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Application of Laplace decomposition method for Burgers-Huxley and Burgers-Fisher equations

Mohammad Reza Yaghouti* and Ali Zabihi

Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran
Emails: yaghouti@guilan.ac.ir, alizabihi90@yahoo.com

Abstract. In this paper, we apply the Laplace decomposition method to obtain a series solutions of the Burgers-Huxley and Burgers-Fisher equations. The technique is based on the application of Laplace transform to nonlinear partial differential equations. The method does not need linearization, weak nonlinearity assumptions or perturbation theory and the nonlinear terms can be easily handled by using the Adomian polynomials. We compare the numerical results of the proposed method with those of some available methods.

Keywords: Laplace decomposition method, Burgers-Huxley Equation, Burgers-Fisher equation.
AMS Subject Classification: 65D20, 65D25.

1 Introduction

The decomposition method has been shown to solve efficiently, easily and accurately a large class of linear and nonlinear ordinary, partial, deterministic or stochastic differential equations. The method is very well suited to physical problems since it does not require unnecessary linearization, perturbation and other restrictive methods and assumptions which may change the problem being solved, sometimes seriously. This paper presents a Laplace transform numerical scheme, based on the decomposition method, for solving nonlinear differential equations. The analysis will be adapted to

*Corresponding author.
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the approximate solution of a class of nonlinear second-order initial-value problems, though the algorithm is well suited for a wide range of nonlinear problems. Khuri in [9] proposed a Laplace Decomposition Method (LDM) for the approximate solution of a class of nonlinear ordinary differential equations. In [12], Yusufoglu developed this method for the solution of Duffing equation. Nasser [10] exploited this method to solve Falkner-Skan equation. LDM was proved to be compatible with the versatile nature of the physical problems and was applied to a wide class of functional equations, respectively, in [7] and [11]. In this paper, the method is implemented for two numerical examples and the numerical results show that the scheme approximates the exact solution with a high degree of accuracy using only few terms of the iterative scheme.

The rest of this paper is organized as follows. In Section 2, we give an analysis of LDM. Section 3 is devoted to the convergence of the Adomian decomposition method for ordinary differential equations. In Section 4, we present convergence of the Adomian decomposition method for partial differential equations. Application of LDM is presented in Section 5. Finally, we give our conclusions in Section 6.

2 Laplace decomposition method

The aim of this section is to discuss the use of LDM for solving partial differential equations written in an operator form

$$L_t u + Ru + Nu = g, \tag{1}$$

with initial condition

$$u(x,0) = f(x), \tag{2}$$

where L_t is considered a first-order partial differential operator, R and N are linear and nonlinear operators, respectively, and g is source term. The method consists of first applying the Laplace transform to both sides of Eq. (1) and then by using initial condition (2), we have

$$\mathcal{L}[L_t u] + \mathcal{L}[Ru] + \mathcal{L}[Nu] = \mathcal{L}[g], \tag{3}$$

using the differentiation property of Laplace transform, we get

$$\mathcal{L}[u] = \frac{f(x)}{s} + \frac{1}{s}\mathcal{L}[g] - \frac{1}{s}\mathcal{L}[Ru] - \frac{1}{s}\mathcal{L}[Nu]. \tag{4}$$

The LDM defines the solutions $u(x,t)$ by the infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n. \tag{5}$$

The nonlinear term N is usually represented by an infinite series of the so-called Adomian polynomials as

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n. \tag{6}$$

The Adomian polynomials can be generated for all forms of nonlinearity. It is determined by the following relation

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right], \quad n = 0, 1, 2, \ldots. \tag{7}$$

Substituting Eqs. (5) and (6) into Eq. (4), gives

$$\mathcal{L}[\sum_{n=0}^{\infty} u_n] = \frac{f(x)}{s} + \frac{1}{s}\mathcal{L}[g] - \frac{1}{s}\mathcal{L}[R(\sum_{n=0}^{\infty} u_n)] - \frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} A_n]. \tag{8}$$

Applying the linearity of the Laplace transform, we define the following recursive formula

$$\mathcal{L}[u_0] = \frac{f(x)}{s} + \frac{1}{s}\mathcal{L}[g], \tag{9}$$

$$\mathcal{L}[u_{k+1}] = -\frac{1}{s}\mathcal{L}[R(u_k)] - \frac{1}{s}\mathcal{L}[A_k], \quad k \geq 1. \tag{10}$$

Therefore, by applying the inverse Laplace transform, we can evaluate u_k $(k \geqslant 0)$.

3 Convergence of the Adomian decomposition method for ordinary differential equations

In this section, we investigate the convergence of ADM for initial-value problems associated with systems of ordinary differential equations.

3.1 Formula of the ADM

In reviewing the basic methodology, we consider an abstract system of nonlinear differential equations

$$\frac{dy}{dt} = f(t,y), \quad y \in \mathbb{R}^d, \quad f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \tag{11}$$

with initial condition $y(0) = y_0 \in \mathbb{R}^d$. Assume that f is analytic near $y = y_0$ and $t = 0$. Solving Eq. (11) is equivalent to solve the Volterra integral equation

$$y(t) = y_0 + \int_0^t f(s, y(s))ds. \tag{12}$$

To set up the Adomian method, consider y in a series of the form

$$y = y_0 + \sum_{n=1}^{\infty} y_n, \tag{13}$$

and write the nonlinear function $f(t, y)$ as the series of functions

$$f(t, y) = \sum_{n=0}^{\infty} A_n(t, y_0, y_1, \dots, y_n). \tag{14}$$

The dependence of A_n on t and y_0 may be non-polynomial. Formally, A_n is obtained by

$$A_n = \frac{1}{n!}\frac{d^n}{d\varepsilon^n} f\Big(t, \sum_{i=0}^{\infty} \varepsilon^i y_i\Big)\Big|_{\varepsilon=0}, \quad n = 0, 1, 2, \dots \tag{15}$$

where ε is a formal parameter. Functions A_n are polynomials in $(y_1, \dots, y_n)$, which are referred to as the Adomian polynomials.

In what follows, we shall consider a scaler differential equation and set $d = 1$. A generalization for $d \geq 2$ is possible but is technically longer.

The first four Adomian polynomials for $d = 1$ are listed as follows

$$\begin{aligned} A_0 &= f(t, y_0), \\ A_1 &= y_1 f'(t, y_0), \\ A_2 &= y_2 f'(t, y_0) + \frac{1}{2} y_1^2 f''(t, y_0), \\ A_3 &= y_3 f'(t, y_0) + y_1 y_2 f''(t, y_0) + \frac{1}{6} y_1^3 f'''(t, y_0), \end{aligned} \tag{16}$$

where primes denote the partial derivatives with respect to y.

It was proven by Abbaoui and Cherruault [1] that the Adomian polynomials A_n are defined by the explicit formula

$$A_n = \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(t, y_0)\Big(\sum_{p_1+\ldots+p_k=n} y_{p_1} \ \dots \ y_{p_k}\Big), \quad n \geq 1, \tag{17}$$

or, in an equivalent form, by

$$A_n = \sum_{|nk|=n} f^{(|k|)}(t, y_0) \frac{y_1^{k_1} \dots y_n^{k_n}}{k_1! \dots k_n!}, \quad n \geq 1, \tag{18}$$

where $|k| = k_1 + \cdots + k_n$, and $|nk| = k_1 + 2k_2 + \cdots + nk_n$. Khelifa and Cherruault in [8] obtained a bound for Adomian polynomials by,

$$|A_n| \leq \frac{(n+1)^n}{(n+1)!} M^{n+1}, \tag{19}$$

where

$$\sup_{t \in J} |f^{(k)}(t, y_0)| \leq M, \tag{20}$$

for a given time interval $J \subset \mathbb{R}$.

