}\{u_{i,0} - p[u_{i,0} + N_i(V_1, \ldots, V_n) \\ -f_i(x_1, \ldots, x_{n-1}, t)]\}\}, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{12}$$

where $V_i(0) = u_i(x_1, x_2, \ldots, x_{n-1}, 0)$, for $i = 1, 2, \ldots, n$. By applying inverse Laplace transform on both sides of (12), we obtain

$$\begin{aligned} V_i = V_i(0) + \mathcal{L}^{-1}\{\tfrac{1}{s^{\mu_i}} \mathcal{L}\{u_{i,0} - p[u_{i,0} + N_i(V_1, \ldots, V_n) \\ -f_i(x_1, \ldots, x_{n-1}, t)]\}\}, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{13}$$

Next, we assume that the solution of system (13) can be written as a power series in embedding parameter p, as follows:

$$V_i = V_{i,0} + pV_{i,1}, \quad i = 1, \dots, n, \tag{14}$$

where $V_{i,0}$ and $V_{i,1}$, $i = 0, \dots, n$, are functions which should be determined. Suppose that the initial approximation of the solutions of Eqs. (8) are in the following form:

$$u_{i,0} = \sum_{j=0}^{\infty} \frac{a_{i,j}(x)}{\Gamma(j\mu_i + 1)}\, t^{j\mu_i}, \quad i = 1, \dots, n, \tag{15}$$

where $a_{i,j}(x)$ for $j = 1, 2, \dots$, $i = 1, \dots, n$ are functions which must be computed. Substituting (14) and (15) into (13) and equating the coefficients of p with the same powers leads to:

$$\begin{aligned} &p^0 : V_{i,0} = V_i(0) + \mathcal{L}^{-1}\{\tfrac{1}{s^{\mu_i}}\mathcal{L}\{\textstyle\sum_{j=0}^{\infty} \frac{a_{i,j}(x)}{\Gamma(j\mu_i+1)}\, t^{j\mu_i}\}\}, \\ &p^1 : V_{i,1} = -\mathcal{L}^{-1}\{\tfrac{1}{s^{\mu_i}}\mathcal{L}\{\textstyle\sum_{n=0}^{\infty} \frac{a_{i,j}(x)}{\Gamma(j\mu_i+1)}\, t^{j\mu_i} + [N_i(V_{i,0}) - f_i(x_1, \dots, x_{n-1}, t)]\}\}. \end{aligned} \tag{16}$$

Now, with vanishing $V_{i,1}$, and by taking $\mu_i = 1$, $i = 1, \dots, n$, we obtain the coefficients $a_{i,j}(x)$, for $j = 1, 2, \dots$, $i = 1, \dots, n$. Therefore the exact solution may be obtained as the following:

$$\begin{aligned} u_i = V_i &= V_i(0) + \mathcal{L}^{-1}\{\tfrac{1}{s^{\mu_i}}\mathcal{L}\{\textstyle\sum_{j=0}^{\infty} \frac{a_{i,j}(x)}{\Gamma(j\mu_i+1)}\, t^{j\mu_i}\}\} \\ &= V_i(0) + \textstyle\sum_{j=0}^{\infty} \frac{a_{i,j}(x)}{\Gamma(j\mu_i+\mu_i+1)}\, t^{j\mu_i+\mu_i}, \quad i = 1, \dots, n. \end{aligned} \tag{17}$$

To show the efficiency and reliability of the method, we apply the LTNHPM to some examples in the next section.

4 Example

In this Section, we shall illustrate the the method above by linear and non linear systems of fractional partial differential equations, two linear and two nonlinear, which have been widely discussed in the literature.

4.1 The homogeneous linear system

First we consider the following the homogeneous linear fractional system:

$$\begin{aligned} &D_t^{\mu} u - v_x + (u + v) = 0, \\ &D_t^{\nu} v - u_x + (u + v) = 0, \end{aligned} \tag{18}$$

subject to the following initial conditions

$$u(x,0) = \sinh(x) \quad v(x,0) = \cosh(x),$$

where $0 < \mu, \nu \leq 1$.
To solve Eq. (18) by the LTNHPM, we construct the following homotopies:

$$\begin{aligned} D_t^{\mu} U &= u_0(x,t) - p[u_0(x,t) - V_x + (U+V)], \\ D_t^{\nu} V &= u_0(x,t) - p[v_0(x,t) - U_x + (U+V)]. \end{aligned} \tag{19}$$

Applying Laplace transform on both sides of (19) and using the property of the Laplace transform, we have

$$\begin{aligned} s^{\mu}\,\mathcal{L}\{U\} - s^{\mu-1}U(x,0) &= \mathcal{L}\{u_0(x,t) - p[u_0(x,t) - V_x + (U+V)]\}, \\ s^{\nu}\,\mathcal{L}\{V\} - s^{\nu-1}V(x,0) &= \mathcal{L}\{v_0(x,t) - p[v_0(x,t) - U_x + (U+V)]\}, \end{aligned} \tag{20}$$

or

$$\begin{aligned} \mathcal{L}\{U\} &= \tfrac{1}{s}U(x,0) + \tfrac{1}{s^{\mu}}\{\mathcal{L}\{u_0(x,t) - p[u_0(x,t) - V_x + (U+V)]\}\}, \\ \mathcal{L}\{V\} &= \tfrac{1}{s}V(x,0) + \tfrac{1}{s^{\nu}}\{\mathcal{L}\{v_0(x,t) - p[v_0(x,t) - U_x + (U+V)]\}\}. \end{aligned} \tag{21}$$

