Journal of Mathematical Modelling
Vol. 1, No. 1, 2013, pp. 68-75

JMM

A modified homotopy perturbation method to periodic solution of a coupled integrable dispersionless equation

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Abstract. In this paper, a reliable approach is introduced to approximate periodic solutions of a system of coupled integrable dispersionless. The system is firstly, transformed into an ordinary differential equation by wave transformation. The solution of ODE is obtained by the homotopy perturbation method. To show the periodic behavior of the solution, a modification based on the Laplace transforms and Pade approximation, known as aftertreatment technique, is proposed. The angular frequencies are compared with the exact frequency. Comparison of the approximated results and exact one shows a good agreement.

Keywords: Homotopy perturbation method, nonlinear ordinary differential equations, coupled dispersionless equations
AMS Subject Classification: 65L03, 65H20.

1 Introduction

Recently, considerable interest has been shown in the study of the dispersionless or quasiclassical limits of integrable equations and hierarchies. The dispersionless hierarchies arise in the analysis of several problems in applied mathematics and physics from the theory of quantum fields and strings to the theory of conformal maps on the complex plane [5, 7, 13]. Various methods, in particular the inverse scattering transformation (IST) have been used in the literature to study dispersionless equations [3, 12, 6].

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Received: 30 September 2013 / Accepted: 19 December 2013.

The generalized coupled dispersionless equations have been solved through the IST method by Konno and kakuhata [2]. In this paper, a new approach for approximations to the time–space periodic solutions of nonlinear evolution equations is introduced. The method is applied to a system of coupled integrable dispersionless. In the study of equations describing wave phenomena, one of the fundamental objects is the traveling wave solution [9] that a solution possesses constant form moving with a fixed velocity and changeless shapes during propagation. The wave transform $\xi = x - ct$ is used to convert a nonlinear PDE to an ODE. By this transformation the PDE is converted to a nonlinear Duffing oscillator. There are various methods to solve nonlinear ordinary differential equations, such as homotopy perturbation method [1, 4, 8, 10]. To show the periodic behavior of the solution, that is the property of the Duffing oscillator, a new approach based on the Laplace transforms and Pade approximation, known as after treatment technique, is introduced.

2 The coupled integrable dispersionless equation

Consider a hierarchy of coupled integrable dispersionless equations

$$\begin{aligned} u_{xt} - (vw)_t &= 0, \\ v_{xt} - 2vu_x &= 0, \\ w_{xt} - 2wu_x &= 0. \end{aligned} \tag{1}$$

By substituting the wave transformation

$$u = u(\xi), \qquad v = v(\xi), \qquad w = w(\xi), \qquad \xi = x - ct, \qquad c \neq 0, \tag{2}$$

Eq. (1) is converted to the following system

$$\begin{cases} cu_{\xi\xi} - (vw)_\xi = 0, \\ cv_{\xi\xi} + 2vu_\xi = 0, \\ cw_{\xi\xi} + 2wu_\xi = 0. \end{cases} \tag{3}$$

Integrating the first equation in Eq. (3), yields

$$u_\xi = \frac{1}{c} vw + k_1, \tag{4}$$

where k_1 is the integration constant. Substituting Eq. (4) into the rest of equations in (3), and letting $k_2 = \frac{2k_1}{c}$ leads to

$$\begin{cases} v_{\xi\xi} + \frac{2}{c^2} v^2 w + k_2 v = 0, \\ w_{\xi\xi} + \frac{2}{c^2} v w^2 + k_2 w = 0. \end{cases} \tag{5}$$

By multiplying w and v in first and second equations in (5), respectively, the relation between w and v will be determined as follows

$$vw_{\xi\xi} - wv_{\xi\xi} = 0, \tag{6}$$

or

$$(vw_{\xi} - wv_{\xi})_{\xi} = 0. \tag{7}$$

This equation yields

$$(\frac{w}{v})_{\xi} = 0 \Rightarrow w = k_3 v, \tag{8}$$

where k_3 is an integration constant. Therefore, from the first equation of (5), a second order ODE over v will be obtained

$$v_{\xi\xi} + k_2 v + \frac{2k_3}{c^2}v^3 = 0, \tag{9}$$

with initial conditions

$$v(0) = A, \qquad v'(0) = 0. \tag{10}$$

Eq. (9) is the canonical form of Duffing oscillator. We now intend to solve Eq. (9) via homotopy perturbation method. Then by Eqs. (8) and (4) the solution of Eq. (3), in terms of ξ, can be obtained. The solution of Eq. (1) will be obtained by substituting $\xi = x - ct$.