Substituting Eqs. (13) and (14) into Eq. (12) gives a recursive equation for y_{n+1} in terms of $(y_0, y_1, \dots, y_n)$ as

$$y_{n+1}(t) = \int_0^t A_n(s, y_0(s), \dots, y_n(s)) ds, \qquad n = 0, 1, 2, \dots \tag{21}$$

Convergence of series (13) obtained by Eq. (21) is a subject of our studies in next section.

3.2 Convergence analysis

From Eq. (15), it is clear that A_n's are some polynomials in terms of $y_1, \dots, y_n$ and thus y_{n+1} is obtained from Eq. (21) explicitly, if we compute A_n. The first proof of convergence of the ADM was given by Cherruault, who used fixed point theorem for abstract functional equations. Furthermore, Boumenir and Gordon in [4] discussed the rate of convergence of the ADM.

The proof of the convergence for ADM for the functional equation

$$y = y_0 + f(y), \quad y \in \mathbb{H}, \tag{22}$$

is given as follows, where $\mathbb{H}$ is a Hilbert space and $f : \mathbb{H} \to \mathbb{H}$. Let $S_n = y_1 + y_2 + \cdots + y_n$, and $f_n(y_0 + S_n) = \sum_{i=0}^n A_i$. ADM is equivalent to determining the sequence $\{S_n\}_{n \in \mathbb{N}}$ defined by

$$S_{n+1} = f_n(y_0 + S_n), \quad S_0 = 0. \tag{23}$$

If the limits

$$S = \lim_{n \to \infty} S_n, \quad f = \lim_{n \to \infty} f_n, \tag{24}$$

exist in a Hilbert space $\mathbb{H}$, then S solves a fixed-point equation $S = f(y_0+S)$ in $\mathbb{H}$. The convergence of the ADM was proved in [5], under the following two conditions

$$||f|| \leq 1, \;\; ||f_n - f|| = \varepsilon_n \to 0 \;\; \text{as} \;\; n \to \infty. \tag{25}$$

These two conditions are rather restrictive. The first condition implies a constraint on the nonlinear function Eq. (22) while, the second condition implies the convergence of the series $\sum_{n=0}^{\infty} A_n$. It is difficult to satisfy the two conditions for a given nonlinear function $f(y)$. In the following, we shall prove convergence of the Adomian method in the context of the ODEs system (11) by using the Cauchy-Kovalevskaya theorem [5]. We only require that the function f be analytic in t and y. Let us start by reviewing the Cauchy-Kovalevskaya theorem for ordinary differential equations.

Theorem 1. *Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a real analytic function in the domain $[-t_0, t_0] \times B_{\delta_0}(y_0)$ for some $t_0 > 0$ and $\delta_0 > 0$. Let $y(t, y_0)$ be a unique solution for $t \in [-t_0, t_0]$ of the initial-value problem*

$$\begin{cases} \frac{dy}{dt} = f(t, y), \\ y(0) = y_0. \end{cases}$$

Then $y(t, y_0)$ is also a real analytic function of t near $t = 0$, that is, there exists $\tau \in (0, t_0)$ such that $y : [-\tau, \tau] \to \mathbb{R}^d$ is a real analytic function.

Remark 1. Existence, uniqueness and continuity dependence on t and y_0 of $y(t, y_0)$ follows from the Picard's method since if f is real analytic, then it is locally Lipschitz (see [4, 5]).

Remark 2. We shall consider and prove Theorem 1 for $d = 1$. Generalization for $d \geq 2$ can be developed by more complicated formulas (For more details the reader is referred to the proof of Cauchy-Kovalevskaya theorem) (see [4, 5]).

Proof of Theorem 1. By Cauchy estimation for a real analytic function in the domain $[-t_0, t_0] \times B_{\delta_0}(y_0)$, there exists a constant $C > 0$ such that

$$1 + |f(0, y_0)| \leq C, \tag{26}$$

$$\sum_{k_1+k_2=k} \frac{1}{k_1! k_2!} |\partial_t^{k_1} \partial_y^{k_2} f(0, y_0)| \leq \frac{C}{a^k}, \quad \forall k \geq 1, k_1, k_2 \geq 0. \tag{27}$$

By the Cauchy estimation (26) and (27), the Taylor series for $f(t,y)$ at $t=0$ and $y=y_0$ is bounded by

$$1+|f(t,y)| \leq C\sum_{k=0}^{\infty}\left(\frac{\rho}{a}\right)^k = \frac{C}{1-\frac{\rho}{a}} = \frac{Ca}{a-\rho} = g(\rho), \tag{28}$$

where $\rho = |t|+|y-y_0| < a$. By the Weierstrass M-Test, the Taylor series for f converges for all

$$|t|+|y-y_0| < a. \tag{29}$$

Therefore, we have

$$1+|f(0,y_0)| \leq C = g(0), \tag{30}$$
$$\sum_{k_1+k_2=k}\frac{1}{k_1!k_2!}|\partial_t^{k_1}\partial_y^{k_2}f(0,y_0)| \leq \frac{C}{a^k} \equiv \frac{1}{k!}g^{(k)}(0), \quad \forall k\geq 1, k_1, k_2 \geq 0.$$

Let us consider a problem for $\rho \in \mathbb{R}_+$

$$\begin{cases} \frac{d\rho}{dt} = g(\rho) = \frac{Ca}{a-\rho}, \\ \rho(0) = 0. \end{cases}$$

This problem has an explicit solution

$$\rho(t) = a - \sqrt{a^2 - 2aCt}, \tag{31}$$

which is an analytic function of t in $|t| < \frac{a}{2C}$. By comparison principle, if

$$\begin{cases} \frac{dy}{dt} = f(t,y), \\ y(0) = y_0, \end{cases}$$

and $1+|f(t,y)| \leq g(|t|+|y(t)-y_0|)$, for all $|t|+|y(t)-y_0| < a$ then

$$|t|+|y(t)-y_0| \leq \rho(t) = a-\sqrt{a^2-2aCt} = \sum_{k=1}^{\infty}\frac{1}{k!}\rho^{(k)}(0)t^k. \tag{32}$$

Therefore, for all $t \geq 0$,

$$|y(t,0)-y_0| \leq t\big(\rho'(0)-1\big) + \sum_{n\geq 2}\frac{1}{k!}\rho^{(k)}(0)t^k, \tag{33}$$

where the Taylor series absolutely converges in $|t| < \frac{a}{2C}$. To prove that $y(t,y_0)$ is analytic in $|t| < \min\left(a, \frac{a}{2C}\right)$, it remains to prove $|y^{(k)}(0,y_0)| \leq \rho^{(k)}(0)$ for any $k \geq 1$. If this is the case, then the Taylor series for $y(t,y_0)$

has a majorant convergent series, such that the Taylor series for $y(t, y_0)$ converges, by the Weierstrass M-Test. To prove that $|y^{(k)}(0, y_0)| \le \rho^{(k)}(0)$, from the ODE system we have

$$\begin{aligned} \frac{d^2y}{dt^2} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}, \\ \frac{d^3y}{dt^3} &= \frac{\partial^2 f}{\partial t^2} + 2f\frac{\partial^2 f}{\partial t\partial y} + \frac{\partial f}{\partial t}\frac{\partial f}{\partial y} + f\Big(\frac{\partial f}{\partial y}\Big)^2 + f^2\frac{\partial^2 f}{\partial y^2}, \end{aligned} \tag{34}$$

and therefore

$$\begin{aligned} |\frac{d^2y}{dt^2}| &\le |\frac{\partial f}{\partial t}| + |\frac{\partial f}{\partial y}||f|, \\ &\le g'(0)(1+|f|) \le g(0)g'(0) = \frac{d^2\rho}{dt^2}(0), \\ |\frac{d^3y}{dt^3}| &\le |\frac{\partial^2 f}{\partial t^2}| + 2|\frac{\partial^2 f}{\partial y\partial t}||f| + |\frac{\partial f}{\partial t}||\frac{\partial f}{\partial y}| + |f||\frac{\partial f}{\partial y}|^2 + |f|^2|\frac{\partial^2 f}{\partial y^2}|, \\ &\le g''(0)(1+|f|)^2 + (g'(0))^2(1+|f|), \\ &\le g^2(0)g''(0) + (g'(0))^2 g(0) = \frac{d^3\rho}{dt^3}(0). \end{aligned} \tag{35}$$