Operating with the inverse Laplace transform on both sides of (21) and from the initial conditions $U(x,0) = \sinh(x)$ and $V(x,0) = \cosh(x)$, we derive

$$\begin{aligned} U &= \sinh(x) + \mathcal{L}^{-1}\{\tfrac{1}{s^{\mu}}\{\mathcal{L}\{u_0(x,t) - p[u_0(x,t) - V_x + (U+V)]\}\}\}, \\ V &= \cosh(x) + \mathcal{L}^{-1}\{\tfrac{1}{s^{\nu}}\{\mathcal{L}\{v_0(x,t) - p[v_0(x,t) - U_x + (U+V)]\}\}\}. \end{aligned} \tag{22}$$

Suppose the solutions of system Eq. (22) have the following form:

$$\begin{aligned} U(x,t) &= U_0(x,t) + pU_1(x,t), \\ V(x,t) &= V_0(x,t) + pV_1(x,t), \end{aligned} \tag{23}$$

where $U_i(t,x)$, $0 \leq i \leq 1$, and $V_i(t,x)$, $0 \leq i \leq 1$, are functions which

should be determined.
Substituting Eq. (23) into Eq. (22), collecting the same powers of p, and equating each coefficient of p to zero yields:

$$p^0: \quad \begin{cases} U_0(x,t) = \sinh(x) + \mathcal{L}^{-1}\{\frac{1}{s^\mu}\{\mathcal{L}\{u_0(x,t)\} \\ V_0(x,t) = \cosh(x) + \mathcal{L}^{-1}\{\frac{1}{s^\nu}\{\mathcal{L}\{v_0(x,t)\}\}, \end{cases}$$

$$p^1: \quad \begin{cases} U_1(x,t) = -\mathcal{L}^{-1}\{\frac{1}{s^\mu}\{\mathcal{L}\{u_0(x,t) - V_{0_x} + (U_0 + V_0)\} \\ V_1(x,t) = -\mathcal{L}^{-1}\{\frac{1}{s^\nu}\{\mathcal{L}\{v_0(x,t) - U_{0_x} + (U_0 + V_0)\}\}. \end{cases}$$

Assume $u_0(x,t) = \sum_{n=0}^{\infty} \frac{a_n(x)}{\Gamma(n\mu+1)}\ t^{n\mu}$, $v_0(x,t) = \sum_{n=0}^{\infty} \frac{b_n(x)}{\Gamma(n\nu+1)}\ t^{n\nu}$. Now if we set $U_1(x,t) = 0$ and $\mu = \nu = 1$, then we have

$$\begin{aligned} U_1(x,t) = {} & (-a_0\,(x) - \cosh\,(x))t + (-\tfrac{1}{2}\,a_1\,(x) - \tfrac{1}{2}\,a_0\,(x) - \tfrac{1}{2}\,b_0\,(x) \\ & +\tfrac{1}{2}\,\tfrac{d}{dx}b_0\,(x))t^2 + (-\tfrac{1}{6}\,a_2\,(x) - \tfrac{1}{6}\,a_1\,(x) - \tfrac{1}{6}\,b_1\,(x) + \tfrac{1}{6}\,\tfrac{d}{dx}b_1\,(x))t^3 \\ & +(-\tfrac{1}{24}\,a_3\,(x) - 124\,a_2\,(x) - 124\,b_2\,(x) + 124\,\tfrac{d}{dx}b_2\,(x))t^4 + \cdots \\ = {} & 0, \end{aligned}$$

and by setting $V_1(x,t) = 0$ and $\mu = \nu = 1$ we obtain

$$\begin{aligned} V_1(x,t) = {} & (-b_0\,(x) - \sinh\,(x))t + (-\tfrac{1}{2}\,b_1\,(x) - \tfrac{1}{2}\,b_0\,(x) - \tfrac{1}{2}\,a_0\,(x) \\ & +\tfrac{1}{2}\,\tfrac{d}{dx}a_0\,(x))t^2 + (-\tfrac{1}{6}\,b_2\,(x) - \tfrac{1}{6}\,b_1\,(x) - \tfrac{1}{6}\,a_1\,(x) + \tfrac{1}{6}\,\tfrac{d}{dx}a_1\,(x))t^3 \\ & +(-\tfrac{1}{24}\,b_3\,(x) - \tfrac{1}{24}\,b_2\,(x) - \tfrac{1}{24}\,a_2\,(x) + \tfrac{1}{24}\,\tfrac{d}{dx}a_2\,(x))t^4 + \cdots = 0. \end{aligned}$$

It can be easily shown that

$$\begin{aligned} a_0(x) = a_2(x) = a_4(x) = \cdots = -\cosh\,(x)\,, \\ a_1(x) = a_3(x) = a_5(x) = \cdots = \sinh\,(x)\,, \\ b_0(x) = b_2(x) = b_4(x) = \cdots = -\sinh\,(x)\,, \\ b_1(x) = b_3(x) = b_5(x) = \cdots = \cosh\,(x)\,. \end{aligned}$$

Therefore, we gain the solutions of Eq. (18) as:

$$\begin{aligned} u(x,t) &= \sinh(x)\left(1 + \tfrac{t^{2\mu}}{\Gamma(2\mu+1)} + \cdots\right) - \cosh(x)\left(\tfrac{t^{\mu}}{\Gamma(\mu+1)} + \tfrac{t^{3\mu}}{\Gamma(3\mu+1)} + \cdots\right), \\ v(x,t) &= \cosh(x)\left(1 + \tfrac{t^{2\nu}}{\Gamma(2\nu+1)} + \cdots\right) - \sinh(x)\left(\tfrac{t^{\nu}}{\Gamma(\nu+1)} + \tfrac{t^{3\nu}}{\Gamma(3\nu+1)} + \cdots\right), \end{aligned} \tag{24}$$

which are the exact solutions. Now, if we put $\mu = \nu = 1$ in (24), we obtain $u(x,t) = \sinh(x-t)$, $v(x,t) = \cosh(x-t)$ which are the exact solutions of the given nonlinear system (18) for $\mu = \nu = 1$.