3 The solution procedure

3.1 Homotopy perturbation method

Using the HPM, the following homotopy for Eq. (9) can be constructed

$$v'' + p(k_2 v + \frac{2k_3}{c^2}v^3) = 0, \tag{11}$$

where primes denotes derivatives with respect to ξ with initial conditions $v(0) = A$ and $v(0) = 0$. When the homotopy parameter p varies from 0 to 1, Eq. (11) varies from $v'' = 0$ to original differential equation (9), or from initial approximation to an exact solution of Eq. (9). The solution of Eq. (11) can be expressed as a power series over p

$$v = v_0 + v_1 p + v_2 p^2 + \cdots . \tag{12}$$

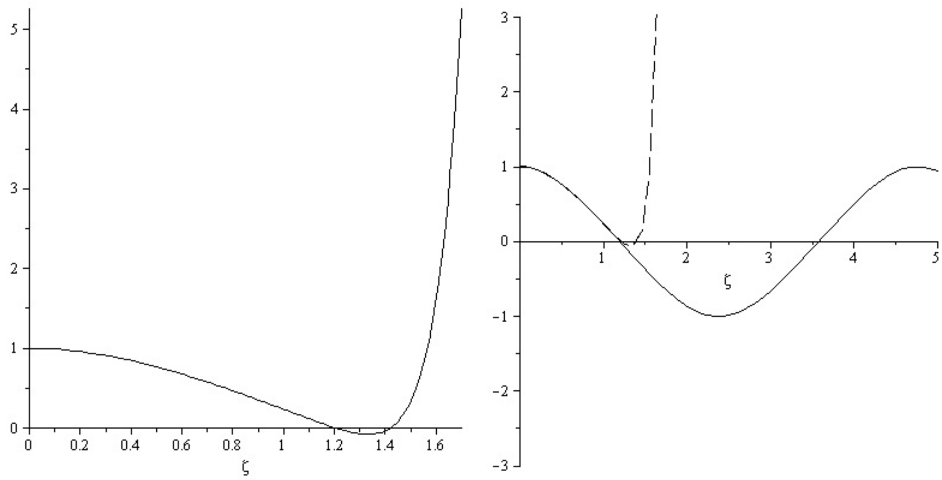

Figure 1: (Left) Plot of $v(\xi)$ by HPM. (Right) Comparison of modified HPM and HPM by solid line and dash line respectively.

Substituting Eq. (12) into Eq. (11) and equating the coefficients of the terms with identical powers, a series of linear equations of the following form will be obtained

$$\begin{array}{l} p^0 : v_0''(\xi) = 0, \quad v_0(0) = A, \quad v_0'(0) = 0, \\ p^1 : v_1''(\xi) + k_2 v_0(\xi) + \frac{2k_3}{c^2} \times v_0^3(\xi) = 0, \quad v_1(0) = 0, \quad v_1'(0) = 0, \\ p^2 : v_2''(\xi) + k_2 v_1(\xi) + \frac{2k_3}{c^2} \times 3v_0^2(\xi) v_1(\xi) = 0, \quad v_2(0) = 0, \quad v_2'(0) = 0, \\ \vdots \end{array} \tag{13}$$

By solving equations in (13), $v_0(\xi)$, $v_1(\xi)$, $v_2(\xi)$, ... can be determined and the series solution (12) will be entirely determined. The m'th order approximation solution can be considered as follows

$$v(\xi) = \sum_{i=0}^{m} v_i(\xi).$$

3.2 Improvement of the solution by an easy modification

The solution of this method for some values of parameters, say $k_1 = k_3 = 0.5$, $k_2 = 1$, $c = 1$, and $A = 1$ is as follows

$$\begin{aligned} v(\xi) = 1 - \xi^2 + 0.33333\xi^4 - 0.14444\xi^6 + 0.06389\xi^8 - 0.02728\xi^{10} \\ +0.01173\xi^{12} - 0.00505\xi^{14} + 0.00217\xi^{16}, \end{aligned} \tag{14}$$

and is shown in Fig. 1. It is clear that the solution (14) does not exhibit the periodic behavior which is characteristic of oscillatory systems, and this truncated series solution can be diverge rapidly, over a small region. To overcome this lack and to improve the accuracy of the method, an aftertreatment technique [11], can be applied as follows.

In the first step, applying the Laplace transformation to the series solution (14), yields

$$L(v(\xi)) = \frac{1}{s} - \frac{2}{s^3} + \frac{8}{s^5} - \frac{104}{s^7} + \frac{2576}{s^9} - \frac{99007.99990}{s^{11}} + \frac{5.61747 \times 10^6}{s^{13}} - \frac{4.4 \times 10^8}{s^{15}} + \frac{4.54078 \times 10^{10}}{s^{17}}. \tag{15}$$

For simplicity, let $s = \frac{1}{p}$

$$L(v(\xi)) = p - 2p^3 + 8p^5 - 104p^7 + 2576p^9 - 99007.99990p^{11} + 5.61747 \times 10^6 p^{13} - 4.4 \times 10^8 p^{15} + 4.54078 \times 10^{10} p^{17}. \tag{16}$$