Generally

$$y^{(k+1)}(0, y_0) = P_k(f)_{t=0, y=y_0}, \tag{36}$$

where $P_k(f)$ is a polynomial of f and its partial derivatives up to k^{th} order evaluated at $t = 0$ and $y = y_0$. Since $P_k(f)$ has positive coefficients and by Eq. (29) we obtain

$$\begin{aligned} |y^{(k+1)}(0, y_0)| &= |P_k(f)|_{t=0,y=y_0} \le P_k(|f|)|_{t=0,y=y_0} \\ &\le P_k(1+|f|)|_{t=0,y=y_0} \le P_k(g)|_{\rho=0} = \rho^{(k+1)}(0), \ k \ge 0, \end{aligned} \tag{37}$$

where the last identity follows from $\frac{d\rho}{dt} = g(\rho)$. Thus, the statement of the theorem is proved. □

We can now state the main result of this subsection.

Theorem 2. *Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a real analytic function in the domain $[-t_0, t_0] \times B_{\delta_0}(y_0)$ for some $t_0 > 0$ and $\delta_0 > 0$. Let $y_n(t)$ be defined by the recurrence Eq.* (21)*. There exists a $\tau \in [0, t_0]$ such that the n^{th} partial sum of the Adomian series* (13) *converges to the solution $y(t, y_0)$ of the Volterra Eq.* (12) *in $C([-\tau, \tau], \mathbb{R}^d)$.*

Proof. Similar to Theorem 1 we prove the theorem $d = 1$. From the iteration of the Adomian method, we set

$$y_{k+1} = \int_0^t A_k(s, y_0(s), \ldots, y_n(s)),\ k \geq 0, \tag{38}$$

where

$$A_k = \frac{1}{k!}\frac{d^k}{d\varepsilon^k} f\Big(t, y_0 + \sum_{m=1}^{\infty} \varepsilon^m y_m\Big)\Big|_{\varepsilon=0}. \tag{39}$$

For $k = 0$ we have

$$|y_1(t)| \leq \int_0^t |f(s, y_0)| ds \leq g(0)t \equiv \rho'(0)t, \tag{40}$$

and for $k = 1$,

$$|y_2(t)| \leq \int_0^t |f'(s, y_0)||y_1(s)| ds \leq \frac{t^2}{2} g'(0)g(0) \equiv \frac{t^2}{2}\rho''(0). \tag{41}$$

By induction assume that

$$|y_n(t)| \leq \frac{1}{n!} t^n \rho^{(n)}(0). \tag{42}$$

We shall prove

$$|y_{n+1}(t)| \leq \frac{1}{(n+1)!} t^{n+1} \rho^{(n+1)}(0). \tag{43}$$

Let

$$Y_n(t) = \sum_{m=0}^{n} \varepsilon^m y_m(t), \tag{44}$$

where $\varepsilon > 0$ is a formal parameter. Then,

$$\begin{aligned} |Y_n(t)| &\leq \sum_{m=0}^{n} \varepsilon^m |y_m(t)| \leq \sum_{m=0}^{n} \frac{\varepsilon^m t^m \rho^{(m)}(0)}{m!} \\ &= \sum_{m=0}^{\infty} \frac{\varepsilon^m t^m \rho^{(m)}(0)}{m!} - \sum_{m=n+1}^{\infty} \frac{\varepsilon^m t^m \rho^{(m)}(0)}{m!}. \end{aligned} \tag{45}$$

Let $m - (n+1) = l$ then

$$Y_n(t) = \rho(\varepsilon t) - \varepsilon^{n+1} t^{n+1} \sum_{l=0}^{\infty} \frac{\varepsilon^l t^l}{(1+l+n)!} \rho^{1+l+n}(0). \tag{46}$$

Therefore, there exists a C^∞ function $\tilde{Y}_n(t)$ on $[-\tau,\tau]$ such that

$$Y_n(t) = \rho(\varepsilon t) - \varepsilon^{n+1} t^{n+1} \tilde{Y}_n(t), \quad \forall t \in [-\tau,\tau], \tag{47}$$

where τ is defined by Theorem 1. The first few estimates of Adomian polynomials are given by

$$\begin{aligned} |A_0| &\le C = g(0) = \rho'(0), \\ |A_1| &\le |f'||y_1| \le \frac{C}{a} g(0)t = tg(0)g'(0) = t\rho''(0), \\ |A_2| &\le |f'||y_2| + \frac{1}{2}|f''||y_1^2| \le \frac{C}{a}|y_2| + \frac{C}{a^2}|y_1|^2, \\ &\le \frac{t^2}{2}\big(g(0)(g'(0))^2 + g''(0)(g(0))^2\big) = \frac{t^2}{2}\rho'''(0). \end{aligned} \tag{48}$$

To estimate $A_n(t)$ in general case, we use the formula

$$A_n(t) = \frac{1}{n!}\frac{d^n}{d\varepsilon^n} f(t, Y_n(t))|_{\varepsilon=0},$$

and compute

$$|A_n(t)| \le \frac{1}{n!}\Big|\frac{d^n}{d\varepsilon^n} f(t, Y_n)\Big|_{\varepsilon=0} \le \frac{t^n}{n!}\Big|\frac{d^n}{d\mu^n} f(t,\rho(\mu))\Big|_{\mu=\varepsilon t=0} \tag{50}$$

$$\le \frac{t^n}{n!}|P_n(\rho(0))| \le \frac{t^n}{n!}\rho^{(n+1)}(0), \tag{51}$$

where the last inequality is obtained in Eq. (37). Using the iterative formula (38), we finally obtain

$$|y_{n+1}(t)| \le \frac{1}{(n+1)!} t^{n+1} \rho^{(n+1)}(0). \tag{52}$$

Therefore the Adomian series is majorant by the same power series as the analytic solution in Theorem 1. By the Weierstrass M-Test, the Adomian series converges. Moreover, it follows from Eq. (38) that the series (14) for Aomian polynomials converges as well, such that the Adomian series solves the same Volterra integral Eq. (12) in $C([-\tau,\tau],\mathbb{R})$. By uniqueness of solutions, the Adomian series is equivalent to the solution $y(t, y_0)$ of the Volterra equation (12). □

3.3 Rate of the convergence

In this subsection, a simple method is introduced to determine the rate of convergence of the ADM. Using this method, we give a bound for the error of the Adomian decomposition series.

Theorem 3. *Under the same condition as in Theorem 2, the rate of convergence is exponential in the sense that there exists $C_0 > 0$ such that*

$$E_n \leq C_0\Big(\frac{2C\tau}{a}\Big)^{n+1}, \quad n \geq 1 \tag{53}$$

for all $\tau < \frac{a}{2C}$, where

$$E_n = \left\| y - \sum_{m=0}^{n} y_m \right\|, \tag{54}$$

and the parameters a and C are defined in Cauchy estimation (26) and (27).