4.2 The inhomogeneous linear system

We next consider the inhomogeneous linear fractional system:

$$\begin{aligned} D_t^{\mu} u - v_x - (u-v) &= -2, \\ D_t^{\nu} v + u_x - (u-v) &= -2, \end{aligned} \tag{25}$$

subject to the following initial conditions

$$u(x,0) = 1 + e^x \quad v(x,0) = -1 + e^x,$$

where $0 < \mu, \nu \leq 1$.
To solve Eq. (25) by the LTNHPM, we construct the following homotopies

$$\begin{aligned} D_t^{\mu} U &= u_0(x,t) - p[u_0(x,t) - V_x - (U-V) + 2], \\ D_t^{\nu} V &= v_0(x,t) - p[v_0(x,t) + U_x - (U-V) + 2]. \end{aligned} \tag{26}$$

Applying Laplace transform on both sides of (26), we have

$$\begin{aligned} s^{\mu}\, \mathcal{L}\{U\} - s^{\mu-1} U(x,0) &= \mathcal{L}\{u_0(x,t) - p[u_0(x,t) - V_x - (U-V) + 2]\}, \\ s^{\nu}\, \mathcal{L}\{V\} - s^{\nu-1} V(x,0) &= \mathcal{L}\{v_0(x,t) - p[v_0(x,t) + U_x - (U-V) + 2]\}, \end{aligned} \tag{27}$$

or

$$\begin{aligned} \mathcal{L}\{U\} &= \tfrac{1}{s} U(x,0) + \tfrac{1}{s^{\mu}}\{\mathcal{L}\{u_0(x,t) - p[u_0(x,t) - V_x - (U-V) + 2]\}\}, \\ \mathcal{L}\{V\} &= \tfrac{1}{s} V(x,0) + \tfrac{1}{s^{\nu}}\{\mathcal{L}\{v_0(x,t) - p[v_0(x,t) + U_x - (U-V) + 2]\}\}. \end{aligned} \tag{28}$$

By applying inverse Laplace transform on both sides of (28) and from the initial conditions $U(x,0) = 1 + e^x$ and $V(x,0) = -1 + e^x$, we derive

$$\begin{aligned} U &= 1 + e^x + \mathcal{L}^{-1}\{\tfrac{1}{s^{\mu}}\{\mathcal{L}\{u_0(x,t) - p[u_0(x,t) - V_x - (U-V) + 2]\}\}\}, \\ V &= -1 + e^x + \mathcal{L}^{-1}\{\tfrac{1}{s^{\nu}}\{\mathcal{L}\{v_0(x,t) - p[v_0(x,t) + U_x - (U-V) + 2]\}\}\}. \end{aligned} \tag{29}$$

Suppose the solutions of system Eq. (29) have the following form:

$$\begin{aligned} U(x,t) &= U_0(x,t) + pU_1(x,t), \\ V(x,t) &= V_0(x,t) + pV_1(x,t), \end{aligned} \tag{30}$$

where $U_i(t,x)$, $0 \leq i \leq 1$, and $V_i(t,x)$, $0 \leq i \leq 1$, are functions which should be determined.
Substituting Eq. (30) into Eq. (29), collecting the same powers of p, and equating each coefficient of p to zero yields:

$$p^0: \quad \begin{cases} U_0(x,t) = 1 + e^x + \mathcal{L}^{-1}\{\frac{1}{s^\mu}\{\mathcal{L}\{u_0(x,t)\} \\ \\ V_0(x,t) = -1 + e^x + \mathcal{L}^{-1}\{\frac{1}{s^\nu}\{\mathcal{L}\{v_0(x,t)\}\}, \end{cases}$$

$$p^1: \quad \begin{cases} U_1(x,t) = -\mathcal{L}^{-1}\{\frac{1}{s^\mu}\{\mathcal{L}\{u_0(x,t) - V_{0_x} - (U_0 - V_0) + 2\} \\ \\ V_1(x,t) = -\mathcal{L}^{-1}\{\frac{1}{s^\nu}\{\mathcal{L}\{v_0(x,t) + U_{0_x} - (U_0 - V_0) + 2\}\}. \end{cases}$$

Assume $u_0(x,t) = \sum_{n=0}^{\infty} \frac{a_n(x)}{\Gamma(n\mu+1)} t^{n\mu}$, $v_0(x,t) = \sum_{n=0}^{\infty} \frac{b_n(x)}{\Gamma(n\nu+1)} t^{n\nu}$. Now if we set $U_1(x,t) = 0$ and $\mu = \nu = 1$, then we have

$$\begin{aligned} U_1(x,t) = (-a_0(x) + \mathrm{e}^x)t + (-\tfrac{1}{2} a_1(x) + \tfrac{1}{2} a_0(x) + \tfrac{1}{2} \tfrac{d}{dx} b_0(x) - \tfrac{1}{2} b_0(x))t^2 \\ +(-\tfrac{1}{6} a_2(x) + \tfrac{1}{6} a_1(x) + \tfrac{1}{6} \tfrac{d}{dx} b_1(x) - \tfrac{1}{6} b_1(x))t^3 \\ +(-\tfrac{1}{24} a_3(x) + \tfrac{1}{24} a_2(x) + \tfrac{1}{24} \tfrac{d}{dx} b_2(x) - \tfrac{1}{24} b_2(x))t^4 + \cdots = 0, \end{aligned}$$

and by assuming $V_1(x,t) = 0$ and $\mu = \nu = 1$ we deduce

$$\begin{aligned} V_1(x,t) = (-e^x - b_0(x))t + (-\tfrac{1}{2} b_1(x) - \tfrac{1}{2} b_0(x) - \tfrac{1}{2} \tfrac{d}{dx} a_0(x) + \tfrac{1}{2} a_0(x))t^2 \\ +(-\tfrac{1}{6} b_2(x) - \tfrac{1}{6} b_1(x) - \tfrac{1}{6} \tfrac{d}{dx} a_1(x) + \tfrac{1}{6} a_1(x))t^3 \\ +(-\tfrac{1}{24} b_3(x) - \tfrac{1}{24} b_2(x) - \tfrac{1}{24} \tfrac{d}{dx} a_2(x) + \tfrac{1}{24} a_2(x))t^4 + \cdots = 0. \end{aligned}$$