By pade approximation of series (16), the following rational approximation will be obtained

$$[\frac{8}{8}] = \frac{p + 194.00004\, p^3 + 9164.00336p^5 + 99856.05229p^7}{1 + 196.00004\, p^2 + 9548.00343p^5 + 1.17488 \times 10^5 p^7 + 1.76400 \times 10^5 p^8}. \tag{17}$$

After recalling $p = \frac{1}{s}$

$$[\frac{8}{8}] = \frac{99856.05230s + 9164.00336s^3 + 194.00004s^5 + s^7}{1.76400 \times 10^5 + 1.17488 \times 10^5 s^2 + 9548.00343s^4 + 196.00003s^6 + s^8}. \tag{18}$$

By using the inverse Laplace transformation to (18), a periodic approximate solution will be obtained as

$$v(\xi) = 0.98208 \cos(1.31845\, t) + 0.01770 \cos(3.99163\, t) + 0.00022 \cos(7.02844\, t) + 4.70476 \times 10^{-7} \cos(11.35472\, t). \tag{19}$$

Fig. 1 shows the comparison of modified approximate solution obtained by Laplace transformation technique and the solution obtained by HPM. It obviously shows the periodic behavior of the solution and the improvement of convergency.

4 Results and discussions

To show the effectiveness of the proposed method for nonlinear oscillator (9), the comparison of the approximate frequencies for $k_1 = k_3 = 0.5$,

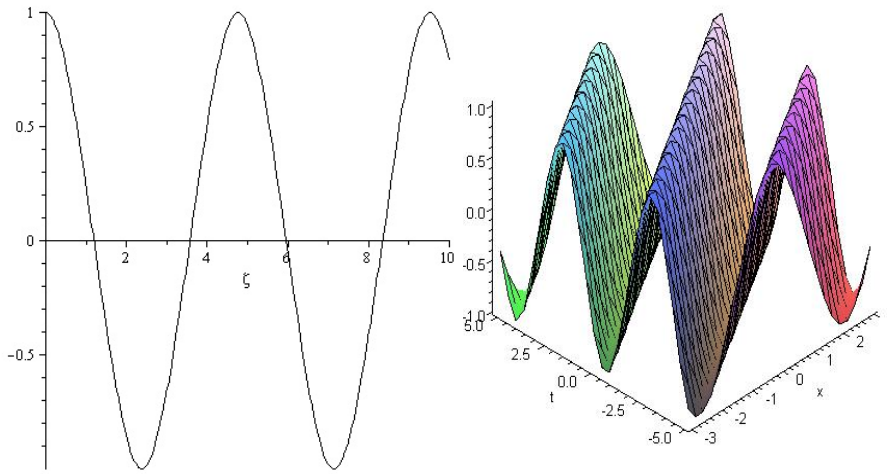

Figure 2: (Left) The solution of (9) with respect to ξ ($v(\xi)$). (Right) The solution of (1) ($v(x,t)$) for *A=1*

$k_2 = 1$, and $c = 1$, for various amount of A, and the exact frequency are shown in Table 1. The exact frequency of the oscillator (9) is

$$\omega_{exact} = \frac{\pi}{2}(\int_0^A [k_2(A - v^2) - \frac{k_3}{c^2}(A^4 - v^4)]^{-\frac{1}{2}} dv)^{-1}.$$

Fig 2. shows the solution of (9) ($v(\xi)$), and the solution of (1) ($v(x,t)$). The periodic solution of Eq. (1), can be obtained by Eqs. (4) and (8), and transformation (2), as follows

$$\begin{aligned} v(\xi) &= 0.98208\cos(1.31845\,t) + 0.01770\cos(3.99163\,t) \\ &\quad + 0.00022\,\cos(7.02844\,t) + 4.70476\times 10^{-7}\cos(11.35472\,t), \\ u(\xi) &= 0.5\,v(\xi), \\ u(\xi) &= \int_0^{\xi} v(y)w(y)dy + \frac{\xi}{2}, \end{aligned}$$

with $\xi = x - ct$.

5 Conclusion

Modified homotopy perturbation method has been applied to three coupled integrable dispersionless equations. It has been shown that the traditional homotopy perturbation method does not show the periodic behavior of

Table 1: Comparison of the approximate frequencies for $k_1 = k_3 = 0.5$, $k_2 = 1$, and $c = 1$.

A	Approximate frequency	ω_{exact}
0.5	1.08917	1.08916
1	1.31845	1.31778
2	1.98315	1.97602
5	4.36349	4.35746
10	8.56621	8.53359

the solution. But by a modification, a periodic solution has been obtained. The comparison of the approximate frequencies and exact one shows a good agreement. Computations were performed by Maple 12.

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