Proof. By Theorem 2, we have

$$|y_{n+1}(t)| \leq \frac{t^{n+1}\rho^{(n+1)}(0)}{(n+1)!}, \quad \forall t \in [0,T], \tag{55}$$

such that

$$||y_{n+1}|| \leq \frac{\tau^{n+1}\rho^{(n+1)}(0)}{(n+1)!}, \tag{56}$$

where the norm $||.||$ in $C([-\tau,\tau],\mathbb{R}^d)$ is defined by

$$||y|| = \sup_{|t|<t_0} |y(t)|. \tag{57}$$

Since $\rho(t)$ is explicitly given by

$$\rho(t) = a - \sqrt{a^2 - 2Cat}, \tag{58}$$

we have

$$\rho^{(n)}(0) = \frac{(2n-3)!C^n}{a^{n-1}}. \tag{59}$$

By Theorem 2, the Adomian series $y(t) = \sum_{m=0}^{\infty} y_m(t)$ converges and the error is estimated by

$$E_n = \Big|\Big| \sum_{j=n+1}^{\infty} y_i \Big|\Big| \leq \sum_{j=n+1}^{\infty} ||y_i|| \leq \sum_{j=n+1}^{\infty} \frac{\tau^j \rho^{(j)}(0)}{j!} \leq \sum_{j=n+1}^{\infty} \frac{a}{j!}\Big(\frac{C\tau}{a}\Big)^j (2j-3)!. \tag{60}$$

Let $k = j-(n+1)$, then

$$E_n \leq a\Big(\frac{2C\tau}{a}\Big)^{n+1} \sum_{k=0}^{\infty} \frac{(2k+2n-1)!}{2^{k+n+1}(k+n+1)!}\Big(\frac{2C\tau}{a}\Big)^k. \tag{61}$$

Since

$$\frac{(2k+2n-1)!}{2^{k+n+1}(k+n+1)!} \leq \frac{1}{2k+2n} \leq 1, \quad \forall n \geq 1,\ k \geq 1, \tag{62}$$

we obtain

$$E_n \leq a\Big(\frac{2C\tau}{a}\Big)^{n+1} \sum_{k=0}^{\infty} \Big(\frac{2C\tau}{a}\Big)^k = \frac{a\Big(\frac{2C\tau}{a}\Big)^{n+1}}{1-\frac{2C\tau}{a}}, \tag{63}$$

for all $\tau < \frac{a}{2C}$. The theorem is proved with $C_0 = \frac{a}{1-\frac{2C\tau}{a}}$. □

4 Convergence of the Adomian decomposition method for partial differential equations

In this section, we analyze the convergence of the ADM for nonlinear partial differential equations in the form

$$u_t = L(u) + N(u), \tag{64}$$

where L is an unbounded differential operator from a Banach space X to a Banach space Y, $(X \subseteq Y)$, and $N(u)$ is a nonlinear function that maps an element of X to an element of X.

Let $E(t)$ be a fundamental solution operator associated with the linear Cauchy problem

$$\begin{cases} v_t = Lv, \\ v(0) = f \in X, \end{cases}$$

such that $v(t) = E(t)f$. For symbolic notations, we write $E(t) = e^{tL}$. In what follows, we shall assume that

$$||E(t)f||_X \leq C||f||_X. \tag{65}$$

For instance, if $L \equiv i\partial_x^2$, then the above linear initial-value Cauchy problem defines the Schrödinger equation which is solved in the Fourier transform form as

$$v(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi^2 t + i\xi x} \hat{f}(\xi) d\xi, \quad \forall (x,t) \in \mathbb{R}^2, \tag{66}$$

where

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x) dx, \quad \forall \xi \in \mathbb{R}. \tag{67}$$

Therefore, $E(t)$ is defined in the Fourier transform form by $\hat{E}(t) = e^{-i\xi^2 t}$. By the Parseval's identity, $E(t)$ preserves the H^s-norm in the sense that

$$\begin{aligned} ||E(t)f||^2_{H^s} &= \frac{1}{2\pi}\int_{\mathbb{R}}(1+\xi^2)^s\left|\widehat{E(t)f}\right|^2 d\xi, \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}(1+\xi^2)^s\left|\widehat{E(t)}\right|^2\left|\hat{f}\right|^2 d\xi, \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}(1+\xi^2)^s\left|\hat{f}\right|^2 d\xi, \\ &= ||f||^2_{H^s}, \end{aligned} \tag{68}$$

such that the assumption (65) holds with $C = 1$.

By Duhamel's principle, the initial-value problem (64) can be reformulated as an integral equation

$$u(t) = E(t)f + \int_0^t E(t-s)N(u(s))ds. \tag{69}$$

Remark 3. If $L : X \to Y$, $N : X \to X$, and $||E(t)f||_X \leq C||f||_X$ for some $C > 0$, then there exists a unique fixed-point of the integral Eq. (69) in space $C([0,T],X)$ for a sufficiently small $T > 0$, which corresponds to a unique solution of the PDE problem (64) in space $u(t) \in C([0,T],X) \cap C^1([0,T],Y)$.

To set up the Adomian method, define

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{70}$$

where $u_0(t) = E(t)f$ and

$$u_{n+1}(t) = \int_0^t E(t-s)A_n(u_0(s),\ldots,u_n(s))ds, \quad n \geq 0, \tag{71}$$

in which A_n is the same Adomian polynomials as before introduced and generated from an analytic function $N(u)$.

We would like to prove convergence of the Adomian series (69) in space X.

Theorem 4. *Let $N : X \to X$ be a real analytic function in the ball $B_a(f) \subset X$ for some radius $a > 0$. Assume that $L : X \to Y$ satisfies $||E(t)f||_x \leq C||f||_X$ for some $C > 0$. Let $u_0(t) = E(t)f$ and $u_n(t)$ for*

$n \geq 1$ be defined by the recurrence Eq. (70). *There exists a $T > 0$ such that the n^{th} partial sum of the Adomian series* (70) *converges to the solution u of Eq.* (69) *in $C([0,T], X)$.*

Proof. Assume that $N(u)$ is analytic in $u \in X$. Then, by Cauchy estimation, there exist $a > 0$, and $b > 0$ such that

$$\left|\left|\partial_u^k N(f)\right|\right|_X \leq \frac{bk!}{a^k}, \quad k \geq 0. \tag{72}$$

The Taylor series for $N(u)$ at $u = f$

$$N(u) = \sum_{k=0}^{\infty} \frac{1}{k!}[\partial_u^k N(f)](u-f)^k, \tag{73}$$

converges for any $||u - f||_X < a$, and moreover, we obtain that

$$\begin{aligned} ||N(u)||_X &\leq \sum_{k=0}^{\infty} \frac{1}{k!}\frac{bk!}{a^k}||u-f||_X^k, \\ &\leq \frac{b}{1 - \frac{||u-f||_X}{a}}, \\ &= \frac{ba}{a-\rho} \equiv g(\rho), \end{aligned} \tag{74}$$

where $\rho = ||u - f||_X < a$. It is now clear that $\left|\left|\partial_u^k N(f)\right|\right| \leq g^{(k)}(0)$ for any $k \geq 0$.

From Eq. (71), we find that $||u_0 - f||_X \leq (C+1)||f||_X \equiv a$, and

$$\begin{aligned} ||u_1||_X &\leq \int_0^t ||E(t-s)A_0||_X ds \leq C\int_0^t ||A_0||_X ds, \\ &\leq Cg(0)t = Ct\rho'(0), \\ ||u_2||_X &\leq \int_0^t ||E(t-s)A_1||_X ds \leq C\int_0^t ||A_1||_X ds, \\ &\leq C^2 g'(0)g(0)t = C^2\frac{t^2}{2}\rho''(0). \end{aligned} \tag{75}$$

By induction one can see that

$$||u_{n+1}(t)||_X \leq \frac{C^{n+1}}{(n+1)!t^{n+1}\rho^{(n+1)}(0)}. \tag{76}$$

Therefore, the Adomian series in X is majorant by the convergent power series for $\rho(t) = a - \sqrt{a^2 - 2abCt}$ for any $t \in [0,T]$ for $T < \frac{a}{2bC}$, in full correspondence with the proof Theorem 2. □

5 Numerical experiments

In this section, we give two examples to illustrate the LDM method.