It can be easily shown that

$$a_i(x) = e^x, \qquad b_i(x) = (-1)^{i+1} e^x, \quad i = 0, 1, 2, \ldots.$$

Therefore, we gain the solutions of Eq. (25) as

$$\begin{aligned} u(x,t) &= 1 + e^x(1 + \frac{t^\mu}{\Gamma(\mu+1)} + \frac{t^{2\mu}}{\Gamma(2\mu+1)} + \frac{t^{3\mu}}{\Gamma(3\mu+1)} + \cdots) \\ &= 1 + e^x E_\mu(t^\mu), \\ v(x,t) &= -1 + e^x(1 - \frac{t^\nu}{\Gamma(\nu+1)} + \frac{t^{2\nu}}{\Gamma(2\nu+1)} - \frac{t^{3\nu}}{\Gamma(3\nu+1)} + \cdots) \\ &= -1 + e^x E_\nu(-t^\nu), \end{aligned} \tag{31}$$

which are the exact solutions. Now, if we put $\mu = \nu = 1$ in (31), we obtain $u(x,t) = 1 + e^{x+t}$, $v(x,t) = -1 + e^{x-t}$ which are the exact solutions of the given nonlinear system (25) for $\mu = \nu = 1$.

4.3 The inhomogeneous nonlinear system

We now consider the inhomogeneous nonlinear fractional system:

$$\begin{aligned} &D_t^{\mu} u + v u_x + u = 1, \\ &D_t^{\nu} v - u v_x - v = 1, \end{aligned} \tag{32}$$

subject to the following initial conditions

$$u(x,0) = e^x \quad v(x,0) = e^{-x},$$

where $0 < \mu, \nu \leq 1$. To solve Eq. (32) by the LTNHPM, we construct the following homotopies:

$$\begin{aligned} &D_t^{\mu} U = u_0(x,t) - p[u_0(x,t) + V U_x + U - 1], \\ &D_t^{\nu} V = v_0(x,t) - p[v_0(x,t) - U V_x - V - 1]. \end{aligned} \tag{33}$$

Applying Laplace transform on both sides of (33) and using the property of the Laplace transform, we have

$$\begin{aligned} &s^{\mu}\, \mathcal{L}\{U\} - s^{\mu-1} U(x,0) = \mathcal{L}\{u_0(x,t) - p[u_0(x,t) + V U_x + U - 1]\}, \\ &s^{\nu}\, \mathcal{L}\{V\} - s^{\nu-1} V(x,0) = \mathcal{L}\{v_0(x,t) - p[v_0(x,t) - U V_x - V - 1]\}, \end{aligned} \tag{34}$$

or

$$\begin{aligned} &\mathcal{L}\{U\} = \tfrac{1}{s} U(x,0) + \tfrac{1}{s^{\mu}} \{\mathcal{L}\{u_0(x,t) - p[u_0(x,t) + V U_x + U - 1]\}\}, \\ &\mathcal{L}\{V\} = \tfrac{1}{s} V(x,0) + \tfrac{1}{s^{\nu}} \{\mathcal{L}\{v_0(x,t) - p[v_0(x,t) - U V_x - V - 1]\}\}. \end{aligned} \tag{35}$$

Operating with the inverse Laplace transform on both sides of (35) and from the initial conditions $U(x,0) = e^x$ and $V(x,0) = e^{-x}$, we derive

$$\begin{aligned} &U = e^x + \mathcal{L}^{-1}\{\tfrac{1}{s^{\mu}} \{\mathcal{L}\{u_0(x,t) - p[u_0(x,t) + V U_x + U - 1]\}\}\}, \\ &V = e^{-x} + \mathcal{L}^{-1}\{\tfrac{1}{s^{\nu}} \{\mathcal{L}\{v_0(x,t) - p[v_0(x,t) - U V_x - V - 1]\}\}\}. \end{aligned} \tag{36}$$

Suppose the solutions of system Eq. (36) have the following form:

$$\begin{aligned} &U(x,t) = U_0(x,t) + p U_1(x,t), \\ &V(x,t) = V_0(x,t) + p V_1(x,t), \end{aligned} \tag{37}$$

where $U_i(t,x)$, $0 \le i \le 1$, and $V_i(t,x)$, $0 \le i \le 1$, are functions which should be determined.
Substituting Eq. (37) into Eq. (36), collecting the same powers of p, and equating each coefficient of p to zero yields

$$p^0: \quad \begin{cases} U_0(x,t) = e^x + \mathcal{L}^{-1}\{\frac{1}{s^\mu}\{\mathcal{L}\{u_0(x,t)\} \\ \\ V_0(x,t) = e^{-x} + \mathcal{L}^{-1}\{\frac{1}{s^\nu}\{\mathcal{L}\{v_0(x,t)\}\}, \end{cases}$$

$$p^1: \quad \begin{cases} U_1(x,t) = -\mathcal{L}^{-1}\{\frac{1}{s^\mu}\{\mathcal{L}\{u_0(x,t) + V_0 U_{0_x} + U_0 - 1\} \\ \\ V_1(x,t) = -\mathcal{L}^{-1}\{\frac{1}{s^\nu}\{\mathcal{L}\{v_0(x,t) - U_0 V_{0_x} - V_0 - 1\}\}. \end{cases}$$