Example 1. ([3]) Consider the generalized Burgers-Fisher equation

$$u_t + \alpha u^\sigma u_x - u_{xx} = \beta u(1-u^\sigma), \qquad \forall \quad 0 \leqslant x \leqslant 1,\ t \geqslant 0,$$

where the initial condition is given by

$$u(x,0) = (\frac{1}{2} - \frac{1}{2}\tanh[\frac{\alpha\sigma x}{2(\sigma+1)}])^{\frac{1}{\sigma}}.$$

The exact solution of Eq. (11) is given by

$$u(x,t) = (\frac{1}{2} - \frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}(x - (\frac{\alpha}{\sigma+1} + \frac{\beta(\sigma+1)}{\alpha})t)])^{\frac{1}{\sigma}},$$

where α, $\beta \geq 0$ and $\sigma > 0$ are given parameters. Taking the Laplace transform on both sides of Eq. (11), then, by using the differentiation property of Laplace transform and initial condition (12) gives

$$\begin{aligned}\mathcal{L}[u] &= \frac{1}{s}(\frac{1}{2} - \frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x])^{\frac{1}{\sigma}} - \alpha\frac{1}{s}\mathcal{L}[u^\sigma u_x] + \frac{1}{s}\mathcal{L}[u_{xx}] \\ &\quad + \beta\frac{1}{s}\mathcal{L}[u] - \beta\frac{1}{s}\mathcal{L}[uu^\sigma].\end{aligned}$$

The LDM defines the solutions $u(x,t)$ by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Inserting this series into both sides of Eq. (13) yields

$$\begin{aligned}\mathcal{L}[\sum_{n=0}^{\infty} u_n] &= \frac{1}{s}(\frac{1}{2} - \frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x])^{\frac{1}{\sigma}} - \alpha\frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} A_n], \\ &\quad + \frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} u_{nxx}] + \beta\frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} u_n] - \beta\frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} B_n],\end{aligned}$$

where A_n and B_n are the so-called Adomian polynomials defined by Eq. (7) that represent the nonlinear terms $u^\sigma u_x$ and uu^σ, respectively (see [10]). Now we define the following recursive formula

$$\begin{aligned}\mathcal{L}[u_0] &= \frac{1}{s}(\frac{1}{2} - \frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x])^{\frac{1}{\sigma}}, \\ \mathcal{L}[u_{k+1}] &= -\alpha\frac{1}{s}\mathcal{L}[A_k] + \frac{1}{s}\mathcal{L}[u_{kxx}] + \beta\frac{1}{s}\mathcal{L}[u_k] - \beta\frac{1}{s}\mathcal{L}[B_k], \qquad k \geqslant 0.\end{aligned}$$

Taking the inverse Laplace transform of both sides of Eq. (15), we get

$$\begin{aligned} u_0(x,t) &= (\frac{1}{2}-\frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x])^{\frac{1}{\sigma}}, \\ u_1(x,t) &= \frac{1}{4}(\frac{1}{2}-\frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x])^{-2+\frac{1}{\sigma}}t[4\beta\frac{1}{2}-\frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x]^2 \\ &\quad +(1-\frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x])^2 \\ &\quad \times(\frac{(\frac{(-1-\frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x])\alpha^2\sigma^2(-1+\sigma-\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x](\sigma+1))}{2(\sigma+1)^2})+\cdots}{\sigma^2})], \end{aligned}$$

and so on. Using Eq. (14), the series solution is therefore given by

$$\begin{aligned} u(x,t) &= (\frac{1}{2}-\frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x])^{\frac{1}{\sigma}} \\ &\quad +\frac{1}{4}(\frac{1}{2}-\frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x])^{-2+\frac{1}{\sigma}}t[4\beta\frac{1}{2}-\frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x]^2 \\ &\quad +(1-\frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x])^2 \\ &\quad \times(\frac{(\frac{(-1-\frac{1}{2}\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x])\alpha^2\sigma^2(-1+\sigma-\tanh[\frac{\alpha\sigma}{2(\sigma+1)}x](\sigma+1))}{2(\sigma+1)^2})+\cdots}{\sigma^2})]. \end{aligned}$$

The comparison between the absolute errors of the solution for Eq. (11), by LDM, Adomian Decomposition method (ADM) [3], Differential Transform method (DTM) and Variational Iteration Method (VIM) are shown in Tables 1-3 for different values of α and β for $\sigma = 1, 2$. Also we plot the evolution results for the approximate solutions derived by LDM and the exact solution in Fig. 1, respectively. Also, Fig. 2 shows the absolute errors of the computed solutions for different values of α and β for $\sigma = 1$ and Fig. 3 shows numerical results for $\sigma = 2$.

Example 2. ([3]) Consider the generalized Burgers-Huxley equation

$$u_t + \alpha u^\sigma u_x - u_{xx} = \beta u(1-u^\sigma)(u^\sigma-\gamma), \quad 0 \leqslant x \leqslant 1, \quad t \geqslant 0,$$

subject to the initial condition

$$u(x,0) = (\frac{\gamma}{2}+\frac{\gamma}{2}\tanh[A_1 x])^{\frac{1}{\sigma}}.$$

Note that the exact solution of Eq. (19) is given by

$$u(x,t) = (\frac{\gamma}{2}+\frac{\gamma}{2}\tanh[A_1(x-A_2 t)])^{\frac{1}{\sigma}},$$

Table 1: Absolute errors for $\alpha = \beta = 0.001$, $\sigma = 1$, by using 5-terms ADM, one iteration of VIM, 5-terms of DTM and 2-terms of LDM for Example 1.

x	t	ADM	VIM	DTM	LDM
0.1	0.005	9.68619E-6	7.55579E-17	2.49989E-6	3.19327E-28
	0.001	1.93724E-6	3.10571E-18	4.99975E-7	5.19268E-31
	0.010	1.93724E-5	2.91807E-16	4.99982E-6	5.00494E-27
0.5	0.005	9.68691E-6	3.88214E-16	2.49939E-6	1.62271E-27
	0.001	1.93738E-6	1.56119E-17	4.99875E-7	2.60468E-30
	0.010	1.93738E-5	1.54243E-15	4.99875E-6	2.58591E-27
0.9	0.005	9.68619E-6	7.00870E-16	2.49989E-6	2.92609E-27
	0.001	1.93738E-6	2.81182E-17	4.99775E-7	4.69010E-30
	0.010	1.93738E-5	2.79305E-15	4.99782E-6	4.67132E-26

Table 2: Absolute errors for $\alpha = \beta = 1$, $\sigma = 2$, by using 5-terms ADM, one iteration of VIM, 5-terms of DTM and 2-terms of LDM for Example 1.

x	t	ADM	VIM	DTM	LDM
0.1	0.0005	1.40177E-3	2.49584E-8	4.31483E-4	2.49584E-8
	0.0001	2.80396E-4	9.97913E-10	1.62647E-4	9.97913E-10
	0.0010	2.80301E-3	9.98871E-8	7.67572E-4	9.98871E-8
0.5	0.0005	1.34526E-3	1.46692E-8	2.36200E-3	1.46692E-8
	0.0001	2.69094E-4	5.86290E-10	2.14691E-3	5.86290E-10
	0.0010	2.69000E-3	5.87371E-8	2.63089E-3	5.87371E-8
0.9	0.0005	1.27699E-3	3.75350E-9	5.97983E-3	2.92609E-27
	0.0001	2.55438E-4	1.49658E-10	5.81437E-3	1.49658E-10
	0.0010	2.55346E-3	1.50741E-8	6.18665E-3	1.50741E-8

Table 3: Absolute errors for $\alpha = 1$, $\beta = 0$, $\sigma = 1$, by using 5-terms ADM, one iteration of VIM, 5-terms of DTM and 2-terms of LDM for Example 1.

x	t	ADM	VIM	DTM	LDM
0.1	0.005	6.34216E-8	4.83815E-9	1.13584E-13	1.26299E-15
	0.001	2.02886E-6	1.19482E-10	1.58893E-13	2.02888E-18
	0.010	6.42801E-5	1.19190E-8	1.07702E-13	2.01066E-14
0.5	0.005	5.66705E-8	2.38739E-8	1.23392E-8	6.07712E-15
	0.001	1.8471E-6	9.56179E-10	1.26967E-8	9.73050E-18
	0.010	1.8471E-6	9.53427E-8	1.18923E-8	9.71452E-14
0.9	0.005	4.12803E-8	4.10692E-8	7.56982E-7	9.91363E-15
	0.001	1.37967E-6	1.64382E-9	7.68925E-7	1.58668E-17
	0.010	4.75268E-5	9.53427E-8	7.42046E-7	1.58555E-13

where

$$\begin{aligned} A_1 &= \frac{-\alpha\sigma + \sigma\sqrt{\alpha^2 - 4\beta(1+\sigma)}}{4(1+\sigma)}\gamma, \\ A_2 &= \frac{\gamma\alpha}{(1+\sigma)} - \frac{(1+\sigma-\gamma)(-\alpha + \sqrt{\alpha^2 - 4\beta(1+\sigma)})}{2(1+\sigma)}, \end{aligned}$$

where α, β, γ and σ are parameters, $\beta \geqslant 0$, $\sigma > 0$, $\gamma \in (0, 1)$.

Using the differentiation property of Laplace transform we get

$$\begin{aligned} \mathcal{L}[u] &= \frac{1}{s}(\frac{\gamma}{2} + \frac{\gamma}{2}\tanh[A_1 x])^{\frac{1}{\sigma}} - \alpha\frac{1}{s}\mathcal{L}[u^\sigma u_x] + \frac{1}{s}\mathcal{L}[u_{xx}] - \beta\gamma\frac{1}{s}\mathcal{L}[u] \\ &\quad + \beta(1+\gamma)\frac{1}{s}\mathcal{L}[u^{1+\sigma}] - \beta\frac{1}{s}\mathcal{L}[u^{2\sigma}]. \end{aligned}$$

The second step in LDM is that we represent the solution as an infinite series given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n.$$

By substituting Eq. (22) in Eq. (21), we get

$$\begin{aligned} \mathcal{L}[\sum_{n=0}^{\infty} u_n] &= \frac{1}{s}(\frac{\gamma}{2} + \frac{\gamma}{2}\tanh[A_1 x])^{\frac{1}{\sigma}} - \alpha\frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} A_n] + \frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} u_{nxx}] \\ &\quad - \beta\gamma\frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} u_n] + \beta(1+\gamma)\frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} B_n] - \beta\frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} C_n], \end{aligned}$$

where A_n, B_n and C_n are so-called Adomian polynomials by Eq. (7) that represent the nonlinear terms $u^\sigma u_x$, $u^{1+\sigma}$ and $u^{2\sigma}$, respectively. Matching both sides of Eq. (25), we have the following relation

$$\begin{aligned}\mathcal{L}[u_0] &= \frac{1}{s}(\frac{\gamma}{2}+\frac{\gamma}{2}\tanh[A_1x])^{\frac{1}{\sigma}},\\ \mathcal{L}[u_1] &= \frac{1}{s}\mathcal{L}[-\alpha A_0 + u_{0xx} - \beta\gamma u_0 + \beta(1+\gamma)B_0 - \beta C_0].\end{aligned}$$

In general the recursive relation is given by

$$\begin{aligned}\mathcal{L}[u_{k+1}] &= -\alpha\frac{1}{s}\mathcal{L}[A_k] + \frac{1}{s}\mathcal{L}[u_{kxx}] - \beta\gamma\frac{1}{s}\mathcal{L}[u_k]\\ &\quad +\beta(1+\gamma)\frac{1}{s}\mathcal{L}[B_k] - \beta\frac{1}{s}\mathcal{L}[C_k].\end{aligned} \tag{78}$$

Using the above recurrence relation and by using the inverse Laplace transform of both sides of it, we have

$$\begin{aligned}u_0(x,t) &= (\frac{\gamma}{2}+\frac{\gamma}{2}\tanh[A_1x])^{\frac{1}{\sigma}},\\ u_1(x,t) &= \frac{1}{2}t[-2\beta(\frac{\gamma}{2}(1+\tanh[A_1x])^{\frac{1}{\sigma}})^{2\sigma+1}\\ &\quad+\frac{2(\frac{\gamma}{2}(1+\tanh[A_1x])^{\frac{1}{\sigma}})^{\sigma+1}(A_1\alpha(\tanh[A_1x]-1)+\beta\sigma(1+\gamma))}{\sigma}\\ &\quad+\frac{\gamma(1+\tanh[A_1x])^{\frac{1}{\sigma}}(A_1^2(\tanh[A_1x]-1)(-1+\sigma+\tanh[A_1x](\sigma+1))-\beta\gamma\sigma^2)}{\sigma^2}],\end{aligned}$$

and so on. Using Eq. (14), the series solution are therefore given by

$$\begin{aligned}u(x,t) &= (\frac{\gamma}{2}+\frac{\gamma}{2}\tanh[A_1x])^{\frac{1}{\sigma}}\\ &\quad+\frac{1}{2}t[-2\beta(\frac{\gamma}{2}(1+\tanh[A_1x])^{\frac{1}{\sigma}})^{2\sigma+1}\\ &\quad+\frac{2(\frac{\gamma}{2}(1+\tanh[A_1x])^{\frac{1}{\sigma}})^{\sigma+1}(A_1\alpha(\tanh[A_1x]-1)+\beta\sigma(1+\gamma))}{\sigma}\\ &\quad+\frac{\gamma(1+\tanh[A_1x])^{\frac{1}{\sigma}}(A_1^2(\tanh[A_1x]-1)(-1+\sigma+\tanh[A_1x](\sigma+1))-\beta\gamma\sigma^2)}{\sigma^2}].\end{aligned}$$

In Tables 4, 5 and 6, we present the absolute errors of the computed solution for Eq. (18), by DTM, ADM, VIM (given in [5]), together with those of LDM for $\sigma = 1$. As the numerical results show, we can conclude that LDM present remarkable accuracy and better in comparison with the other three methods. Also the behavior of the exact solution and the computed solution obtained by LDM is shown in Fig. 4 and Fig. 5. In this figures the absolute

Table 4: Absolute errors for $\alpha = 1$, $\beta = 1$, $\gamma = 0.001$, $\sigma = 1$, by using 5-terms ADM, one iteration of VIM, 5-terms of DTM and 2-terms of LDM for Example 2.

x	t	ADM	VIM	DTM	LDM
0.1	0.05	1.87406E-8	1.87405E-8	1.87406E-8	1.87406E-8
	0.1	3.74812E-8	3.74813E-8	3.74813E-8	3.74812E-8
	1.0	3.74812E-7	3.74812E-7	3.748125E-7	3.74812E-7
0.5	0.05	1.87406E-8	1.87405E-8	1.87406E-8	1.87406E-8
	0.1	3.74812E-8	3.74813E-8	3.74813E-8	3.74812E-8
	1.0	3.74812E-7	3.74813E-7	3.74813E-7	3.74812E-7
0.9	0.05	1.87406E-8	1.87405E-8	1.87406E-8	1.87406E-8
	0.1	3.74812E-8	3.74813E-8	3.74813E-8	3.74812E-8
	1.0	3.74812E-7	3.74813E-7	3.748125E-7	3.74812E-7

Table 5: Absolute errors for $\alpha = 0.1$, $\beta = 0.001$, $\gamma = 0.1$, $\sigma = 1$, by using 5-terms ADM, one iteration of VIM, 5-terms of DTM and 2-terms of LDM for Example 2.

x	t	ADM	VIM	DTM	LDM
0.1	0.05	1.3634E-7	11.3608E-7	11.3608E-7	11.3607E-7
	0.10	2.7243E-7	12.7216E-7	12.7216E-7	12.7215E-7
	1.00	2.72200E-6	12.72151E-6	12.72151E-6	12.72150E-7
0.5	0.05	1.3736E-7	1.3608E-7	1.3608E-7	1.3607E-7
	0.10	2.7345E-7	2.7216E-7	2.7216E-7	2.7216E-7
	1.00	2.72302E-6	2.72151E-6	2.72151E-6	2.72150E-6
0.9	0.05	1.3838E-7	1.3608E-7	1.3608E-7	1.3607E-7
	0.10	2.7447E-7	2.7216E-7	2.7216E-7	2.7215E-7
	1.00	2.72404E-6	2.72151E-6	2.72151E-7	2.72150E-6