Assume $u_0(x,t) = \sum_{n=0}^{\infty} \frac{a_n(x)}{\Gamma(n\mu+1)} t^{n\mu}$, $v_0(x,t) = \sum_{n=0}^{\infty} \frac{b_n(x)}{\Gamma(n\nu+1)} t^{n\nu}$. Now if we set $U_1(x,t) = 0$ and $\mu = \nu = 1$, then we have

$$\begin{aligned} U_1(x,t) = {} & (-a_0(x) - \mathrm{e}^x)t \\ & +(-\tfrac{1}{2}\,a_1(x) - \tfrac{1}{2}\left(\tfrac{d}{dx}a_0(x)\right)\mathrm{e}^{-x} - \tfrac{1}{2}\,a_0(x) - \tfrac{1}{2}\,b_0(x)\,\mathrm{e}^x)t^2 \\ & +(-\tfrac{1}{3}\,b_0(x)\tfrac{d}{dx}a_0(x) - \tfrac{1}{6}\,a_2(x) - \tfrac{1}{6}\left(\tfrac{d}{dx}a_1(x)\right)\mathrm{e}^{-x} \\ & -\tfrac{1}{6}\,a_1(x) - \tfrac{1}{6}\,b_1(x)\,\mathrm{e}^x)t^3 \\ & +(-\tfrac{1}{8}\,b_0(x)\tfrac{d}{dx}a_1(x) - \tfrac{1}{8}\,b_1(x)\tfrac{d}{dx}a_0(x) - \tfrac{1}{24}\,a_3(x) \\ & -\tfrac{1}{24}\left(\tfrac{d}{dx}a_2(x)\right)\mathrm{e}^{-x} - \tfrac{1}{24}\,a_2(x) - \tfrac{1}{24}\,b_2(x)\,\mathrm{e}^x)t^4 + \cdots = 0, \end{aligned}$$

and by taking $V_1(x,t) = 0$ and $\mu = \nu = 1$ we see that

$$\begin{aligned} V_1(x,t) = {} & (-b_0(x) + \mathrm{e}^{-x})t + (-\tfrac{1}{2}\,\mathrm{e}^{-x}a_0(x) - \tfrac{1}{2}\,b_1(x) + \tfrac{1}{2}\,\mathrm{e}^x\tfrac{d}{dx}b_0(x) \\ & +\tfrac{1}{2}\,b_0(x))t^2 + (-\tfrac{1}{6}\,\mathrm{e}^{-x}a_1(x) + \tfrac{1}{3}\left(\tfrac{d}{dx}b_0(x)\right)a_0(x) - \tfrac{1}{6}\,b_2(x) \\ & +\tfrac{1}{6}\,\mathrm{e}^x\tfrac{d}{dx}b_1(x) + \tfrac{1}{6}\,b_1(x))t^3 \\ & +(-\tfrac{1}{24}\,\mathrm{e}^{-x}a_2(x) + \tfrac{1}{8}\left(\tfrac{d}{dx}b_1(x)\right)a_0(x) + \tfrac{1}{8}\,a_1(x)\tfrac{d}{dx}b_0(x) \\ & -\tfrac{1}{24}\,b_3(x) + \tfrac{1}{24}\,b_2(x) + \tfrac{1}{24}\,\mathrm{e}^x\tfrac{d}{dx}b_2(x))t^4 + \cdots = 0. \end{aligned}$$

It can be easily shown that

$$a_i(x) = (-1)^{i+1}e^x, \qquad b_i(x) = e^{-x}, \quad i = 0, 1, 2, \cdots.$$

Therefore, we gain the solutions of Eq. (32) as:

$$\begin{aligned} u(x,t) &= e^x(1-\frac{t^\mu}{\Gamma(\mu+1)}+\frac{t^{2\mu}}{\Gamma(2\mu+1)}-\frac{t^{3\mu}}{\Gamma(3\mu+1)}+\cdots) \\ &= 1+e^x E_\mu(-t^\mu), \\ v(x,t) &= e^{-x}(1+\frac{t^\nu}{\Gamma(\nu+1)}+\frac{t^{2\nu}}{\Gamma(2\nu+1)}+\frac{t^{3\nu}}{\Gamma(3\nu+1)}+\cdots) \\ &= e^{-x}E_\nu(t^\nu), \end{aligned} \tag{38}$$

which are the exact solutions. Now, if we put $\mu=\nu=1$ in (38), we obtain $u(x,t)=e^{x-t}$, $v(x,t)=e^{-x+t}$ which are the exact solutions of the given nonlinear system (32) for $\mu=\nu=1$.

4.4 The homogeneous nonlinear system

We finally examine the homogeneous nonlinear fractional system:

$$\begin{aligned} &D_t^\mu u+v_x w_y-v_y w_x=-u, \\ &D_t^\alpha v+u_x w_y+u_y w_x=v, \\ &D_t^\nu w+u_x v_y+u_y v_x=w, \end{aligned} \tag{39}$$

subject to the following initial conditions

$$u(x,y,0)=e^{x+y} \quad v(x,y,0)=e^{x-y}, \quad w(x,y,0)=e^{-x+y},$$

where $0<\mu,\alpha,\nu\leq 1$. To solve Eq. (39) by the LTNHPM, we construct the following homotopies

$$\begin{aligned} D_t^\mu U &= u_0(x,y,t)-p[u_0(x,y,t)+V_xW_y-V_yW_x+U], \\ D_t^\alpha V &= v_0(x,y,t)-p[v_0(x,y,t)+U_xW_y+U_yW_x-V], \\ D_t^\nu W &= w_0(x,y,t)-p[w_0(x,y,t)+U_xV_y+U_yV_x-W]. \end{aligned} \tag{40}$$