Table 6: Absolute errors for $\alpha = 0.01$, $\beta = 0$,$\gamma = 0.0001$, $\sigma = 1$, by using 5-terms ADM, one iteration of VIM, 5-terms of DTM and 2-terms of LDM for Example 2.

x	t	ADM	VIM	DTM	LDM
0.1	0.05	2E-14	2E-14	2E-14	1.87E-14
	0.10	4E-14	3E-14	3E-14	3.74E-14
	1.00	3.7E-13	3.7E-13	3.7E-13	3.74E-14
0.5	0.05	2E-14	2E-14	2E-14	1.87E-14
	0.10	4E-14	3E-14	3E-14	3.74E-14
	1.00	3.7E-13	3.7E-13	3.7E-13	3.74E-14
0.9	0.05	2E-14	2E-14	2E-14	1.87E-14
	0.10	4E-14	3E-14	3E-14	3.74E-14
	1.00	3.7E-13	3.7E-13	3.7E-13	3.74E-14

error of the computed solution for different values of α, β and γ for $\sigma = 1$ has been displayed.

In addition, the evolution results for the approximate solutions obtained by LDM, and the exact solutions of Eq. (18) are given in Fig. 6 for $\sigma = 2$, respectively and Fig. 5 shows absolute errors for different values of α, β and γ when $\sigma = 2$. Also, Tables 7, 8 and 9, show the absolute errors for Eq. (18), by DTM, ADM, and VIM (reported in [4]), are compared with those of the LDM for $\sigma = 2$.

6 Conclusion

The LDM has been successfully applied to find an approximate solution of the generalized Burgers-Huxley equation and generalized Burgers-Fisher equation. The results reveal that the LDM is more effective and accuracy compared to ADM , VIM, DTM. Also illustrate that LDM is a powerful tool to search for solutions of various nonlinear problems. An excellent agreement between the present and existing solutions is achieved. The proposed scheme can be applied for other nonlinear equations of physics applications.

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Table 7: Absolute errors for $\alpha = \beta = 1, \gamma = 0.001, \sigma = 2$, by using 5-terms ADM, one iteration of VIM, 5-terms of DTM and 2-terms of LDM for Example 2.

x	t	ADM	VIM	DTM	LDM
0.1	0.1	1.74857E-6	1.74992E-6	1.74982E-6	4.85677E-7
	0.2	3.49911E-6	3.49983E-6	3.49962E-6	9.71369E-7
	0.3	5.24970E-6	5.24971E-6	5.24940E-6	1.45708E-6
	0.4	7.00033E-6	6.99958E-6	6.99916E-6	1.94281E-6
	0.5	8.75099E-6	8.74942E-6	8.74890E-6	2.42855E-6
0.3	0.1	1.74459E-6	1.74983E-6	1.74954E-8	4.85645E-7
	0.2	3.49506E-6	3.49967E-6	3.49962E-6	9.71327E-7
	0.3	5.24557E-6	5.24948E-6	5.24915E-6	1.45701E-6
	0.4	6.99612E-6	6.99927E-6	6.99881E-6	1.94272E-6
	0.5	8.74670E-6	8.74903E-6	8.74848E-6	2.42845E-6
0.5	0.1	1.74459E-6	1.74976E-6	1.74954E-8	4.85633E-7
	0.2	3.49101E-6	3.49953E-6	3.49918E-6	9.71285E-7
	0.3	5.24145E-6	5.24925E-6	5.24881E-6	1.45695E-6
	0.4	6.99193E-6	6.99897E-6	6.99843E-6	1.94264E-6
	0.5	8.74243E-6	8.74866E-6	8.74801E-6	2.42642E-6

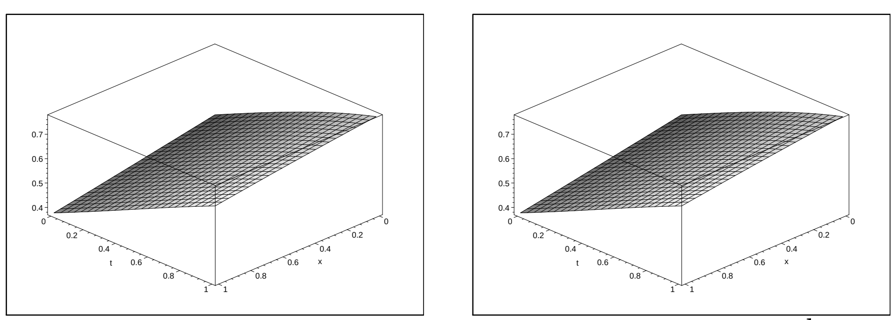

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Figure 1: The behavior of $u(x, t)$: (a) exact solution, (b) the approximate solution obtained by LD method when $\alpha = \beta = 1$ and $\sigma = 1$ for Example 1.

Table 8: Absolute errors for $\alpha = \gamma = 0.01$, $\beta = 0.001$, $\sigma = 2$, by using 5-terms ADM, one iteration of VIM, 5-terms of DTM and 2-terms of LDM for Example 2.

x	t	ADM	VIM	DTM	LDM
0.1	0.1	1.2954E-6	1.2911E-6	1.2874E-6	9.3044E-7
	0.2	2.5864E-6	2.5821E-6	2.5748E-6	1.8608E-6
	0.3	3.8773E-6	3.8732E-6	3.8621E-6	2.7913E-6
	0.4	5.1683E-6	5.1641E-6	5.1496E-6	3.7217E-6
	0.5	6.4593E-6	6.4552E-6	6.4370E-6	4.6522E-6
0.3	0.1	1.3043E-6	1.2909E-6	1.2858E-6	9.3036E-7
	0.2	2.5951E-6	2.5820E-6	2.5731E-6	1.8607E-6
	0.3	3.8860E-6	3.8728E-6	3.8603E-6	2.7911E-6
	0.4	5.1769E-6	5.1638E-6	5.1477E-6	3.7214E-6
	0.5	6.4677E-6	6.4546E-6	1.2827E-6	4.6518E-6
0.5	0.1	1.3131E-6	1.2908E-6	1.74954E-8	9.3029E-7
	0.2	2.6037E-6	2.5817E-6	2.5700E-6	1.8605E-6
	0.3	3.8945E-6	3.8724E-6	3.8570E-6	2.7908E-6
	0.4	5.1853E-6	5.1633E-6	5.1444E-6	3.7211E-6
	0.5	6.4760E-6	6.4541E-6	6.4315E-6	4.6514E-6

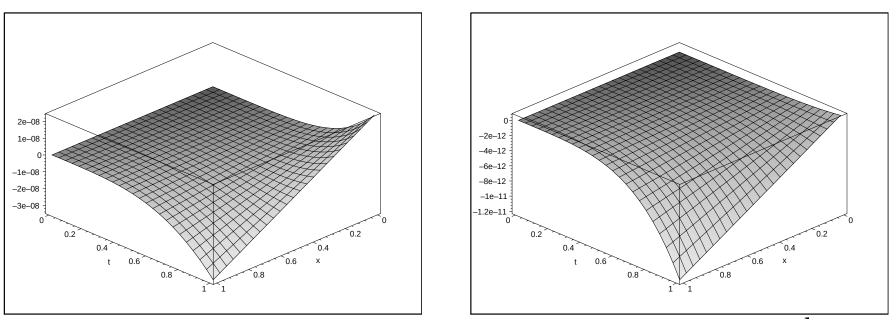

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Figure 2: The absolute errors of $u(x,t)$ with exact solution when $\alpha = \beta = 0.1$ for $\sigma = 1$ and when $\alpha = 0.1$ and $\beta = 0.01$ for $\sigma = 1$, respectively, (a) and (b) for Example 1.

Table 9: Absolute errors for $\alpha = 0.01$, $\beta = \gamma = 0.0001$, $\sigma = 2$, by using 5-terms ADM, one iteration of VIM, 5-terms of DTM and 2-terms of LDM for Example 2.

x	t	ADM	VIM	DTM	LDM
0.1	0.1	6E-12	6E-12	6E-12	1.5353E-12
	0.2	1.2E-11	1.2E-11	1.3E-11	3.0707E-12
	0.3	1.8E-11	1.8E-11	1.8E-11	4.6061E-12
	0.4	2.2E-11	2.2E-11	2.3E-11	6.1415E-12
	0.5	2.8E-11	2.8E-11	2.8E-11	7.6769E-12
0.3	0.1	6E-12	6E-12	6E-12	9.3036E-7
	0.2	1.1E-11	1.1E-11	1.2E-11	3.0707E-12
	0.3	1.8E-11	1.8E-11	1.8E-11	4.6061E-12
	0.4	2.2E-11	2.3E-11	2.4E-11	6.1415E-12
	0.5	2.9E-11	2.9E-11	2.9E-11	7.6769E-12
0.5	0.1	6E-12	6E-12	6E-12	1.5353E-12
	0.2	1.1E-11	1E-11	1.2E-11	3.0707E-12
	0.3	1.7E-11	1.6E-11	1.7E-11	4.6061E-12
	0.4	2.3E-11	2.2E-11	2.4E-11	6.1415E-12
	0.5	2.9E-11	2.8E-11	2.9E-11	7.6769E-12

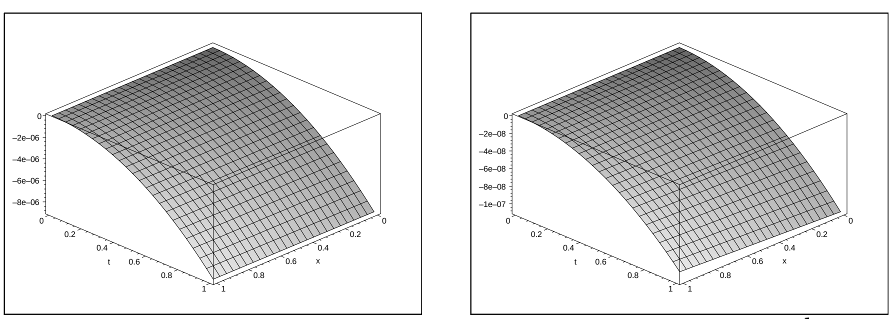

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Figure 3: The absolute errors of $u(x,t)$ with exact solution for $\sigma = 1$ when $\alpha = \beta = 0.01$ and when $\alpha = 0.1$ and $\beta = 0$, respectively (a) and (b) for Example 1.

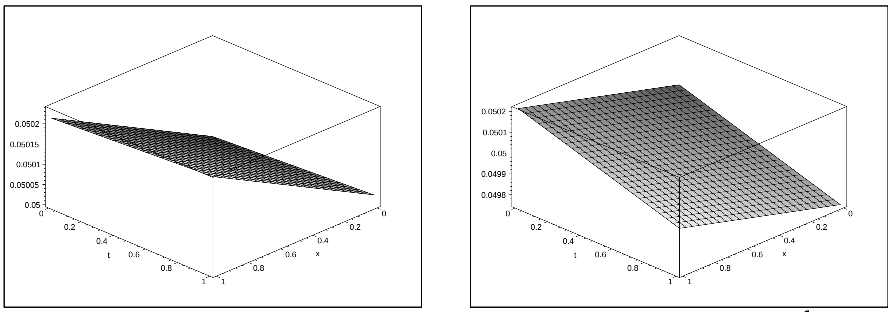

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Figure 4: The evolution results for the generalized Burgers-Huxley equation for $\sigma = 1$ when $\alpha = 1$, $\beta = \gamma = 0.1$: (a) exact solution, (b) the approximate solution obtained by LD method for Example 2.

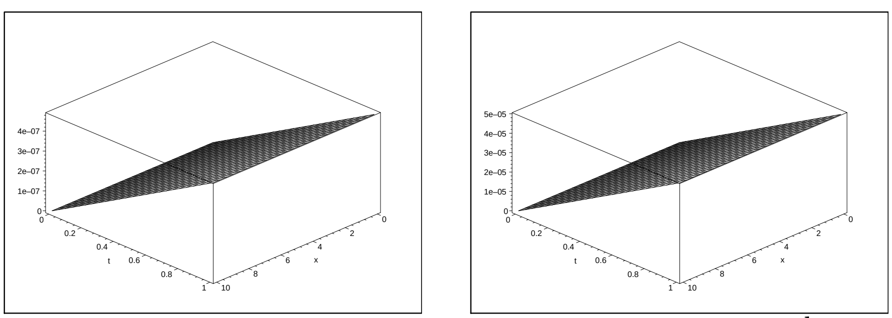

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Figure 5: The absolute errors of $u(x,t)$ with exact solution for $\sigma = 1$ when $\alpha = 0.1$, $\beta = 1$, $\gamma = 0.001$ and when $\alpha = 0.01$ and $\beta = 1$ and $\gamma = 0.01$, respectively (a) and (b) for Example 2.

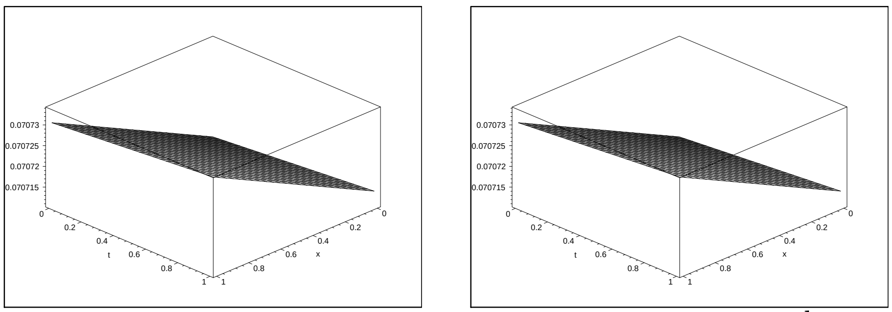

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Figure 6: The evolution results for the generalized Burgers-Huxley equation for $\sigma = 2$ when $\alpha = \beta = \gamma = 0.01$: (a) exact solution, (b) the approximate solution obtained by LD method for Example 2.

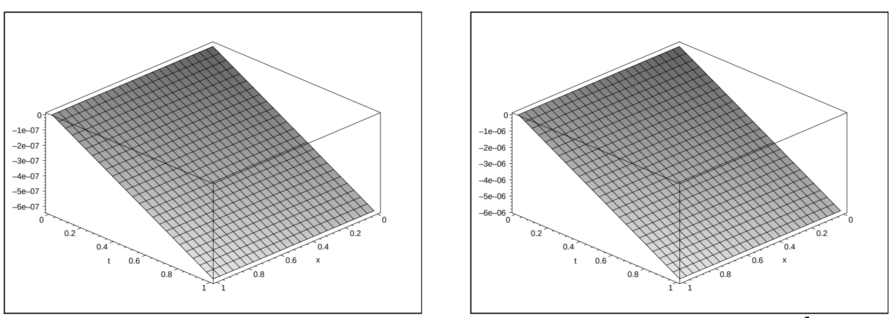

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Figure 7: The absolute errors of the generalized Burgers-Huxley equation for $\sigma = 2$ in comparison with exact solution for $\sigma = 2$ when $\alpha = 0.1$, $\beta = 1$, $\gamma = 0.001$ and when $\alpha = \beta = 0.1$ and $\gamma = 0.01$, respectively (a) and (b) for Example 2.

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