Applying Laplace transform on both sides of (40), we have

$$\begin{aligned} &s^\mu\ \mathcal{L}\{U\}-s^{\mu-1}U(x,y,0)=\mathcal{L}\{u_0(x,y,t)-p[u_0(x,y,t)+V_xW_y-V_yW_x+U]\}, \\ &s^\alpha\ \mathcal{L}\{V\}-s^{\alpha-1}V(x,y,0)=\mathcal{L}\{v_0(x,y,t)-p[v_0(x,y,t)+U_xW_y+U_yW_x-V]\}, \\ &s^\nu\ \mathcal{L}\{W\}-s^{\nu-1}W(x,y,0)=\mathcal{L}\{w_0(x,y,0)-p[w_0(x,y,t)+U_xV_y+U_yV_x-W]\}, \end{aligned} \tag{41}$$

or

$$\begin{aligned}
&\mathcal{L}\{U\} = \tfrac{1}{s}U(x,y,0) + \tfrac{1}{s^{\mu}}\{\mathcal{L}\{u_0(x,y,t) - p[u_0(x,y,t) + V_xW_y - V_yW_x + U]\}\},\\
&\mathcal{L}\{V\} = \tfrac{1}{s}V(x,y,0) + \tfrac{1}{s^{\alpha}}\{\mathcal{L}\{v_0(x,y,t) - p[v_0(x,y,t) + U_xW_y + U_yW_x - V\}\},\\
&\mathcal{L}\{W\} = \tfrac{1}{s}W(x,y,0) + \tfrac{1}{s^{\nu}}\{\mathcal{L}\{w_0(x,y,t) - p[w_0(x,y,t) + U_xV_y + U_yV_x - W]\}\}.
\end{aligned} \tag{42}$$

By applying inverse Laplace transform on both sides of (42) and from the initial conditions $U(x,y,0) = e^{x+y}$, $V(x,y,0) = e^{x-y}$ and $W(x,y,0) = e^{-x+y}$, we derive

$$\begin{aligned}
&U = e^{x+y} + \mathcal{L}^{-1}\{\tfrac{1}{s^{\mu}}\{\mathcal{L}\{u_0(x,y,t) - p[u_0(x,y,t) + V_xW_y - V_yW_x + U]\}\}\},\\
&V = e^{x-y} + \mathcal{L}^{-1}\{\tfrac{1}{s^{\alpha}}\{\mathcal{L}\{v_0(x,y,t) - p[v_0(x,y,t) + U_xW_y + U_yW_x - V]\}\}\},\\
&V = e^{-x+y} + \mathcal{L}^{-1}\{\tfrac{1}{s^{\nu}}\{\mathcal{L}\{w_0(x,y,t) - p[w_0(x,y,t) + U_xV_y + U_yV_x - W]\}\}\}.
\end{aligned} \tag{43}$$

Suppose the solutions of system Eq. (43) have the following form:

$$\begin{aligned}
U(x,y,t) &= U_0(x,y,t) + pU_1(x,y,t),\\
V(x,y,t) &= V_0(x,y,t) + pV_1(x,y,t),\\
W(x,y,t) &= W_0(x,y,t) + pW_1(x,y,t),
\end{aligned} \tag{44}$$

where $U_i(x,y,t)$, $V_i(x,y,t)$ and $W_i(x,y,t)$ for $0 \le i \le 1$, are functions which should be determined.
Substituting Eq. (44) into Eq. (43), collecting the same powers of p, and equating each coefficient of p to zero yields

$$p^0: \quad \begin{cases} U_0(x,y,t) = e^{x+y} + \mathcal{L}^{-1}\{\frac{1}{s^{\mu}}\{\mathcal{L}\{u_0(x,y,t)\} \\ V_0(x,y,t) = e^{x-y} + \mathcal{L}^{-1}\{\frac{1}{s^{\alpha}}\{\mathcal{L}\{v_0(x,y,t)\}\} \\ W_0(x,y,t) = e^{-x+y} + \mathcal{L}^{-1}\{\frac{1}{s^{\nu}}\{\mathcal{L}\{w_0(x,y,t)\}\}, \end{cases}$$

$$p^1: \quad \begin{cases} U_1(x,y,t) = -\mathcal{L}^{-1}\{\frac{1}{s^{\mu}}\{\mathcal{L}\{u_0(x,y,t) + V_{0_x}W_{0_y} - V_{0_y}W_{0_x} + U_0\} \\ V_1(x,y,t) = -\mathcal{L}^{-1}\{\frac{1}{s^{\alpha}}\{\mathcal{L}\{v_0(x,y,t) + U_{0_x}W_{0_y} + U_{0_y}W_{0_x} - V_0\}\}, \\ W_1(x,y,t) = -\mathcal{L}^{-1}\{\frac{1}{s^{\nu}}\{\mathcal{L}\{w_0(x,y,t) + U_{0_x}V_{0_y} + U_{0_y}V_{0_x} - W_0\}\}. \end{cases}$$

Assume

$$u_0(x,y,t) = \sum_{n=0}^{\infty} \frac{a_n(x,y)}{\Gamma(n\mu+1)}\, t^{n\mu}, \qquad v_0(x,y,t) = \sum_{n=0}^{\infty} \frac{b_n(x,y)}{\Gamma(n\alpha+1)}\, t^{n\alpha}$$

and

$$w_0(x,y,t) = \sum_{n=0}^{\infty} \frac{c_n(x,y)}{\Gamma(n\nu+1)}\, t^{n\nu}.$$

Now if we set $U_1(x,y,t) = 0$ and $\mu = \alpha = \nu = 1$, then we have

$$\begin{aligned}
U_1(x,y,t) = {} & (-a_0\,(x,y) - \mathrm{e}^{x+y})t + (-\tfrac{1}{2}\,a_0\,(x,y) - \tfrac{1}{2}\,a_1\,(x,y) \\
& -\tfrac{1}{2}\,\left(\tfrac{\partial}{\partial x}c_0\,(x,y) + \tfrac{\partial}{\partial y}c_0\,(x,y)\right)\mathrm{e}^{x-y} \\
& -\tfrac{1}{2}\,\left(\tfrac{\partial}{\partial x}b_0\,(x,y) + \tfrac{\partial}{\partial y}b_0\,(x,y)\right)\mathrm{e}^{-x+y})t^2 \\
& +(-\tfrac{1}{6}\,a_2\,(x,y) - \tfrac{1}{6}\,a_1\,(x,y) - \tfrac{1}{3}\left(\tfrac{\partial}{\partial x}b_0\,(x,y)\right)\tfrac{\partial}{\partial y}c_0\,(x,y) \\
& +\tfrac{1}{3}\,\left(\tfrac{\partial}{\partial y}b_0\,(x,y)\right)\tfrac{\partial}{\partial x}c_0\,(x,y) - \tfrac{1}{6}\,\left(\tfrac{\partial}{\partial y}c_1\,(x,y) + \tfrac{\partial}{\partial x}c_1\,(x,y)\right)\mathrm{e}^{x-y} \\
& -\tfrac{1}{6}\,\left(\tfrac{\partial}{\partial x}b_1\,(x,y) + \tfrac{\partial}{\partial y}b_1\,(x,y)\right)\mathrm{e}^{-x+y})t^3 + \cdots = 0,
\end{aligned}$$

and by taking $V_1(x,y,t) = 0$ and $\mu = \alpha = \nu = 1$ then

$$\begin{aligned}
V_1(x,y,t) = {} & (-b_0\,(x,y) + \mathrm{e}^{x-y})t + (\tfrac{1}{2}\,\left(-\tfrac{\partial}{\partial x}a_0\,(x,y) + \tfrac{\partial}{\partial y}a_0\,(x,y)\right)\mathrm{e}^{-x+y} \\
& -\tfrac{1}{2}\,b_1\,(x,y) + \tfrac{1}{2}\,b_0\,(x,y) - \tfrac{1}{2}\,\left(\tfrac{\partial}{\partial x}c_0\,(x,y) + \tfrac{\partial}{\partial y}c_0\,(x,y)\right)\mathrm{e}^{x+y})t^2 \\
& +(-\tfrac{1}{3}\,\left(\tfrac{\partial}{\partial y}a_0\,(x,y)\right)\tfrac{\partial}{\partial x}c_0\,(x,y) - \tfrac{1}{3}\,\left(\tfrac{\partial}{\partial x}a_0\,(x,y)\right)\tfrac{\partial}{\partial y}c_0\,(x,y) \\
& -\tfrac{1}{6}\,b_2\,(x,y) + \tfrac{1}{6}\,b_1\,(x,y) + \tfrac{1}{6}\,\left(-\tfrac{\partial}{\partial x}a_1\,(x,y) + \tfrac{\partial}{\partial y}a_1\,(x,y)\right)\mathrm{e}^{-x+y} \\
& -\tfrac{1}{6}\,\left(\tfrac{\partial}{\partial y}c_1\,(x,y) + \tfrac{\partial}{\partial x}c_1\,(x,y)\right)\mathrm{e}^{x+y})t^3 + \cdots = 0,
\end{aligned}$$

also, if assume that $W_1(x,y,t) = 0$ and Considering $\mu = \alpha = \nu = 1$, then we have

$$
\begin{aligned}
W_1(x,y,t) = {} & (-c_0\,(x,y) + \mathrm{e}^{-x+y})t + (\tfrac{1}{2}\,\left(\tfrac{\partial}{\partial x}a_0\,(x,y) - \tfrac{\partial}{\partial y}a_0\,(x,y)\right)\mathrm{e}^{x-y} \\
& +\tfrac{1}{2}\,c_0\,(x,y) - \tfrac{1}{2}\,c_1\,(x,y) - \tfrac{1}{2}\,\left(\tfrac{\partial}{\partial x}b_0\,(x,y) + \tfrac{\partial}{\partial y}b_0\,(x,y)\right)\mathrm{e}^{x+y})t^2 \\
& +(\tfrac{1}{6}\,\left(\tfrac{\partial}{\partial x}a_1\,(x,y) - \tfrac{\partial}{\partial y}a_1\,(x,y)\right)\mathrm{e}^{x-y} + \tfrac{1}{6}\,c_1\,(x,y) - \tfrac{1}{6}\,c_2\,(x,y) \\
& -\tfrac{1}{3}\,\left(\tfrac{\partial}{\partial y}a_0\,(x,y)\right)\tfrac{\partial}{\partial x}b_0\,(x,y) - \tfrac{1}{3}\,\left(\tfrac{\partial}{\partial x}a_0\,(x,y)\right)\tfrac{\partial}{\partial y}b_0\,(x,y) \\
& -\tfrac{1}{6}\,\left(\tfrac{\partial}{\partial x}b_1\,(x,y) + \tfrac{\partial}{\partial y}b_1\,(x,y)\right)\mathrm{e}^{x+y})t^3 + \cdots = 0,
\end{aligned}
$$

It can be easily shown that

$$
\begin{aligned}
a_i(x,y) &= (-1)^{i+1}e^{x+y}, \\
b_i(x,y) &= e^{x-y}, \qquad\qquad i = 0,1,2,\ldots, \\
c_i(x,y) &= e^{-x+y},
\end{aligned}
$$

Therefore, we gain the solutions of Eq. (39) as:

$$
\begin{aligned}
u(x,y,t) &= e^{x+y}(1 - \tfrac{t^{\mu}}{\Gamma(\mu+1)} + \tfrac{t^{2\mu}}{\Gamma(2\mu+1)} - \tfrac{t^{3\mu}}{\Gamma(3\mu+1)} + \cdots) = e^{x+y}E_{\mu}(-t^{\mu}), \\
u(x,y,t) &= e^{x-y}(1 + \tfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots) = e^{x-y}E_{\alpha}(t^{\alpha}), \qquad (45) \\
w(x,y,t) &= e^{-x+y}(1 + \tfrac{t^{\nu}}{\Gamma(\nu+1)} + \tfrac{t^{2\nu}}{\Gamma(2\nu+1)} + \tfrac{t^{3\nu}}{\Gamma(3\nu+1)} + \cdots) = e^{-x+y}E_{\nu}(t^{\nu}),
\end{aligned}
$$

which are the exact solutions. Now, if we put $\mu = \alpha = \nu = 1$ in (45), we obtain $u(x,y,t) = e^{x+y-t}$, $v(x,y,t) = e^{x-y+t}$ and $w(x,y,t) = e^{-x+y+t}$ which are the exact solutions of the given nonlinear system (39) for $\mu = \alpha = \nu = 1$. The results for the exact solution $u(x,y,t)$ of system (39) are obtained using LTNHPM, for $\mu = 0.5, 0.8$ and 1 are shown in Figure 1. The results for the exact solution $v(x,y,t)$ of system (39) are obtained using LTNHPM, for $\alpha = 0.5, 0.8$ and 1 are shown in Figure 2. The results for the exact solution $w(x,y,t)$ of system (39) are obtained using LTNHPM, for $\nu = 0.5, 0.8$ and 1 are shown in Figure 3.

5 Conclusion

In this paper, we have introduced a combination of Laplace transform and new homotopy perturbation methods for solving systems of fractional partial differential equations which we called LTNHPM. In this technique, the

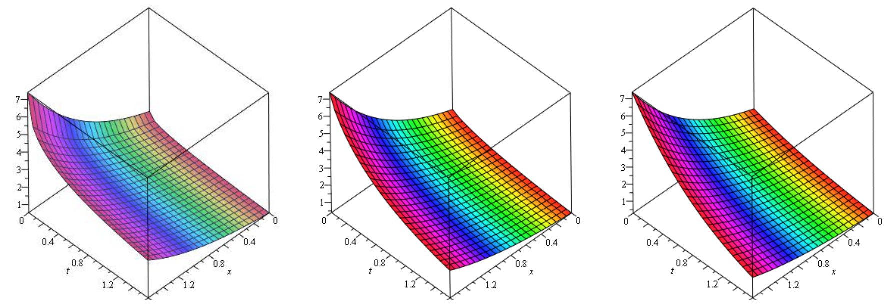

Figure 1: The surface indicates the solution $u(x, y, t)$ for the system (39), for $y = 0.4$ where (a) $\mu = 0.5$, (b) $\mu = 0.8$ and (c) $\mu = 1$.

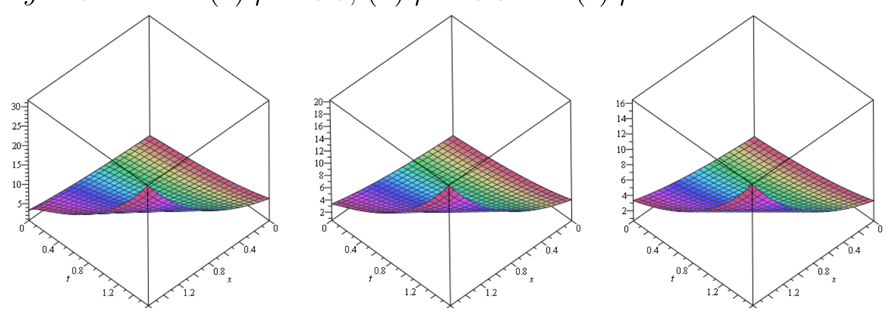

Figure 2: The surface indicates the solution $v(x, y, t)$ for the system (39), for $y = 0.4$ where (a) $\alpha = 0.5$, (b) $\alpha = 0.8$ and (c) $\alpha = 1$

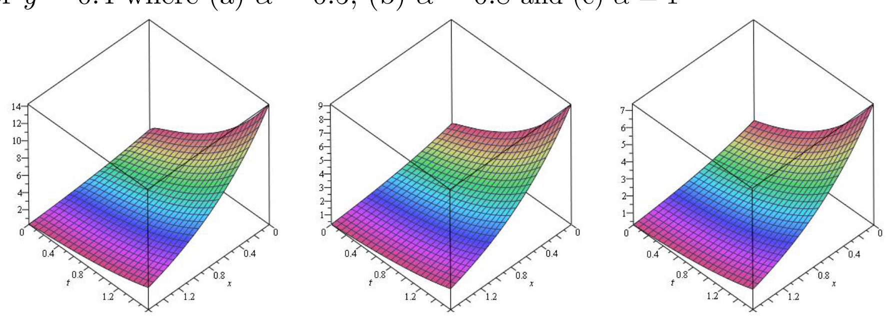

Figure 3: The surface indicates the solution $w(x, y, t)$ for the system (39), for $y = 0.4$ where (a) $\nu = 0.5$, (b) $\nu = 0.8$ and (c) $\nu = 1$.

solution is considered as a Taylor series expansion converges rapidly to the exact solution of the equation. As shown in the four examples of this paper, a clear conclusion can be drawn from the results that the LTNHPM provides an efficient method to handle linear and nonlinear partial differential equations of fractional order. We point out that the corresponding analytical and numerical solutions are obtained using Maple 13.

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